Practice Questions:

Question 1:

a)

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 2 \\ 2 & 4 & 2 & 2 \\ 0 & 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Using the augmented matrix of the coefficients we have:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & b_1 \\ 0 & 0 & 0 & 2 & b_2 \\ 2 & 4 & 2 & 2 & b_3 \\ 0 & 2 & 0 & 3 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & b_1 \\ 0 & 0 & 0 & 2 & b_2 \\ 0 & 0 & 0 & -6 & b_3 - 2b_1 \\ 0 & 2 & 0 & 3 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & b_1 \\ 0 & 0 & 0 & 2 & b_2 \\ 0 & 0 & 0 & -6 & b_3 - 2b_1 \\ 0 & 2 & 0 & 0 & b_4 - \frac{3}{2}b_2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & b_1 \\ 0 & 2 & 0 & 0 & b_4 - \frac{3}{2}b_2 \\ 0 & 0 & 0 & 2 & b_2 \\ 0 & 0 & 0 & -6 & b_3 - 2b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & b_1 \\ 0 & 2 & 0 & 0 & b_4 - \frac{3}{2}b_2 \\ 0 & 0 & 0 & 2 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 + 3b_2 \end{array}\right]$$

The system of linear equations has infinite solutions if and only if:

$$b_3 - 2b_1 + 3b_2 = 0$$

b)

$$\begin{bmatrix} 3 & 5 & 9 \\ 1 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Using cramer's law we have:

$$x = \frac{det\begin{bmatrix} 1 & 5 & 9 \\ 2 & 2 & 8 \\ 1 & 0 & 7 \end{bmatrix}}{det\begin{bmatrix} 3 & 5 & 9 \\ 1 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}} = \frac{1 * (5 * 8 - 2 * 9) + 7 * (1 * 2 - 5 * 2)}{7(3 * 2 - 5 * 1)} = \frac{22 - 56}{7} = \frac{-34}{7}$$

$$y = \frac{det\begin{bmatrix} 3 & 1 & 9 \\ 1 & 2 & 8 \\ 0 & 1 & 7 \end{bmatrix}}{det\begin{bmatrix} 3 & 5 & 9 \\ 1 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}} = \frac{-1 * (3 * 8 - 1 * 9) + 7 * (3 * 2 - 1 * 1)}{7(3 * 2 - 5 * 1)} = \frac{-15 + 35}{7} = \frac{20}{7}$$

$$z = \frac{det\begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}}{det\begin{bmatrix} 3 & 5 & 9 \\ 1 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}} = \frac{1 * (3 * 2 - 1 * 5)}{7(3 * 2 - 5 * 1)} = \frac{1}{7}$$

c)

$$Ax = b \ \ and \ \ Ax_1 = b \ , \quad Ax_2 = b$$

$$x^{\lambda} = \lambda x_1 + (1-\lambda)x_2$$

$$Ax^{\lambda} = A(\lambda x_1 + (1-\lambda)x_2) = A\lambda x_1 + A(1-\lambda)x_2$$

Since λ , $(1-\lambda)$ are real numbers:

$$Ax^{\lambda} = \lambda Ax_1 + (1-\lambda)Ax_2 = \lambda b + (1-\lambda)b = b$$

Question 2:

$$\max_{x,y} f(x,y) = x^{\frac{1}{3}}y^{\frac{1}{2}} - \alpha x - y$$

a)

$$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{2}} - \alpha = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}x^{\frac{1}{3}}y^{-\frac{1}{2}} - 1 = 0$$

So

$$y^{\frac{1}{2}} = 3\alpha x^{\frac{2}{3}}$$

$$x^{\frac{1}{3}} = 2y^{\frac{1}{2}}$$

So the only critical point is $(x^* = 0, y^* = 0)$

b) We should form the Hessian matrix

$$H = \begin{bmatrix} -\frac{2}{9}x^{-\frac{5}{3}}y^{\frac{1}{2}} & \frac{1}{6}x^{-\frac{2}{3}}y^{-\frac{1}{2}} \\ \frac{1}{6}x^{-\frac{2}{3}}y^{-\frac{1}{2}} & -\frac{1}{4}x^{\frac{1}{3}}y^{-\frac{3}{2}} \end{bmatrix}$$

Calculating the determinant of the Hessian Matrix we have:

$$\det(H) = \frac{1}{36}x^{-\frac{4}{3}}y^{-1} > 0$$

And the first element of the matrix is negative, SO the hessian matrix is negative definite and the function f is strictly concave. Consequently, the point $(x^* = 0, y^* = 0)$ is maximum.

c)

x^*, y^* are endogenous variables
α is an exogenous variable

$$f_1 = \frac{\partial f}{\partial x} = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{2}} - \alpha = 0$$

$$f_2 = \frac{\partial f}{\partial y} = \frac{1}{2}x^{\frac{1}{3}}y^{-\frac{1}{2}} - 1 = 0$$

So

$$D_y f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

We calculated the determinant of this matrix before and we know that it is equal to

$$\frac{1}{36} x^{*-\frac{4}{3}} y^{*-1} > 0$$

With the assumption that $x^*, y^* > 0$. Moreover, at the point (x^*, y^*) the gradient of the matrix is zero so we have two constraints of the implicit function theorem.

d)

We also know that

$$D_x f = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Using cramer's rule we have:

$$\begin{bmatrix} -\frac{2}{9} x^{-\frac{5}{3}} y^{\frac{1}{2}} & \frac{1}{6} x^{-\frac{2}{3}} y^{-\frac{1}{2}} \\ \frac{1}{6} x^{-\frac{2}{3}} y^{-\frac{1}{2}} & -\frac{1}{4} x^{\frac{1}{3}} y^{-\frac{3}{2}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} d\alpha = 0$$

$$dx^* = \frac{\det \begin{bmatrix} -1 & \frac{1}{6} x^{*-\frac{2}{3}} y^{*-\frac{1}{2}} \\ 0 & -\frac{1}{4} x^{*\frac{1}{3}} y^{*-\frac{3}{2}} \end{bmatrix}}{\frac{1}{36} x^{*-\frac{4}{3}} y^{*-1}} = \frac{\frac{1}{4} x^{*\frac{1}{3}} y^{*-\frac{3}{2}}}{\frac{1}{36} x^{*-\frac{4}{3}} y^{*-1}} d\alpha$$

AND

$$dy^* = \frac{\det \begin{bmatrix} -\frac{2}{9} x^{*-\frac{5}{3}} y^{*\frac{1}{2}} & -1 \\ \frac{1}{6} x^{*-\frac{2}{3}} y^{*-\frac{1}{2}} & 0 \end{bmatrix}}{\frac{1}{36} x^{*-\frac{4}{3}} y^{*-1}} = \frac{\frac{1}{6} x^{*-\frac{2}{3}} y^{*-\frac{1}{2}}}{\frac{1}{36} x^{*-\frac{4}{3}} y^{*-1}} d\alpha$$

Question 3:

$$\max_{x,y} \frac{1}{2} \ln x + \ln y$$

$$s, t: x + 2y = 2 \,, x, y > 0$$

a)

$$L = \frac{1}{2}\ln x + \ln y - \mu(x + 2y - 2)$$

First order conditions:

$$\frac{\partial L}{\partial x} = \frac{1}{2x} - \mu = 0$$

$$\frac{\partial L}{\partial y} = \frac{1}{y} - 2\mu = 0$$

$$\frac{\partial L}{\partial \mu} = x + 2y - 2 = 0$$

b)

The feasible set is bounded but it is not closed.

The constraint function is linear so the feasible set is convex and because the utility function is strictly increasing and quasiconcave so the first order K-T conditions are sufficient to solve the problem.

c)

From the first two constraints and the fact that $x, y > 0$:

$$x = y$$

Using the third condition:

$$3x^* = 3y^* = 2 \rightarrow x^* = y^* = \frac{2}{3}$$

And

$$\mu^* = \frac{3}{4}$$

d)

If we have a utility maximization problem with one equality constraint then

$$\frac{\partial v(p, w)}{\partial w} = \mu$$

Where v is the value function (the indirect utility function) and can be derived when you solve the problem and insert the optimal values x^*, y^* into the utility function.

According to the envelope theorem, if the income increased by one unit, the maximal utility will be increased equal to the amount of the multiplier, so in here if we increase income by one, the maximal utility will be increased by $\frac{3}{4}$ unit.

Question 4:

$$x_{t+1} = x_t + 2y_t$$

$$y_{t+1} = -x_t + 4y_t$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}^{t+1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

We want to write the A matrix in the form:

$$A = PDP^{-1}$$

Where D is diagonal matrix.

We first need to obtain the eigenvalues of A.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = 0 \rightarrow \lambda_1 = 2\,, \lambda_2 = 3$$

For $\lambda_1 = 2$:

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda_1 = 3$:

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \; AND \;\; D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

So after all

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = PD^tP^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2^{t+1} - 3^t & -2^{t+1} + 2 * 3^t \\ 2^t - 3^t & -2^t + 2 * 3^t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

And since $\lambda_1, \lambda_2 > 1$, the steady state of the system is not stable.