Mathematics for Economists

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Solutions to the problem set 5:

Question 1:

$$\max_{x,y} u(x,y) = x^{\alpha} y^{1-\alpha}$$

$$\text{Subject to} \quad p_x x + p_y y \le w$$

a)

We know that the feasible set is bounded and closed so it is compact. Moreover the optimization function is continuous so according to the Weiertrass theorem the maximum exists.

All constraint functions are linear and the feasible set is convex, so if u is strictly increasing and quasiconcave, the KT conditions are necessary and sufficient for an optimum.

For the last part, since:

$$\lim_{x \to 0} \frac{\partial u}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} = \infty$$

And

$$\lim_{y \to 0} \frac{\partial u}{\partial y} = \alpha y^{\alpha-1} x^{1-\alpha} = \infty$$

K-T conditions implies that at optimum $x, y > 0$, so all the solution points are interior. Moreover, since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are always strictly positive then λ_0 is positive and the budget constraint is binding.

b)

$$MRS_{x,y} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{p_x}{p_y} = \frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha) x^{\alpha} y^{-\alpha}} = \frac{\alpha y}{(1-\alpha) x} = \frac{p_x}{p_y}$$

$$y = x . \frac{p_x}{p_y} \frac{1-\alpha}{\alpha}$$

c)

From the budget constraint we have $\quad p_x x + p_y y = w$

$$p_x x + p_y x . \left(\frac{p_x (1-\alpha)}{p_y \alpha} \right) = w \Rightarrow x^* = \frac{\alpha w}{p_x}$$

And

$$y^* = \frac{(1-\alpha) w}{p_y}$$

$$\lambda_0 = \frac{\partial u}{\partial x} . \frac{1}{p_x} = \frac{\alpha x^{\alpha-1} y^{1-\alpha}}{p_x} > 0$$

We got the last inequality from the fact that $x, y, \alpha \; and \; p_x$ are positive elements.

d)

$$u^* = (x^*)^\alpha (y^*)^{1-\alpha} = \left(\frac{\alpha w}{p_x}\right)^\alpha . \left(\frac{(1-\alpha)w}{p_y}\right)^{1-\alpha} = w\left(\frac{\alpha}{p_x}\right)^\alpha . \left(\frac{(1-\alpha)}{p_y}\right)^{1-\alpha}$$

e)

$$v(p,w) = \max_{px=w} u(x,y) = w\left(\frac{\alpha}{p_x}\right)^\alpha . \left(\frac{(1-\alpha)}{p_y}\right)^{1-\alpha}$$

From Roy's identity:

$$x(p,w) = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial w}} = \frac{\alpha w}{p_x}$$

$$y(p,w) = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial w}} = \frac{(1-\alpha)w}{p_y}$$

Question 2:

a) The consumer can save in the first period but cannot borrow so the constraints are:

$$0 \le c_1 \le w_1 \quad and \quad 0 \le c_2 + c_1 \le w_2 + w_1$$

b) Now the consumer can borrow at the first stage:

$$0 \le c_1 \le w_1 + b \quad and \quad 0 \le c_2 + c_1 \le w_2 + w_1$$

c) The problem in the case of the identical utility functions:

$$\max_{c_1,c_2} u(c_1) + u(c_2)$$
$$\text{s.t: } c_1 \le w_1 + b$$
$$c_2 + c_1 \le w_2 + w_1$$

First we form the Lagrangian:

$$L = u(c_1) + u(c_2) - \lambda_1(c_1 - w_1 - b) - \lambda_2(c_1 + c_2 - w_1 - w_2)$$

Now we write the focs:

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial c_2} = u'(c_2) - \lambda_2 = 0$$
$$\lambda_1(c_1 - w_1 - b) = 0$$
$$\lambda_2(c_1 + c_2 - w_1 - w_2) = 0$$
$$c_1 \le w_1 + b$$
$$c_2 \le w_2 + w_1 - c_1$$
$$c_1, c_2 > 0$$
$$\lambda_1, \lambda_2 \ge 0$$

According to assumptions, $u'(c_t) > 0$ so $\lambda_2 > 0$ so the second constraint is binding and:

$$c_1 + c_2 = w_1 + w_2$$

If we assume that $\lambda_1 \neq 0$ so the first constraint is also binding and $c_1 = w_1 + b$

And

$$c_2 = w_2 - b$$

In the case of $w_2 < b$, c_2 will be negative and it is not a valid solution so $\lambda_1 = 0$, then from focs:

$$u'(c_2) = u'(c_1)$$

And because u' is strictly increasing $c_1 = c_2 = c$

So

$$c^* = \frac{w_1 + w_2}{2}$$

d)

Now the problem is:

$$\max_{c_1, c_2} u(c_1) + \delta u(c_2)$$

$$\text{s.t: } c_1 \leq w_1 + b$$

$$c_2 \leq w_2 + w_1 - c_1$$

All of the steps are the same as the part a and at the end:

$$u'(c_1) = \delta u'(c_2)$$

Considering the facts that:

$$\delta \in (0,1) \;\; and \;\; u''(c) < 0$$

We conclude that:

$$c_1 > c_2$$

e)

According to part d when the consumer is impatient and $u_2(c) = \delta u_1(c)$ and $\delta \in (0,1)$, marginal utility of consumer in the second period will be lower than the one in the first period.

Question 3:

$$\max_{c_1, c_2} \ln(c_1) + \delta \ln(c_2)$$

$$c_2 \leq w_2 + (1+r)(w_1 - c_1)$$

a)

$$L = \ln(c_1) + \delta \ln(c_2) - \mu(c_2 - w_2 - (1+r)(w_1 - c_1))$$

$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1} - \mu(1+r) = 0$$

$$\frac{\partial L}{\partial c_2} = \frac{\delta}{c_2} - \mu = 0$$

$$\mu\left(c_2 - w_2 - (1+r)(w_1 - c_1)\right) = 0$$

$$c_2 \leq w_2 + (1+r)(w_1 - c_1)$$

$$c_1, c_2 > 0$$

b) The constraint function is linear so the feasible set is convex. Moreover, the utility function is strictly increasing and quasiconcave so the first order K-T conditions are sufficient to solve the problem.

c)

$$\frac{\partial L}{\partial c_2} = \frac{\delta}{c_2} - \mu = 0 \Rightarrow \mu \neq 0 \Rightarrow budget\ constraint\ is\ binding$$

Moreover

$$\frac{\delta}{c_2} = \mu = \frac{1}{c_1(1+r)} \Rightarrow c_2 = c_1 \delta (1+r)$$

Putting this into the budget constraint we have:

$$c_1\delta(1+r) = w_2 + (1+r)(w_1 - c_1) \Rightarrow c_1^* = \frac{w_2 + w_1(1+r)}{(1+\delta)(1+r)}$$

$$c_2^* = \frac{\delta(w_2 + w_1(1+r))}{(1+\delta)}$$

d)

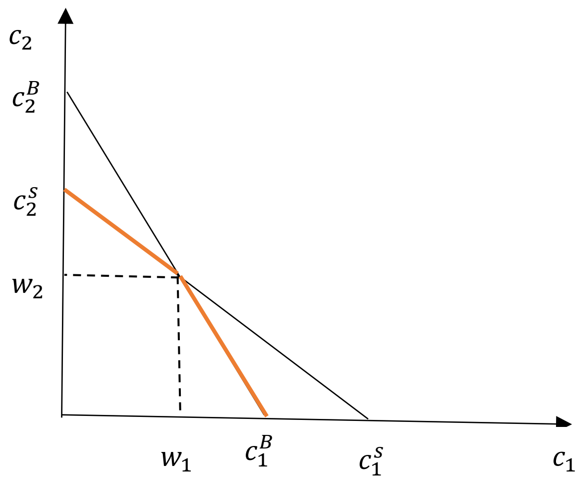
$$c_2 = c_1\delta(1+r) \text{ and } c_1, c_2 > 0$$

If $c_1^* < c_2^*$, then $\delta(1+r) > 1$

Question 4:

a)

$$\max_{c_1,c_2} \ln(c_1) + \delta \ln(c_2)$$
$$w_1 - c_1 < 0 \to c_2 \le w_2 + (1+\bar{r})(w_1 - c_1)$$
$$w_1 - c_1 > 0 \to c_2 \le w_2 + (1+\underline{r})(w_1 - c_1)$$



b)

$$1 + \underline{r} \le MRS_{(c_1=w_1, c_2=w_2)} \le 1 + \bar{r}$$

Otherwise it is not going to be optimum.

C,d)

$$if \;\; 1 + \underline{r} < MRS(c_1 = w_1, c_2 = w_2) < 1 + \bar{r}$$

According to this condition you are at the optimum and there is no way (by saving or borrowing) you could increase your utility.

$$if \quad MRS(c_1 = w_1, c_2 = w_2) > 1 + \bar{r}$$

High MRS means that we prefer to borrow money to increase c_1 and decrease c_2. According to the figure in part a, when MRS is high at the point $(c_1 = w_1, c_2 = w_2)$ we are at the bottom of the $1 + \bar{r}$ budget constraint line. We can increase our utility by borrowing and decreasing c_2 to increase the consumption in period one c_1.

$$if \quad 1 + \underline{r} > MRS(c_1 = w_1, c_2 = w_2)$$

Low MRS means that we want to save money to increase c_2 and decrease c_1. (we are at the bottom of the $1 + \underline{r}$ budget line)

e)

Before solving this problem we should consider that this is exactly the same as question 3. The only difference is that now we should solve the problem for two different cases with two different budget lines, so at the end one of them will be our final solution for the optimization problem. According to the results of the question 3 we have:

For points where $c_1 < w_1 \; and \; c_2 > w_2$:

$$c_{1,1}^* = \frac{w_2 + w_1(1 + \bar{r})}{(1 + \delta)(1 + \bar{r})}$$

$$c_{2,1}^* = \frac{\delta(w_2 + w_1(1 + \bar{r}))}{(1 + \delta)}$$

For points where $c_1 > w_1 \; and \; c_2 < w_2$:

$$c_{1,2}^* = \frac{w_2 + w_1(1 + \underline{r})}{(1 + \delta)(1 + \underline{r})}$$

$$c_{2,2}^* = \frac{\delta(w_2 + w_1(1 + \underline{r}))}{(1 + \delta)}$$

Only one of these solutions will be valid (according to the conditions over c_1, c_2) at the end.

Question 5:

a)

We will use the results of the question 3.

$$c_1^* = \frac{w_2 + w_1(1 + r)}{2(1 + r)}$$

$$c_2^* = \frac{w_2 + w_1(1 + r)}{2}$$

And we assume that $\delta = 1$.

For students $w_1^s = 0, w_2^s = 70$, so

$$c_1^s = \frac{35}{(1+r)} \quad and \quad c_2^s = 35$$

$$b = c_1^s = \frac{35}{(1+r)}$$

And total borrowing is:

$$nb = \frac{35n}{(1+r)}$$

For trust fund kids:

$$c_1^t = 50 \quad and \quad c_2^t = 50(1+r)$$

$$S = saving\ for\ each\ person = 100 - 50 = 50$$

And total saving is equal to

$$mS = 50m$$

b)

$$\frac{35n}{(1+r)} = 50m \Rightarrow r^* = \frac{7n}{10m} - 1$$

c) Considering also w_1^t we have:

$$\frac{35n}{(1+r)} = m(w_1^t - 50) \Rightarrow r^* = \frac{35n}{m(w_1^t - 50)} - 1$$

So increasing w_1^t will decrease equilibrium interest rate. So by increasing w_1^t, c_1^s will be increased but c_2^s will be constant. On the other hand, c_1^t will be increased and c_2^t is equal to:

$$c_2^* = \frac{w_1(1+r)}{2}$$

$$r^* = \frac{35n}{m(w_1 - 50)} - 1$$

So:

$$c_2^* = \frac{w_1 * 35n}{2m(w_1 - 50)} = A\frac{w_1}{(w_1 - 50)}$$

$$\frac{dc_2^*}{dw_1} = A\frac{-50}{(w_1 - 50)^2}$$

Which is always negative for $w_1 \neq 50$, so by increasing w_1, c_2^* will be decreased, but what about over all? Does the trust fund kid increase his welfare by lending money to the student?

The total amount of consumption after lending money is:

$$c_1^t + c_2^t = \frac{w_1}{2} + \frac{w_1(1+r)}{2} = \frac{w_1}{2}(2+r)$$

And using the fact that:

$$r^* = \frac{35n}{m(w_1 - 50)} - 1$$

We can conclude that:

$$C^t = \frac{w_1}{2}\left(\frac{35n}{m(w_1 - 50)} + 1\right)$$

Without lending money, one trust fund kid would consume exactly w_1. So for the kid to be better off with lending money we should have:

$$\frac{w_1}{2}\left(\frac{35n}{m(w_1 - 50)} + 1\right) > w_1$$

So

$$\frac{35n}{m(w_1 - 50)} > 1 \rightarrow w_1 < 50 + \frac{35n}{m}$$