

CHEM-E7190  
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Example II



Aalto University

# Linearisation of nonlinear models: Example

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Example II

## Example II

Linearisation of nonlinear models

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## Example II

We consider two irreversible chemical reactions in a perfectly mixed chemical reactor

- $A \rightarrow B \rightarrow C$
- $2A \rightarrow D$

The two reactions compete to convert species A, species B is the desired product

The chemical reactor operates in liquid-phase

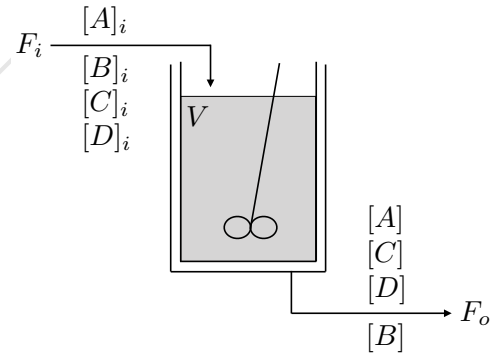
- ↪ Assume constant volume,  $V \neq V(t)$
- ↪ Assume constant density,  $\rho \neq \rho(t)$

Assume constant temperature

- ↪  $T(t) \neq T(t)$

Volumetric flow-rates

- ↪  $F_i(t)$  and  $F_o(t)$

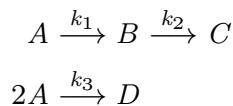


Our interest is in understanding the dynamics of the concentrations inside the reactor

- ↪ The concentration of species A, B, C and D, as a function of time
- $[A](t)$ ,  $[B](t)$ ,  $[C](t)$ , and  $[D](t)$  (molar concentrations,  $[\text{mol lt}^{-1}]$ )

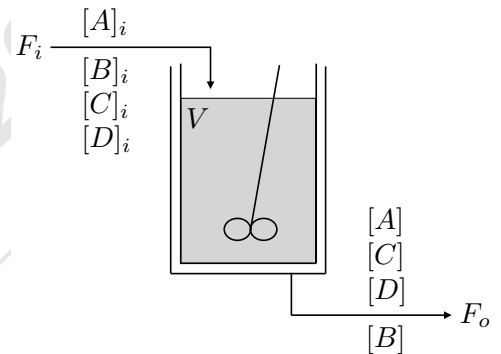
## Example II (cont.)

Reaction rate constants (per unit volume)



Assume that only component A is fed

- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$



The total material balance, under the assumption of a constant volume in the tank

### Total mass balance

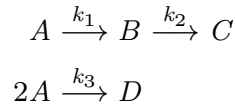
$$\frac{dV(t)}{dt} = F_i(t) - F_o(t) = 0$$

As a result, we simplify notation

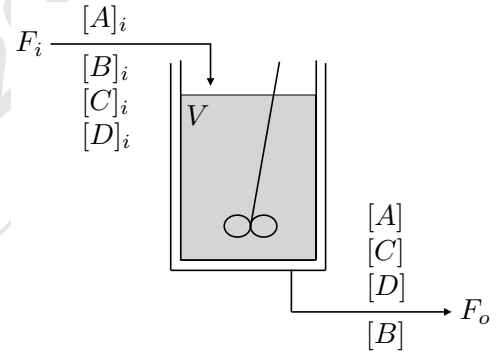
- $V(t) = \text{constant} = V$
- $F_i(t) = F_o(t) = F(t)$

Example II

Example II (cont.)



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$



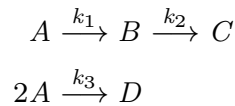
Mass balance for component A

$$\frac{d}{dt} V[A](t) = F(t)[A]_i(t) - F(t)[A](t) - Vk_1[A](t) - Vk_3[A]^2(t)$$

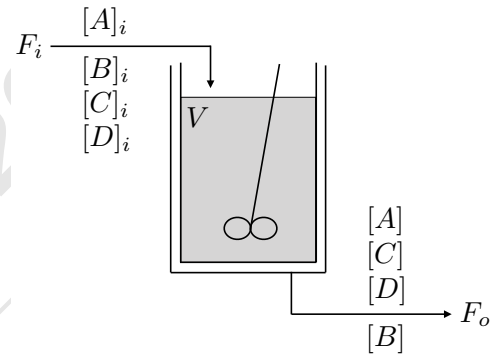
$$= \frac{F(t)}{V} ([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t)$$

Example II

Example II (cont.)



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$



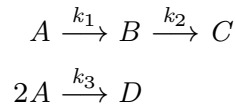
Mass balance for component B, C, and D

$$\frac{d}{dt} [B](t) = \frac{F(t)}{V} \underbrace{([B]_i(t) - [B](t))}_{=0} + k_1[A](t) - k_2[B](t)$$

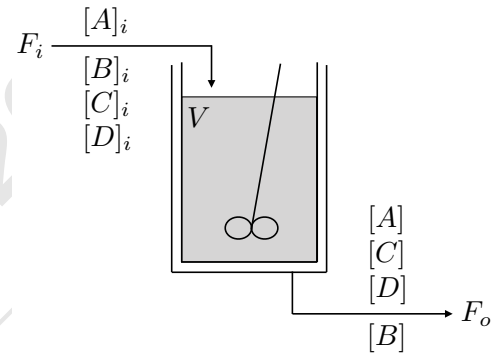
$$\frac{d}{dt} [C](t) = \frac{F(t)}{V} \underbrace{([C]_i(t) - [C](t))}_{=0} + k_2[B](t)$$

$$\frac{d}{dt} [D](t) = \frac{F(t)}{V} \underbrace{([D]_i(t) - [D](t))}_{=0} + \frac{1}{2} k_3[A]^2(t)$$

## Example II (cont.)



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$



Putting things together, we get the dynamics of the state-space model of the reactor

$$\frac{d}{dt}[A](t) = \frac{F(t)}{V}([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t)$$

$$\frac{d}{dt}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t) \quad k_1 = 5/6 \text{ [min}^{-1}\text{]}$$

$$\frac{d}{dt}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t) \quad k_2 = 5/3 \text{ [min}^{-1}\text{]}$$

$$\frac{d}{dt}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t) \quad k_3 = 1/6 \text{ [lt(mol}^{-1}\text{min}^{-1}\text{)]}$$

## Example II (cont.)

$$\frac{d}{dt}[A](t) = \frac{F(t)}{V}([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t)$$

$$\frac{d}{dt}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$

$$\frac{d}{dt}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$

$$\frac{d}{dt}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

↪ State variables,  $x(t)$

$$x(t) = \begin{bmatrix} [A](t) \\ [B](t) \\ [C](t) \\ [D](t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

## Example II (cont.)

$$\begin{aligned}\frac{d}{dt}[A](t) &= \frac{F(t)}{V}([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t) \\ \frac{d}{dt}[B](t) &= -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t) \\ \frac{d}{dt}[C](t) &= -\frac{F(t)}{V}[C](t) + k_2[B](t) \\ \frac{d}{dt}[D](t) &= -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)\end{aligned}$$

↪ Input variables,  $u(t)$

$$u(t) = \begin{bmatrix} F_i(t) \\ [A]_i(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

## Example II (cont.)

$$\begin{aligned}\frac{d}{dt}[A](t) &= \frac{F(t)}{V}([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t) \\ \frac{d}{dt}[B](t) &= -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t) \\ \frac{d}{dt}[C](t) &= -\frac{F(t)}{V}[C](t) + k_2[B](t) \\ \frac{d}{dt}[D](t) &= -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)\end{aligned}$$

↪ Parameters,  $\theta_x$

$$\theta_x = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ V \end{bmatrix} = \begin{bmatrix} \theta_{x,1} \\ \theta_{x,2} \\ \theta_{x,3} \\ \theta_{x,4} \end{bmatrix}$$

## Example II (cont.)

$$\begin{aligned}\frac{d}{dt}[A](t) &= \frac{F(t)}{V}([A]_i(t) - [A](t)) - k_1[A](t) - k_3[A]^2(t) \\ \frac{d}{dt}[B](t) &= -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t) \\ \frac{d}{dt}[C](t) &= -\frac{F(t)}{V}[C](t) + k_2[B](t) \\ \frac{d}{dt}[D](t) &= -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)\end{aligned}$$

Using the control notation, we get

$$\begin{aligned}\frac{dx_1(t)}{dt} &= \frac{u_1(t)}{\theta_{x,4}}(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - k_3x_1^2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t) \\ \frac{dx_3(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t) \\ \frac{dx_4(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)\end{aligned}$$

## Example II (cont.)

$$\begin{aligned}\frac{dx_1(t)}{dt} &= \underbrace{\frac{u_1(t)}{\theta_{x,4}}(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - k_3x_1^2(t)}_{f_1(x, u|\theta_x)} \\ \frac{dx_2(t)}{dt} &= \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)}_{f_2(x, u|\theta_x)} \\ \frac{dx_3(t)}{dt} &= \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t)}_{f_3(x, u|\theta_x)} \\ \frac{dx_4(t)}{dt} &= \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)}_{f_4(x, u|\theta_x)}\end{aligned}$$

$$\rightsquigarrow \dot{x}(t) = f(x(t), u(t)|\theta_x)$$

## Example II (cont.)

$$\begin{aligned}\frac{dx_1(t)}{dt} &= \frac{u_1(t)}{\theta_{x,4}}(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - k_3x_1^2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t) \\ \frac{dx_3(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t) \\ \frac{dx_4(t)}{dt} &= -\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)\end{aligned}$$

Suppose that we are capable of measuring the concentration of B, we then also have

$$y(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{g(x(t), u(t) | \theta_x)} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_D \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

## Example II (cont.)

The dynamics are a set of nonlinear equations, the measurement equation is linear

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \frac{u_1(t)}{\theta_{x,4}}(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - k_3x_1^2(t) \\ -\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t) \\ -\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t) \\ -\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

To be able to proceed with the tools of linear systems theory, we need to linearise

- Approximate nonlinearities with first-order Taylor series expansions
- About some convenient steady-state point,  $(x^{SS}, u^{SS})$

$$\begin{aligned}x^{SS} &= [x_1^{SS} \quad x_2^{SS} \quad x_2^{SS} \quad x_2^{SS}]^T = [A]^{SS} [B]^{SS} [C]^{SS} [D]^{SS}{}^T \\ u^{SS} &= [u_1^{SS} \quad u_2^{SS}]^T = [F_i^{SS} \quad [A]_i^{SS}]^T\end{aligned}$$

## Example II (cont.)

How to determine the steady-state point associated to a desirable operating conditions?

- By simulation, integrate the model until stationarity is reached
- By optimisation, solve  $f(x, u) = 0$  with respect to  $x$  and  $u$

## Example II (cont.)

Sometimes, it can also be worked out from the model equations at steady-state ( $x_{SS}, u_{SS}$ )

At steady-state all derivative are zero, for component  $[A]$  we thus have

$$\begin{aligned} \frac{d[A](t)}{dt} &= \frac{F_i^{SS}}{V} ([A]_i^{SS} - [A](t)) - k_1[A](t) - k_3[A]^2(t) \\ &= -k_3[A]^2(t) - [A](t) \left( \frac{F_i^{SS}}{V} + k_1 \right) + \frac{F_i^{SS}}{V} [A]_i^{SS} \\ &= 0 \end{aligned}$$

We get the second-order equation in the variable  $[A](t)$ ,

$$k_3[A]^2(t) + \left( \frac{F_i^{SS}}{V} + k_1 \right) [A](t) - \frac{F_i^{SS}}{V} [A]_i^{SS} = 0$$

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Second-order equation:  $ax^2 + bx + c = 0$  with solutions  $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



## Example II (cont.)

$$\underbrace{k_3[A]^2(t)}_{ax^2} + \underbrace{\left(\frac{F_i^{SS}}{V} + k_1\right)[A](t)}_{bx} - \underbrace{\frac{F_i^{SS}}{V}[A]_i}_{-c} = 0$$

The steady-state values for  $[A]$ , given  $F_i^{SS}$  and  $[A]_i^{SS}$

$$[A]_{1,2}^{SS} = \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} \pm \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}}$$

We need to consider only the root where  $[A]$  is positive,

$$[A]^{SS} = \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}}$$

## Example II (cont.)

Proceeding similarly for component  $[B]$ , we can write

$$\begin{aligned} \frac{d[B](t)}{dt} &= -[B](t) \left(\frac{F_i^{SS}}{V} + k_2\right) + k_1 \underbrace{[A](t)}_{[A]^{SS}} \\ &= 0 \end{aligned}$$

We get the first-order equation in  $[B](t)$

$$[B](t) \left(\frac{F_i^{SS}}{V} + k_2\right) - k_1[A]^{SS} = 0$$

The steady-state value for  $[B]$ ,

$$[B]^{SS} = \frac{k_1[A]^{SS}}{\left(\frac{F_i^{SS}}{V} + k_2\right)}$$

given  $F_i^{SS}$ ,  $[A]_i^{SS}$ , and  $[A]^{SS}$

## Example II (cont.)

Substituting  $[A]^{SS}$ , we get

$$[B]^{SS} = \frac{k_1[A]^{SS}}{\left(\frac{F^{SS}}{V} + k_2\right)}$$

$$= k_1 \left( \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V} [A]_i^{SS}}{2k_3}} \right) \left( \frac{F^{SS}}{V} + k_2 \right)$$

## Example II (cont.)

For component  $[C]$ , we have

$$\frac{d[C](t)}{dt} = -[C](t) \left( \frac{F_i^{SS}}{V} \right) + k_2 \underbrace{[B](t)}_{[B]^{SS}}$$

$$= 0$$

We get the equation,

$$\left( \frac{F_i^{SS}}{V} \right) [C](t) - k_2 [B]^{SS} = 0$$

The steady-state value for  $[C]$ ,

$$[C]^{SS} = \frac{k_2 [B]^{SS}}{\left( \frac{F_i^{SS}}{V} \right)}$$

given  $F_i^{SS}$ ,  $[A]_i^{SS}$ ,  $[A]^{SS}$ , and  $[B]^{SS}$

## Example II (cont.)

Example II

Substituting  $[B]^{SS}$ , we get

$$[C]^{SS} = \frac{k_2[B]^{SS}}{\left(\frac{F_i^{SS}}{V}\right)}$$

$$= k_2 \frac{\left( \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V} [A]_i^{SS}}{2k_3}} \right)}{\left(\frac{F_i^{SS}}{V} + k_2\right)}$$

## Example II (cont.)

Example II

Finally, for component  $[D]$  we have

$$\frac{d[D](t)}{dt} = -[D](t) \left(\frac{F_i^{SS}}{V}\right) + \frac{1}{2} k_3 \underbrace{[A]^2(t)}_{([A]^{SS})^2}$$

$$= 0$$

We get the equation,

$$\left(\frac{F_i^{SS}}{V}\right) [D](t) - \frac{1}{2} k_3 ([A]^{SS})^2 = 0$$

The steady-state value for  $[D]$ ,

$$[D]^{SS} = \frac{\frac{1}{2} k_3 ([A]^{SS})^2}{\left(\frac{F_i^{SS}}{V}\right)}$$

given  $F_i^{SS}$ ,  $[A]_i^{SS}$ ,  $[A]^{SS}$ , and  $[B]^{SS}$

## Example II (cont.)

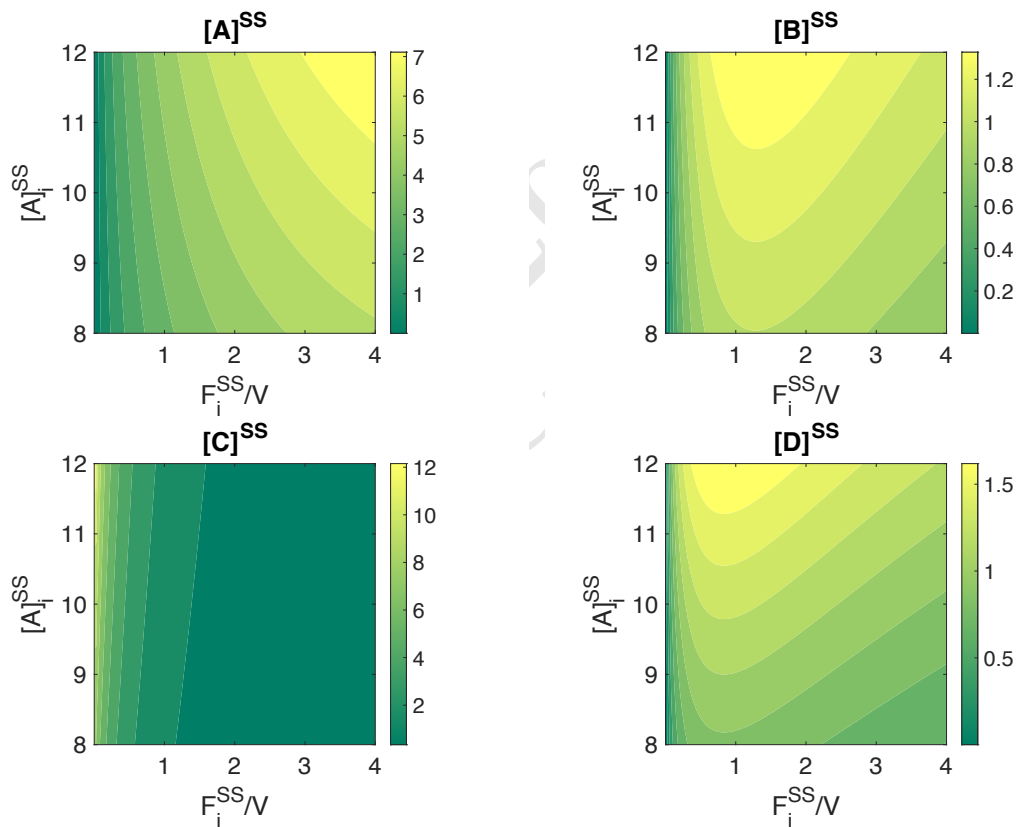
Example II

Substituting  $[A]^{SS}$ , we get

$$[D]^{SS} = \frac{1}{2}k_3 \frac{\left( \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V} [A]_i^{SS}}{2k_3}} \right)^2}{\left(\frac{F_i^{SS}}{V}\right)}$$

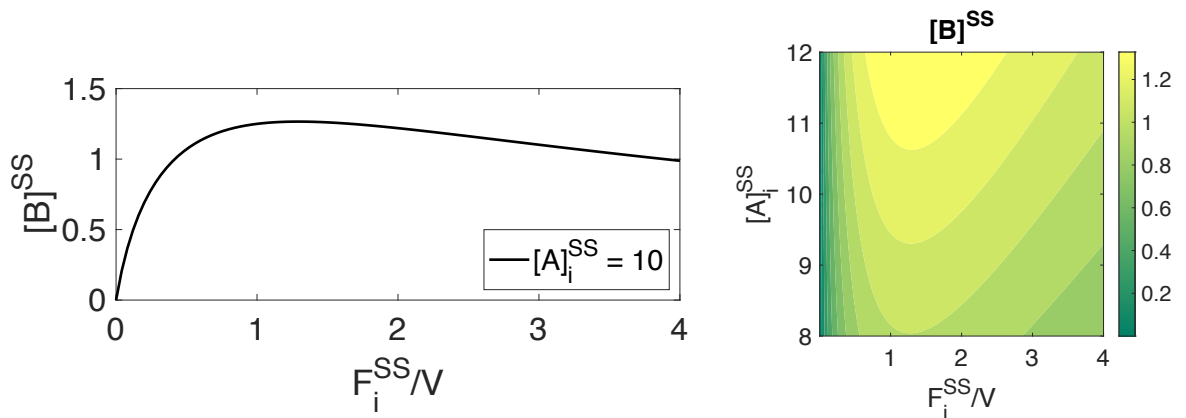
## Example II (cont.)

Example II



## Example II (cont.)

Where would you operate the reactor if told that the feed composition is  $[A]_i^{SS} = 10$ ?



## Example II (cont.)

We could define desirable operating conditions

$$u^{SS} = \begin{bmatrix} \frac{F_i^{SS}}{V} = \frac{4}{7} \text{ min}^{-1} \\ [A]_i^{SS} = 10 \text{ mol l}^{-1} \end{bmatrix} = \begin{bmatrix} F_i^{SS} \\ [A]_i^{SS} \end{bmatrix} \rightsquigarrow \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix}$$

Then, determine the corresponding fixed point

$$x^{SS} = \begin{bmatrix} 3.0000 \text{ mol lt}^{-1} \\ 1.1170 \text{ mol lt}^{-1} \\ 3.2580 \text{ mol lt}^{-1} \\ 1.3125 \text{ mol lt}^{-1} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix} \rightsquigarrow [x_1^{SS} \quad x_2^{SS} \quad x_2^{SS} \quad x_2^{SS}]^T$$

Note that we replaced the first input variable (the feed flow-rate,  $F_i(t)$ )

- We will use the space-velocity  $F_i(t)/V$ , instead
- No difference, as the volume  $V$  is constant

## Example II (cont.)

Given a steady-state point  $((x_1^{SS}, x_2^{SS}, x_3^{SS}, x_4^{SS}), (u_1^{SS}, u_2^{SS}))$ , we linearise the model

We start by defining the deviation variables, for both state- and input variables

- For the state variables, we have

$$x'(t) = \begin{bmatrix} x_1(t) - x_1^{SS} \\ x_2(t) - x_2^{SS} \\ x_3(t) - x_3^{SS} \\ x_4(t) - x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A](t) - [A]^{SS} \\ [B](t) - [B]^{SS} \\ [C](t) - [C]^{SS} \\ [D](t) - [D]^{SS} \end{bmatrix}$$

- For the input variables, we have

$$u'(t) = \begin{bmatrix} u_1(t) - u_1^{SS} \\ u_2(t) - u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i(t)/V - F_i^{SS}/V \\ [A]_i(t) - [A]_i^{SS} \end{bmatrix}$$

Then proceed by computing the Jacobians of dynamics at steady-state  $(x_{SS}, u_{SS})$

↪ State matrix  $A$  and input matrix  $B$

↪  $\dot{x}'(t) = Ax'(t) + Bu'(t)$

## Example II (cont.)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \underbrace{u_1(t)(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - \theta_{x,3}x_1^2(t)}_{f_1} \\ \underbrace{-u_1(t)x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)}_{f_2} \\ \underbrace{-u_1(t)x_3(t) + \theta_{x,2}x_2(t)}_{f_3} \\ \underbrace{-u_1(t)x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)}_{f_4} \end{bmatrix}$$

↪  $A =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}_{SS} = \begin{bmatrix} -u_1 - \theta_{x,1} - 2\theta_{x,3}x_1 & 0 & 0 & 0 \\ \theta_{x,1} & -u_1 - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1 & 0 \\ \theta_{x,3}x_1 & 0 & 0 & -u_1 \end{bmatrix}_{SS}$$

## Example II (cont.)

We get,

$$\begin{aligned}
 A &= \begin{bmatrix} -u_1 - \theta_{x,1} - 2\theta_{x,3}x_1 & 0 & 0 & 0 \\ \theta_{x,1} & -u_1 - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1 & 0 \\ \theta_{x,3}x_1 & 0 & 0 & -u_1 \end{bmatrix}_{SS} \\
 &= \begin{bmatrix} -u_1^{SS} - \theta_{x,1} - 2\theta_{x,3}x_1^{SS} & 0 & 0 & 0 \\ \theta_{x,1} & -u_1^{SS} - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1^{SS} & 0 \\ \theta_{x,3}x_1^{SS} & 0 & 0 & -u_1^{SS} \end{bmatrix} \\
 &= \begin{bmatrix} -(4/7) - (5/6) - 2 \times (1/6) \times 3 & 0 & 0 & 0 \\ (5/6) & -(4/7) - (5/3) & 0 & 0 \\ 0 & (5/3) & (-4/7) & 0 \\ (1/6) \times 3 & 0 & 0 & -(4/7) \end{bmatrix}
 \end{aligned}$$

We used  $\theta_x = [\theta_{x,1} \ \theta_{x,2} \ \theta_{x,3}]^T = [k_1 \ k_2 \ k_3]^T = [(5/6) \ (5/3) \ (1/6)]^T$  and

$$\begin{aligned}
 x^{SS} &= \begin{bmatrix} x_1^{SS} \\ x_2^{SS} \\ x_3^{SS} \\ x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 1.1170 \\ 3.2580 \\ 1.3125 \end{bmatrix} \\
 u^{SS} &= \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i^{SS} / V \\ [A]_i^{SS} \end{bmatrix} = \begin{bmatrix} 4/7 \\ 10 \end{bmatrix}
 \end{aligned}$$

## Example II (cont.)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \underbrace{u_1(t)(u_2(t) - x_1(t)) - \theta_{x,1}x_1(t) - \theta_{x,3}x_1^2(t)}_{f_1} \\ \underbrace{-u_1(t)x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)}_{f_2} \\ \underbrace{-u_1(t)x_3(t) + \theta_{x,2}x_2(t)}_{f_3} \\ \underbrace{-u_1(t)x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)}_{f_4} \end{bmatrix}$$

$$\begin{aligned}
 \rightsquigarrow B &= \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix}_{SS} = \begin{bmatrix} u_2 - x_1 & u_1 \\ -x_2 & 0 \\ -x_3 & 0 \\ -x_4 & 0 \end{bmatrix}_{SS}
 \end{aligned}$$

## Example II (cont.)

We get,

$$\begin{aligned}
 B &= \begin{bmatrix} u_2 - x_1 & u_1 \\ -x_2 & 0 \\ -x_3 & 0 \\ -x_4 & 0 \end{bmatrix}_{SS} \\
 &= \begin{bmatrix} u_2^{SS} - x_1^{SS} & u_1^{SS} \\ -x_2^{SS} & 0 \\ -x_3^{SS} & 0 \\ -x_4^{SS} & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 10 - 3 & (4/7) \\ -1.1170 & 0 \\ -3.2580 & 0 \\ -1.3125 & 0 \end{bmatrix}
 \end{aligned}$$

We used,

$$\begin{aligned}
 x^{SS} &= \begin{bmatrix} x_1^{SS} \\ x_2^{SS} \\ x_3^{SS} \\ x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 1.1170 \\ 3.2580 \\ 1.3125 \end{bmatrix} \\
 u^{SS} &= \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i^{SS} / V \\ [A]_i^{SS} \end{bmatrix} = \begin{bmatrix} 4/7 \\ 10 \end{bmatrix}
 \end{aligned}$$

