7. STRUCTURAL STABILITY

7.1. General rudiments for the stability analysis

In stability analyses traditionally, the equilibrium equations of the structure considered will be written in its deformed configuration. This means that the influence of the deformation has to be taken into account in the equilibrium conditions. We call it geometrical non-linearity. Let's consider at first the Green-Lagrange's general non-linear expressions of strain components in the Cartesian co-ordinate system x, y, z with unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ in accordance with the definitions given in equations (1.3), (1.5), (1.8), (1.11)

$$\begin{aligned}
\varepsilon_x &= \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{i}+\frac{1}{2}\frac{\partial \mathbf{u}}{\partial x}\cdot\frac{\partial \mathbf{u}}{\partial x}=\frac{\partial u}{\partial x}+\frac{1}{2}\left((\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial x})^2+(\frac{\partial w}{\partial x})^2\right)\\
\varepsilon_y &= \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{j}+\frac{1}{2}\frac{\partial \mathbf{u}}{\partial y}\cdot\frac{\partial \mathbf{u}}{\partial y}=\frac{\partial v}{\partial y}+\frac{1}{2}\left((\frac{\partial u}{\partial y})^2+(\frac{\partial v}{\partial y})^2+(\frac{\partial w}{\partial y})^2\right)\\
\varepsilon_z &= \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{k}+\frac{1}{2}\frac{\partial \mathbf{u}}{\partial z}\cdot\frac{\partial \mathbf{u}}{\partial z}=\frac{\partial w}{\partial z}+\frac{1}{2}\left((\frac{\partial u}{\partial z})^2+(\frac{\partial v}{\partial z})^2+(\frac{\partial w}{\partial z})^2\right)\\
\gamma_{xy} &= \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{j}+\frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i}+\frac{\partial \mathbf{u}}{\partial x}\cdot\frac{\partial \mathbf{u}}{\partial y}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\\
\gamma_{yz} &= \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{k}+\frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{j}+\frac{\partial \mathbf{u}}{\partial y}\cdot\frac{\partial \mathbf{u}}{\partial z}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}+\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)\\
\gamma_{zx} &= \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{i}+\frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{k}+\frac{\partial \mathbf{u}}{\partial z}\cdot\frac{\partial \mathbf{u}}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}+\left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right)
\end{aligned} \qquad (7.1)$$

If we introduce for the linear parts of strains the notations

$$\begin{aligned}
&e_x=\frac{\partial u}{\partial x},\ e_y=\frac{\partial v}{\partial y},\ e_z=\frac{\partial w}{\partial z},\ e_{xy}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),\\
&e_{yz}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\ e_{zx}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)
\end{aligned} \qquad (7.2)$$

and apply still the rotation components, defined by (1.17)

$$
\begin{aligned}
\omega_1 = \omega_x &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{k} - \frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{j}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\
\omega_2 = \omega_y &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial z}\cdot\mathbf{i} - \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{k}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (7.3) \\
\omega_3 = \omega_z &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{j} - \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}
$$

all the strain components, ε_x for example, can be expressed by using the linear strains and rotations in the form

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[(\frac{\partial u}{\partial x})^2 + \frac{1}{4}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^2 + \frac{1}{4}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})^2 \right] \qquad (7.4) \\
&= e_x + \tfrac{1}{2}e_x^2 + \tfrac{1}{2}\left(e_{xy} + \omega_z\right)^2 + \tfrac{1}{2}\left(e_{zx} - \omega_y\right)^2
\end{aligned}
$$

And the shear strain γ_{xy} correspondingly

$$
\begin{aligned}
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left(\frac{1}{2}\frac{\partial u}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) + \frac{1}{2}\frac{\partial v}{\partial y}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \right. \\
&\quad \left. + \frac{1}{4}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z})(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) \right) \qquad (7.5) \\
&= 2e_{xy} + e_x(e_{xy} - \omega_z) + e_y(e_{xy} + \omega_z) + (e_{zx} - \omega_y)(e_{yz} + \omega_x)
\end{aligned}
$$

Thus, we see that the non-linear parts of the strains can be expressed by the linear strain and rotation components only. Correspondingly, the rest of strain components can be derived similarly, resulting finally in the form

$$
\begin{aligned}
\varepsilon_x &= e_x + \tfrac{1}{2}e_x^2 + \tfrac{1}{2}\left(e_{xy} + \omega_z\right)^2 + \tfrac{1}{2}\left(e_{zx} - \omega_y\right)^2 \\
\varepsilon_y &= e_y + \tfrac{1}{2}e_y^2 + \tfrac{1}{2}\left(e_{xy} - \omega_z\right)^2 + \tfrac{1}{2}\left(e_{yz} + \omega_x\right)^2 \\
\varepsilon_z &= e_z + \tfrac{1}{2}e_z^2 + \tfrac{1}{2}\left(e_{zx} + \omega_y\right)^2 + \tfrac{1}{2}\left(e_{yz} - \omega_x\right)^2 \qquad (7.6) \\
\gamma_{xy} &= 2e_{xy} + e_x(e_{xy} - \omega_z) + e_y(e_{xy} + \omega_z) + (e_{zx} - \omega_y)(e_{yz} + \omega_x) \\
\gamma_{yz} &= 2e_{yz} + e_y(e_{yz} - \omega_x) + e_z(e_{yz} + \omega_x) + (e_{xy} - \omega_z)(e_{zx} + \omega_y) \\
\gamma_{zx} &= 2e_{zx} + e_z(e_{zx} - \omega_y) + e_x(e_{zx} + \omega_y) + (e_{yz} - \omega_x)(e_{xy} + \omega_z)
\end{aligned}
$$

In stability analyses, the strain components $e_x, e_y, e_z, e_{xy}, e_{yz}, e_{zx}$ are assumed to be small as compared to rotation components $\omega_x, \omega_y, \omega_z$. Thus, in non-linear terms the quadratic terms of rotations only will be included in the analysis. The quadratic terms of strains and the terms of 'rotation times strain' are dropped, yielding for strains the expressions

$$
\begin{aligned}
\varepsilon_x &= e_x + \tfrac{1}{2}(\omega_z^2 + \omega_y^2) \\
\varepsilon_y &= e_y + \tfrac{1}{2}(\omega_z^2 + \omega_x^2) \\
\varepsilon_z &= e_z + \tfrac{1}{2}(\omega_y^2 + \omega_x^2) \\
\gamma_{xy} &= 2e_{xy} - \omega_y \omega_x \\
\gamma_{yz} &= 2e_{yz} - \omega_z \omega_y \\
\gamma_{zx} &= 2e_{zx} - \omega_x \omega_z
\end{aligned}
\tag{7.7}
$$

7.2. Flexural Buckling of a straight plane beam

We consider now a column, in which the axial co-ordinate *x* coincides with the column axis, i.e. goes through the centroid of each cross-section plane. Co-ordinates *y* and *z* are the principal axes of the cross-section. Central axial forces load the column at the initial state, only – neither bending, nor torsion exists. In buckling analyses, these loads are compressive. When defining the kinematics for the analysis, an additional degree of freedom to the initial state has to be adopted. This will lead to the homogeneous system of equations with respect to this additional degree of freedom, of which the critical load, as a solution of an eigenvalue problem will be determined. The kinematics adopted for a straight plane beam including in addition to compression (stretching) also bending, in accordance with the Euler-Bernoulli beam theory is

$$
\mathbf{u} = (u - y\frac{\mathrm{d}v}{\mathrm{d}x})\mathbf{i} + v\mathbf{j} \tag{7.8}
$$

in which $u = u(x)$ and $v = v(x)$ are the displacement components in *x*- and *y*-direction, and the rotation of each normal follows the slope of the beam axis. Calculating now the strains, using (7.2) and (7.3), gives just two non-zero components, while $e_y = e_z = e_{xy} = e_{yz} = e_{zx} = \omega_x = \omega_y = 0$

$$
\begin{aligned}
e_x &= \frac{\partial \mathbf{u}}{\partial x}\cdot \mathbf{i} = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = u' - yv'' \\
\omega_z &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial x}\cdot \mathbf{j} - \frac{\partial \mathbf{u}}{\partial y}\cdot \mathbf{i}\right) = \frac{\mathrm{d}v}{\mathrm{d}x} = v'
\end{aligned}
\tag{7.9}
$$

In the continuation to simplify the notations, a prime $(\cdot)'$ will be used for differentiation with respect to the axial co-ordinate *x*. The non-zero strain components are

$$
\begin{aligned}
\varepsilon_x &= e_x + \tfrac{1}{2}(\omega_z^2 + \omega_y^2) = u' - yv'' + \tfrac{1}{2}(v')^2 \\
\varepsilon_y &= e_y + \tfrac{1}{2}(\omega_z^2 + \omega_x^2) = \tfrac{1}{2}(v')^2
\end{aligned}
\tag{7.10}
$$

At the initial loading state, consisting only of axial compression, the normal stress is $\sigma_x^{\mathrm{o}} = N^{\mathrm{o}} / A$, while all the other stress components are zero.

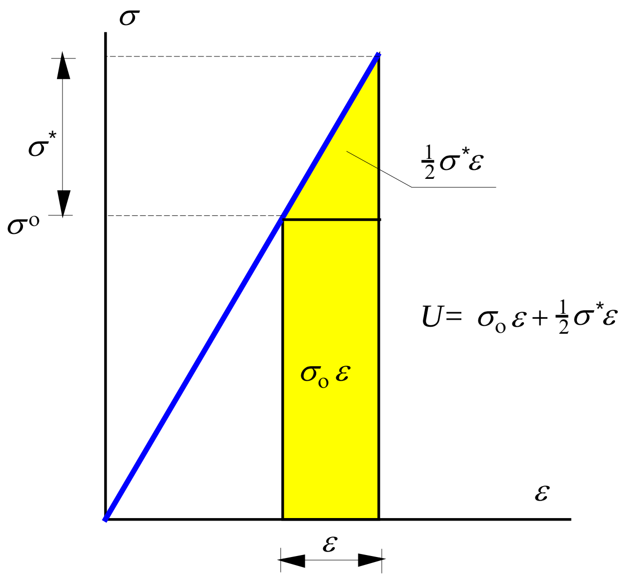


Figure 1. Strain energy by the Euler method.

The procedure we apply follows the linearised theory, called also Euler method (see Figure 1), presented for example by Novozhilov's and Washizu's famous text-books according to which the incremental strain energy of the beam is

$$\begin{aligned} U = U^{\rm o} + U^* &= \int_V (\sigma_x^{\rm o}\varepsilon_x + \tfrac{1}{2}\sigma_x^*\varepsilon_x + \tfrac{1}{2}\sigma_y^*\varepsilon_y)\mathrm{d}V \\ &= \int_V \left[(\sigma_x^{\rm o} + \tfrac{1}{2}\sigma_x^*)(u' - yv'' + \tfrac{1}{2}(v')^2) + \tfrac{1}{4}\sigma_y^*(v')^2\right]\mathrm{d}V \end{aligned} \tag{7.11}$$

Here, $\sigma_y^{\rm o} = 0.$ This can further be split to give

$$U = \int_V \tfrac{1}{2}\sigma_x^*(u' - yv'')\mathrm{d}V + \int_V \sigma_x^{\rm o}(u' - yv'')\mathrm{d}V + \int_V \tfrac{1}{2}\sigma_x^{\rm o}(v')^2\mathrm{d}V + \underbrace{\int_V \tfrac{1}{4}(\sigma_x^* + \sigma_y^*)(v')^2\mathrm{d}V}_{\approx 0} \tag{7.12}$$

The last term can be neglected because incorporating the linear elastic material model into it gives higher (third) order terms of displacement functions. The equation can be written in the form

$$U = U_L + U_{NL_1} + U_{NL_2} \tag{7.13}$$

in which the first term, U_L, is the traditional strain energy expression due to linearized strains and the two following ones, U_{NL_1} and U_{NL_2}, take into account both the non-linear terms of strain components and the initial stresses.

When we take into account the linearly elastic material law in (7.12)

$$\sigma_x^* = E\varepsilon_x = E(u' - yv'' + \tfrac{1}{2}(v')^2) \approx E(u' - yv'') \tag{7.14}$$

the strain energy takes the form

$$U = \frac{1}{2}\int_V E(u' - yv'')^2 \mathrm{d}V + \int_V \sigma_x^{\mathrm{o}}(u' - yv'')\mathrm{d}V + \frac{1}{2}\int_V \sigma_x^{\mathrm{o}}(v')^2 \mathrm{d}V \tag{7.15}$$

in which

$$\begin{aligned}
U_L &= \frac{1}{2}\int_V E(u' - yv'')^2 \mathrm{d}V = \frac{1}{2}\int_0^L \left((EA(u')^2 + EI(v'')^2 \right)\mathrm{d}x \\
U_{NL_1} &= \int_V \sigma_x^{\mathrm{o}}(u' - yv'')\mathrm{d}V = \int_0^L \left(\sigma_x^{\mathrm{o}} A u' \right)\mathrm{d}x \\
U_{NL_2} &= \frac{1}{2}\int_V \sigma_x^{\mathrm{o}}(v')^2 \mathrm{d}V = \frac{1}{2}\int_0^L \left(\sigma_x^{\mathrm{o}} A (v')^2 \right)\mathrm{d}x
\end{aligned} \tag{7.16}$$

Here, we have split the volume integral to be evaluated separately over the cross-section, and along the axis of the column, and taken into account that

$$\int_A \mathrm{d}A = A, \quad \int_A y\mathrm{d}A = S_z = 0, \quad \int_A y^2 \mathrm{d}A = I_z = I \tag{7.17}$$

The potential due to the external loads at the initial stage, the distributed axial load $p_x^{\mathrm{o}}(x)$ and the concentrated loads at the ends of the column P_x^{o}, is

$$V = -\int_0^L p_x^{\mathrm{o}}(x) u \mathrm{d}x - \Big[P_x^{\mathrm{o}} u \Big]_0^L \tag{7.18}$$

Now, the total potential energy is composed of terms

$$\Pi = U + V = U_L + U_{NL_1} + U_{NL_2} + V \tag{7.19}$$

If we consider at first the term

$$U_{NL_1} + V = \int_0^L \left(\sigma_x^{\mathrm{o}} A u' \right)\mathrm{d}x - \int_0^L p_x^{\mathrm{o}}(x) u \mathrm{d}x - \Big[P_x^{\mathrm{o}} u \Big]_0^L \tag{7.20}$$

and find for the stationary value of it. By taking the first variation it is obtained

$$\delta(U_{NL_1} + V) = \int_0^L \left(\sigma_x^o A \delta u' \right) \mathrm{d}x - \int_0^L p_x^o(x) \delta u \mathrm{d}x - \left[P_x^o \delta u \right]_0^L \tag{7.21}$$

While $N^o(x) = \sigma_x^o A$, integrating the first term by parts gives

$$\begin{aligned} \delta(U_{NL_1} + V) &= -\int_0^L \left((\sigma_x^o A)' + p_x^o(x) \right) \delta u \mathrm{d}x + \left[(\sigma_x^o A - P_x^o) \delta u \right]_0^L \\ &= -\int_0^L \left((N^o)' + p_x^o(x) \right) \delta u \mathrm{d}x + \left[(N^o - P_x^o) \delta u \right]_0^L \end{aligned} \tag{7.22}$$

which results in the equilibrium equation of the initial state with boundary conditions

$$\begin{aligned} &\frac{\mathrm{d}N^o}{\mathrm{d}x} + p_x^o(x) = 0, \\ &N^o = P_x^o \text{ or } \delta u = 0 \text{ at } x = 0 \text{ and } x = L \end{aligned} \tag{7.23}$$

This part of equation considers the initial state of the column, and it disappears when the initial state is in equilibrium.

So, we have still the expression

$$\Pi = U_L + U_{NL_2} = \tfrac{1}{2} \int_0^L \left((EA(u')^2 + EI(v'')^2 \right) \mathrm{d}x + \tfrac{1}{2} \int_0^L \left(N^o (v')^2 \right) \mathrm{d}x \tag{7.24}$$

Taking here the first variation gives

$$\delta\Pi = \delta(U_L + U_{NL_2}) = \int_0^L \left((EAu'\delta u' + EIv''\delta v'' + N^o v' \delta v' \right) \mathrm{d}x \tag{7.25}$$

and integrating by parts finally

$$\begin{aligned} \delta\Pi = &-\int_0^L \left(\left(EAu'\right)' \delta u - \left(\left(EIv''\right)'' - \left(N^o v'\right)' \right) \delta v \right) \mathrm{d}x \\ &+ \left[EAu'\delta u + EIv''\delta v' - \left((EIv'')' - N^o v' \right) \delta v \right]_{x=0}^{x=L} \end{aligned} \tag{7.26}$$

Since δu and δv are arbitrary, we get as a result the system of homogeneous differential equations

$$\begin{aligned}&\left(EAu'\right)' = 0\\&\left(EIv''\right)'' - \left(N^{\mathrm{o}}v'\right)' = 0\end{aligned} \tag{7.27}$$

with the boundary conditions

$$\begin{aligned}&EAu' = N = 0 && \text{or} \quad \delta u = 0\\&EIv'' = M_z = 0 && \text{or} \quad \delta v' = 0\\&(EIv'')' - N^{\mathrm{o}}v' = Q_y = 0 && \text{or} \quad \delta v = 0\end{aligned} \tag{7.28}$$

The first one of the equations (7.27) likewise of boundary conditions (7.28) concerns only the initial state of the beam and is thus meaningless. If the cross section of the beam is constant, and we have at the ends as a load a compressive force only, i.e. $N^{\mathrm{o}} = -P$, the equation simplifies to the well-known form

$$EIv'''' + Pv'' = 0 \tag{7.29}$$

which is the homogeneous ordinary differential equation to define the critical compressive load of a beam. Its general solution is

$$v(x) = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 \tag{7.30}$$

where $k^2 = P / EI$. A homogeneous system of equations will be obtained by applying the relevant boundary conditions at the ends of the column.

An alternative formulation for the differential equation (7.27) with boundary conditions (7.28) can be obtained directly by applying the energy integral formulation (7.24), which simplifies among the kinematically admissible functions $v(x)$ to the minimization problem of the functional

$$\Pi = \frac{1}{2}\int_0^L \left(EI(v'')^2 + N^{\mathrm{o}}(v')^2 \right) \mathrm{d}x \tag{7.31}$$

When considering a column in three-dimensional space where the buckling can take place in any one of the directions of the principal axes, the differential equation system (7.27) will be provided with an additional equation

$$\begin{aligned}&\left(EAu'\right)' = 0\\&\left(EI_z v''\right)'' - \left(N^{\mathrm{o}}v'\right)' = 0\\&\left(EI_y w''\right)'' - \left(N^{\mathrm{o}}w'\right)' = 0\end{aligned} \tag{7.32}$$

with $\int_A z^2 \mathrm{d}A = I_y$, and corresponding additional boundary conditions

$$
\begin{aligned}
&EAu' = N = 0 && \text{or} \quad \delta u = 0 \\
&EI_z v'' = M_z = 0 && \text{or} \quad \delta v' = 0 \\
&EI_y w'' = M_y = 0 && \text{or} \quad \delta w' = 0 \\
&(EI_z v'')' - N^o v' = Q_y = 0 && \text{or} \quad \delta v = 0 \\
&(EI_y w'')' - N^o w' = Q_z = 0 && \text{or} \quad \delta w = 0
\end{aligned}
\tag{7.33}
$$

The minimization problem takes the form

$$
\Pi = \frac{1}{2}\int_0^L \left(EI_z (v'')^2 + EI_y (w'')^2 + N^o ((v')^2 + (w')^2) \right) \mathrm{d}x \tag{7.34}
$$

Here, both $v(x)$ and $w(x)$ have to fulfil the requirements due to kinematics.

Illustrative example. Let's consider a beam which is fixed at both ends, and loaded by a centric compressive load *P* at each end. The length of the beam is *L* and the bending stiffness *EI*. We establish the origin of the global coordinate system at the mid-span of the beam so, that it is moving with the deflection of the beam. The homogeneous boundary conditions are thus

$$
\begin{aligned}
&v(0) = 0 \\
&v'(0) = 0 \\
&Q(0) = -EIv'''(0) = 0 \\
&v'(\pm L/2) = 0
\end{aligned}
$$

From the conditions $v'(0) = Q(0) = 0,$ the coefficients of the antisymmetric functions, i.e. $C_1 = C_3 = 0.$ And, from the condition $v(0) = 0 \Rightarrow C_2 + C_4 = 0.$ Then, we end up in the condition $v'(\pm L/2) = 0$ giving

$$
C_2 \sin(\frac{kL}{2}) = 0 \;\Rightarrow\; \frac{kL}{2} = n\pi \;\Rightarrow\; P = \frac{4n^2\pi^2 EI}{L^2} \;\Rightarrow\; P_{cr} = \frac{4\pi^2 EI}{L^2}
$$

If we consider instead, a beam with simply supported ends, the final condition will be replaced by $v''(\pm L/2) = 0$ yielding

$$
C_2 \cos(\frac{kL}{2}) = 0 \;\Rightarrow\; \frac{kL}{2} = \frac{n\pi}{2} \;\Rightarrow\; P = \frac{n^2\pi^2 EI}{L^2} \;\Rightarrow\; P_{cr} = \frac{\pi^2 EI}{L^2}
$$

7.3. Torsional buckling of a straight beam

Torsional buckling is characteristic for the behaviour of beams with thin-walled cross-section. In the torsional buckling, a beam loaded at the initial stage by an axial load only, buckles through mechanisms of torsion and bending, Figure 2. We consider a thin-walled beam, in which the axial co-ordinate *x* coincides with the beam axis, i.e.

goes through the centroid of each cross-section plane. The loading at the initial state consists of centric compression (stretching). Co-ordinates *y* and *z* are the principal axes

Torsional buckling

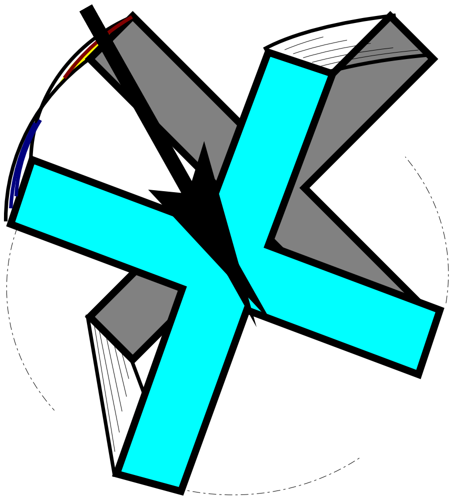


Figure 2. Torsional buckling of a column

of the cross-section. As an additional degree of freedom, the two deflections (*y*- and *z*-directions) and torsion, including the effects of both Saint-Venant's torsion and of Vlasov's warping (sectorial) torsion, is adopted. The kinematics will be defined for the displacement vector of the centre-line of the wall in the cross-section, denoted by $\mathbf{u}_{\text{o}}$. For a straight beam in stretching, bending and torsion the kinematics is defined by

$$\mathbf{u}_{\text{o}} = (u - yv' - zw' - \omega\phi')\mathbf{i} + \left(v - (z - z_v)\phi\right)\mathbf{j} + \left(w + (y - y_v)\phi\right)\mathbf{k} \qquad (7.35)$$

in which $u = u(x)$, $v = v(x)$, $w = w(x)$ and $\phi = \phi(x)$ are the three translation components in the directions of the co-ordinate axes, and the angle of twist, and ω is the sectorial co-ordinate. Co-ordinates (y_v, z_v) define the location of the shear centre of the cross-section. Calculating now the strain and rotation components by applying (7.2) and (7.3) gives, $e_y = e_z = e_{yz} = 0$ and

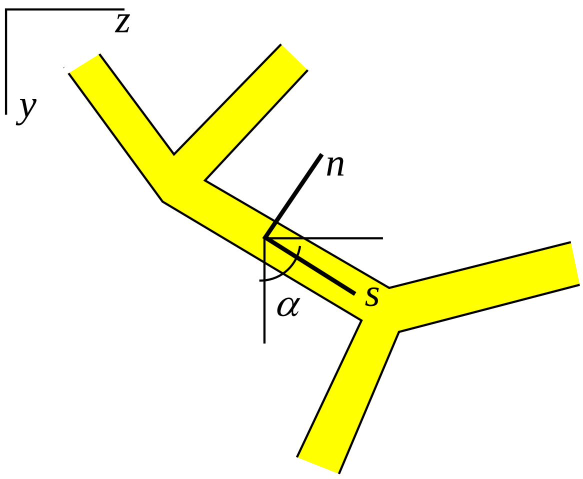


Figure 3. Thin-walled cross-section.

$$
\begin{aligned}
e_x &= u' - yv'' - zw'' - \omega\phi'' \\
e_{xy} &= \frac{1}{2}\left(-(z - z_v) - \frac{\partial \omega}{\partial y} \right)\phi' = 0 \\
e_{zx} &= \frac{1}{2}\left(-\frac{\partial \omega}{\partial z} + (y - y_v) \right)\phi' = 0 \\
\omega_x &= \phi \\
\omega_y &= \frac{1}{2}\left(-\frac{\partial \omega}{\partial z} - (y - y_v) \right)\phi' - w' = -w' - (y - y_v)\phi' \\
\omega_z &= \frac{1}{2}\left(\frac{\partial \omega}{\partial y} - (z - z_v) \right)\phi' + v' = v' - (z - z_v)\phi'
\end{aligned}
\tag{7.36}
$$

Substituting these into the expressions of strains gives

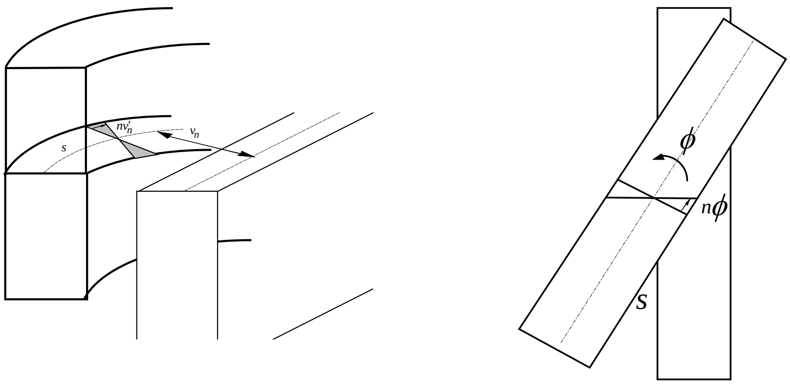
$$
\begin{aligned}
\varepsilon_x &= u' - yv'' - zw'' - \omega\phi'' + \tfrac{1}{2}\left[-w' - (y - y_v)\phi'\right]^2 + \tfrac{1}{2}\left[v' - (z - z_v)\phi'\right]^2 \\
\varepsilon_y &= \tfrac{1}{2}\left[\left[v' - (z - z_v)\phi'\right]^2 + \phi^2\right] \\
\varepsilon_z &= \tfrac{1}{2}\left[\left[-w' - (y - y_v)\phi'\right]^2 + \phi^2\right] \\
\gamma_{xy} &= \left[w' + (y - y_v)\phi'\right]\phi \\
\gamma_{yz} &= \left[w' + (y - y_v)\phi'\right]\left[v' - (z - z_v)\phi'\right] \\
\gamma_{zx} &= \left[-v' + (z - z_v)\phi'\right]\phi
\end{aligned}
\tag{7.37}
$$

The linear strain components $e_{xy} = e_{zx} = 0$, according to the assumptions of the vanishing shear strain components in the warping torsion theory by Vlasov. It can be seen directly of the definition (3.68) of the sectorial co-ordinate

$$
\mathrm{d}\omega = -(z - z_v)\mathrm{d}y + (y - y_v)\mathrm{d}z
$$

of which we get

$$
\begin{aligned}
\frac{\partial \omega}{\partial y} &= -(z - z_v) \\
\frac{\partial \omega}{\partial z} &= (y - y_v)
\end{aligned}
$$



However, these shear strain components do not vanish outside the mid surface of each wall of the cross-section. Thus, the displacement vector defined for the mid surface $\mathbf{u}_o$ (7.35) has to be provided with a component covering the material points outside it. The additional component in the axial direction applies the Euler-Bernoulli beam theory (or Kirchhoff's plate theory), while on the cross-section plane, the rotation of the whole cross-section defines the tangential displacement to the mid surface. In the direction of the thickness of the wall, the wall is assumed to be incompressible. The displacement vector is thus

$$\mathbf{u} = \mathbf{u}_o - n v_n' \mathbf{i} - n\phi \mathbf{e}_s$$

where *n* is the normal co-ordinate to the mid surface of the wall (see Figure 3), and v_n the displacement in this direction, i.e.

$$v_n = -(v - (z - z_v)\phi)\sin\alpha + (w + (y - y_v)\phi)\cos\alpha$$

Inserting this into the definitions of strains gives additional components

$$\varepsilon_x = \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{i} = -n v_n''$$

$$\gamma_{xs} = \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{e}_s + \frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{i} = -n(\phi - \frac{\partial v_n'}{\partial s})$$

The first one is connected to the flexure of each plate locally, and will be dropped out, while the second one is meaningful, leading to Saint-Venant's torsion rigidity of the cross-section. Substituting v_n into it gives

$$\gamma_{xs} = -n\phi' - n\left(\frac{\partial z}{\partial s}\phi' \sin\alpha + \frac{\partial y}{\partial s}\phi' \cos\alpha\right)$$

$$= -n\phi' - n(\phi' \sin^2\alpha + \phi' \cos^2\alpha) = -2n\phi'$$

To combine the shear strain terms, we transform the global shear components γ_{xy} and γ_{zx} in (7.37) onto the components γ_{xs} and γ_{xn}. The transformation is derived when taking into account the transformation between the co-ordinate systems *y*,*z* and *s*,*n*, derived in example 2 (Chapter 4, page 29)

$$\gamma_{xs} = \frac{\partial \mathbf{u}}{\partial s}\cdot\mathbf{i} + \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{e}_s = \left(\frac{\partial \mathbf{u}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \mathbf{u}}{\partial z}\frac{\partial z}{\partial s}\right)\cdot\mathbf{i} + \frac{\partial \mathbf{u}}{\partial x}\cdot(\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k})$$
$$= \left(\frac{\partial \mathbf{u}}{\partial y}\cos\alpha + \frac{\partial \mathbf{u}}{\partial z}\sin\alpha\right)\cdot\mathbf{i} + \frac{\partial \mathbf{u}}{\partial x}\cdot(\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}) =$$
$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\alpha + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\sin\alpha = \gamma_{xy}\cos\alpha + \gamma_{zx}\sin\alpha$$

and γ_{nx} correspondingly, yielding

$$\gamma_{xs} = \gamma_{xy}\cos\alpha + \gamma_{zx}\sin\alpha$$
$$\gamma_{nx} = -\gamma_{xy}\sin\alpha + \gamma_{zx}\cos\alpha$$

we get the final expressions of strains

$$\begin{aligned}
\varepsilon_x &= u' - yv'' - zw'' - \omega\phi'' + \tfrac{1}{2}\left[-w' - (y-y_v)\phi'\right]^2 + \tfrac{1}{2}\left[v' - (z-z_v)\phi'\right]^2 \\
\varepsilon_y &= \tfrac{1}{2}\left[\left[v' - (z-z_v)\phi'\right]^2 + \phi^2\right] \\
\varepsilon_z &= \tfrac{1}{2}\left[\left[-w' - (y-y_v)\phi'\right]^2 + \phi^2\right] \\
\gamma_{xs} &= -2n\phi' + \left[w' + (y-y_v)\phi'\right]\phi\cos\alpha - \left[v' - (z-z_v)\phi'\right]\phi\sin\alpha \\
\gamma_{yz} &= \left[w' + (y-y_v)\phi'\right]\left[v' - (z-z_v)\phi'\right] \\
\gamma_{nx} &= -\left[w' + (y-y_v)\phi'\right]\phi\sin\alpha - \left[v' - (z-z_v)\phi'\right]\phi\cos\alpha
\end{aligned} \tag{7.38}$$

At the initial loading state, consisting only of centric axial compression, the normal stress is $\sigma_x^0 = N^0 / A$, while all the other stress components are zero. Following the same procedure as before, the incremental strain energy of the beam takes the form

$$\begin{aligned}
U &= \int_V (\sigma_x^0\varepsilon_x + \tfrac{1}{2}\sigma_x^*\varepsilon_x + \tfrac{1}{2}\sigma_y^*\varepsilon_y + \tfrac{1}{2}\sigma_z^*\varepsilon_z + \tfrac{1}{2}\tau_{xs}^*\gamma_{xs} + \tfrac{1}{2}\tau_{yz}^*\gamma_{yz} + \tfrac{1}{2}\tau_{xn}^*\gamma_{xn})\mathrm{d}V \\
&\approx \int_V (\sigma_x^0 + \tfrac{1}{2}\sigma_x^*)\left(u' - yv'' - zw'' - \omega\phi'' + \tfrac{1}{2}\left[w' + (y-y_v)\phi'\right]^2 + \tfrac{1}{2}\left[v' - (z-z_v)\phi'\right]^2\right)\mathrm{d}V \\
&+ \int_V \tfrac{1}{2}\tau_{xs}^*(2n\phi')\mathrm{d}V
\end{aligned} \tag{7.39}$$

When we take into account the linearly elastic material law

$$\begin{aligned}
\sigma_x^* &= E\varepsilon_x = E\left[u' - yv'' - zw'' - \omega\phi'' + \tfrac{1}{2}\left[w' + (y-y_v)\phi'\right]^2 + \tfrac{1}{2}\left[v' - (z-z_v)\phi'\right]^2\right] \\
&\approx E(u' - yv'' - zw'' - \omega\phi'') \\
\tau_{xs}^* &= G\gamma_{xs} = 2nG\phi'
\end{aligned} \tag{7.40}$$

this will be

$$
\begin{aligned}
U &= \frac{1}{2}\int_V E(u' - yv'' - zw'' - \omega\phi'')^2 \mathrm{d}V + \frac{1}{2}\int_V G(2n\phi')^2 \mathrm{d}V \\
&+ \int_V \sigma_x^{\mathrm{o}}(u' - yv'' - zw'' - \omega\phi'')\mathrm{d}V \\
&+ \int_V \sigma_x^{\mathrm{o}}\left(\tfrac{1}{2}\left[w' + (y - y_v)\phi'\right]^2 + \tfrac{1}{2}\left[v' - (z - z_v)\phi'\right]^2\right)\mathrm{d}V
\end{aligned}
\tag{7.41}
$$

This is a sum of three parts as

$$U = U_L + U_{NL_1} + U_{NL_2} \tag{7.42}$$

with taking into account that $\sigma_x^{\mathrm{o}} = N^{\mathrm{o}} / A$, when

$$
\begin{aligned}
U_L &= \frac{1}{2}\int_V E(u' - yv'' - zw'' - \omega\phi'')^2 \mathrm{d}V + \frac{1}{2}\int_V G(2n\phi')^2 \mathrm{d}V \\
&= \frac{1}{2}\int_0^L \left(EA(u')^2 + EI_z(v'')^2 + EI_y(w'')^2 + EI_\omega(\phi'')^2 + GI_t(\phi')^2 \right)\mathrm{d}x \\
U_{NL_1} &= \int_V \sigma_x^{\mathrm{o}}(u' - yv'' - zw'' - \omega\phi'')\mathrm{d}V = \int_0^L \left(N^{\mathrm{o}}u'\right)\mathrm{d}x \\
U_{NL_2} &= \int_V \tfrac{1}{2}\sigma_x^{\mathrm{o}}\left(\left[w' + (y - y_v)\phi'\right]^2 + \left[v' - (z - z_v)\phi'\right]^2\right)\mathrm{d}V \\
&= \frac{1}{2}\int_0^L N^{\mathrm{o}}\left[(w')^2 + (v')^2 - 2y_v w'\phi' + 2z_v v'\phi' + r^2(\phi')^2\right]\mathrm{d}x
\end{aligned}
\tag{7.43}
$$

Here in addition to (7.17), the notations

$$\int_A z\mathrm{d}A = 0,\quad \int_A z^2\mathrm{d}A = I_y,\quad \int_A \omega\mathrm{d}A = 0,\quad \int_A \omega^2\mathrm{d}A = I_\omega,\quad \int_A 4n^2\mathrm{d}A = I_t \tag{7.44}$$

are used, and $r^2 = (I_y + I_z)/A + y_v^2 + z_v^2$.

The potential due to the external loads in the axial direction is

$$V = -\int_0^L p_x^{\mathrm{o}}(x)u\mathrm{d}x - \left[P_x^{\mathrm{o}}u\right]_0^L \tag{7.45}$$

Now, the total potential energy is composed of terms

$$\Pi = U + V = U_L + U_{NL_1} + U_{NL_2} + V \tag{7.46}$$

If we consider at first the part

$$U_{NL_1} + V = \int_0^L \left(N^o u' \right) dx - \int_0^L p_x^o(x) u dx - \left[P_x^o u \right]_0^L \tag{7.47}$$

It is exactly the same as in the case of pure flexural buckling, and disappears when the initial structure is in equilibrium fulfilling the equation of equilibrium with boundary conditions

$$\begin{aligned} &\frac{dN^o}{dx} + p_x^o(x) = 0, \\ &N^o = P_x^o \text{ or } \delta u = 0 \text{ at } x = 0 \text{ and } x = L \end{aligned} \tag{7.48}$$

This part of equation disappears when the beam in the initial state is in equilibrium.

So, we have still the equation

$$\begin{aligned} \Pi = U_L + U_{NL_2} = &\frac{1}{2}\int_0^L \left[EA(u')^2 + EI_z(v'')^2 + EI_y(w'')^2 + EI_\omega(\phi'')^2 + GI_t(\phi')^2 \right] dx \\ &+ \frac{1}{2}\int_0^L N^o \left[(w')^2 + (v')^2 - 2y_v w'\phi' + 2z_v v'\phi' + r^2(\phi')^2 \right] dx \end{aligned} \tag{7.49}$$

Taking at first the first variation of the first term U_L results in

$$\delta U_L = \int_0^L \left[(EAu'\delta u' + EI_z v''\delta v'' + EI_y w''\delta w'' + EI_\omega \phi''\delta\phi'' + GI_t \phi'\delta\phi' \right] dx \tag{7.50}$$

and integrating this by parts gives

$$\begin{aligned} \delta U_L = &-\int_0^L \left[(EAu')'\delta u - (EI_z v'')''\delta v - (EI_y w'')''\delta w - \left((EI_\omega \phi'')'' - (GI_t \phi')' \right) \delta\phi \right] dx \\ &+ \left[EAu'\delta u + EI_z v''\delta v' - (EI_z v'')'\delta v + EI_y w''\delta w' - (EI_y w'')'\delta w + \right. \\ &\left. + EI_\omega \phi''\delta\phi' - (EI_\omega \phi'')'\delta\phi + GI_t \phi'\delta\phi \right]_{x=0}^{x=L} \end{aligned} \tag{7.51}$$

The first variation of the second term U_{NL_2} in (7.49) is

$$\delta U_{NL_2} = \int_0^L \left[N^o \left(w'\delta w' + v'\delta v' - y_v(w'\delta\phi' + \phi'\delta w') + z_v(v'\delta\phi' + \phi'\delta v') + r^2\phi'\delta\phi' \right) \right] dx \tag{7.52}$$

which takes after integrating by parts the form

$$\delta U_{NL_2} = -\int_0^L \Big[\Big((N^o w')' - y_v (N^o \phi')' \Big) \delta w + \Big((N^o v')' + z_v (N^o \phi')' \Big) \delta v$$
$$- \Big(y_v (N^o w')' - z_v (N^o v')' - (N^o r^2 \phi')' \Big) \delta\phi \Big] \mathrm{d}x \tag{7.53}$$
$$+ N^o \Big[(w' - y_v \phi') \delta w + (v' + z_v \phi') \delta v - (y_v w' - z_v v' - r^2 \phi') \delta\phi \Big]_{x=0}^{x=L}$$

Combining finally (7.51) and (7.53) results in a homogeneous system of ordinary differential equations with boundary conditions, which are obtained since δu, δv, δw and $\delta\phi$ are arbitrary. We get

$$\begin{aligned}
&\big(EAu'\big)' = 0 \\
&(EI_z v'')'' - (N^o v')' - z_v (N^o \phi')' = 0 \\
&(EI_y w'')'' - (N^o w')' + y_v (N^o \phi')' = 0 \\
&\big(EI_\omega \phi''\big)'' - (GI_t \phi')' + y_v (N^o w')' - z_v (N^o v')' - r^2 \big(N^o \phi'\big)' = 0
\end{aligned} \tag{7.54}$$

with corresponding boundary conditions

$$\begin{aligned}
&EAu' = N = 0 && \text{or} \quad \delta u = 0 \\
&-EI_z v'' = M_z = 0 && \text{or} \quad \delta v' = 0 \\
&-EI_z w'' = M_y = 0 && \text{or} \quad \delta w' = 0 \\
&-EI_\omega \phi'' = B = 0 && \text{or} \quad \delta\phi' = 0 \\
&-(EI_z v'')' + N^o (v' + z_v \phi') = Q_y = 0 && \text{or} \quad \delta v = 0 \\
&-(EI_y w'')' + N^o (w' - y_v \phi') = Q_z = 0 && \text{or} \quad \delta w = 0 \\
&-(EI_\omega \phi'')' + GI_t \phi' - N^o (y_v w' - z_v v' - r^2 \phi') = M_\omega + M_t = 0 && \text{or} \quad \delta\phi = 0
\end{aligned} \tag{7.55}$$

at each end of the beam. The first equation of (7.54) and (7.55) describe the initial state and are not of interest in this context. If the cross section of the beam is not changing in the axial direction of the beam, and we have at the ends as a load a compressive force only, i.e. $N^o = -P$, the system of equations simplifies to the well-known form

$$\begin{aligned}
&EI_z v'''' + Pv'' + z_v P\phi'' = 0 \\
&EI_y w'''' + Pw'' - y_v P\phi'' = 0 \\
&EI_\omega \phi'''' - GI_t \phi'' - y_v Pw'' + z_v Pv'' + r^2 P\phi'' = 0
\end{aligned} \tag{7.56}$$

which is the homogeneous system of equations to define the critical compressive load of the beam.

An alternative formulation for the differential equation (7.54) with boundary conditions (7.55) can be obtained directly of the energy integral formulation (7.49), which simplifies to the minimization problem of the functional

$$\Pi = \frac{1}{2}\int_0^L \left[EI_z(v'')^2 + EI_y(w'')^2 + EI_\omega(\phi'')^2 + GI_t(\phi')^2 \right]\mathrm{d}x \\ + \frac{1}{2}\int_0^L N^{\mathrm{o}} \left[(w')^2 + (v')^2 - 2y_v w'\phi' + 2z_v v'\phi' + r^2(\phi')^2 \right]\mathrm{d}x \tag{7.57}$$

7.4. Combined flexural and torsional buckling

The beam considered in this context is loaded at the initial state by an eccentric compressive load yielding two mutually equal bending moments at each end of the beam. Thus, the bending moment distributions over the beam length are constant. The loading at the initial state is a combination of axial force and bending. For stability analysis, the additional degree of freedom is then torsion. The initial normal stress distribution takes the form

$$\sigma_x^{\mathrm{o}} = \frac{N^{\mathrm{o}}}{A} + \frac{M_z^{\mathrm{o}}}{I_z}y + \frac{M_y^{\mathrm{o}}}{I_y}z = \frac{P_x^{\mathrm{o}}}{A} + \frac{P_x^{\mathrm{o}} e_y}{I_z}y + \frac{P_x^{\mathrm{o}} e_z}{I_y}z \tag{7.58}$$

All the other initial stress components disappear. The latter part of the presentation (7.58) concerns the eccentric axial load, in which co-ordinates e_y, e_z define its location on the cross-section plane. The consideration deviates from the one of the previous section only in the terms U_{NL_1} and U_{NL_2} in (7.43) through the initial normal stress distribution σ_x^{o}, and in V in (7.45), in which the end moments have to be included. The terms in (7.43) will get supplements ΔU_{NL_1} and ΔU_{NL_2}

$$\Delta U_{NL_1} = \int_V \sigma_x^{\mathrm{o}}(u' - yv'' - zw'' - \omega\phi'')\mathrm{d}V = \int_V \left(\frac{M_z^{\mathrm{o}}}{I_z}y + \frac{M_y^{\mathrm{o}}}{I_y}z\right)(u' - yv'' - zw'' - \omega\phi'')\mathrm{d}V \\ = \int_0^L \left(-M_z^{\mathrm{o}}v'' - M_y^{\mathrm{o}}w'' \right)\mathrm{d}x$$

$$\Delta U_{NL_2} = \frac{1}{2}\int_V \sigma_x^{\mathrm{o}} \left[[-w' - (y - y_v)\phi']^2 + [v' - (z - z_v)\phi']^2 \right]\mathrm{d}V \\ = \frac{1}{2}\int_V \left(\left(\frac{M_z^{\mathrm{o}}}{I_z}y + \frac{M_y^{\mathrm{o}}}{I_y}z\right)\left[[-w' - (y - y_v)\phi']^2 + [v' - (z - z_v)\phi']^2 \right] \right)\mathrm{d}V \\ = \int_0^L \left[M_z^{\mathrm{o}}\left(w'\phi' + \beta_y(\phi')^2 \right) - M_y^{\mathrm{o}}\left(v'\phi' - \beta_z(\phi')^2 \right) \right]\mathrm{d}x \tag{7.59}$$

Here, we have utilised the definitions

$$\int_A y\omega \mathrm{d}A = 0, \quad \int_A z\omega \mathrm{d}A = 0 \tag{7.60}$$

In these equations, the notations β_y and β_z i.e. Wagner's coefficients,

$$\begin{aligned}\beta_y &= \frac{1}{2I_z}\int_A y(y^2+z^2)\mathrm{d}A - y_v \\ \beta_z &= \frac{1}{2I_y}\int_A z(y^2+z^2)\mathrm{d}A - z_v\end{aligned} \tag{7.61}$$

are used. The potential of the external load V in (7.45) takes a supplement ΔV

$$\Delta V = -\left[P_x^{\mathrm{o}} e_y v' + P_x^{\mathrm{o}} e_z w'\right]_0^L \tag{7.62}$$

Evaluating the sum $U_{NL_1} + \Delta U_{NL_1} + V + \Delta V$ and taking the first variation of it gives

$$\begin{aligned}\delta(U_{NL_1} + \Delta U_{NL_1} + V + \Delta V) &= \int_0^L \left(N^{\mathrm{o}}\delta u' - M_z^{\mathrm{o}}\delta v'' - M_y^{\mathrm{o}}\delta w''\right)\mathrm{d}x - \int_0^L p_x^{\mathrm{o}}(x)\delta u \mathrm{d}x \\ &- P_x^{\mathrm{o}}\left[\delta u + e_y\delta v' + e_z\delta w'\right]_0^L\end{aligned} \tag{7.63}$$

and integrating by parts

$$\begin{aligned}\delta(U_{NL_1} + \Delta U_{NL_1} + V + \Delta V) &= -\int_0^L\left[\left((N^{\mathrm{o}})' + p_x^{\mathrm{o}}(x)\right)\delta u + \left(M_z^{\mathrm{o}}\right)''\delta v + \left(M_y^{\mathrm{o}}\right)''\delta w\right]\mathrm{d}x \\ +\Big[(N^{\mathrm{o}} - P_x^{\mathrm{o}})\delta u + (M_z^{\mathrm{o}} - P_x^{\mathrm{o}}e_y)\delta v' &- (M_z^{\mathrm{o}})'\delta v + (M_y^{\mathrm{o}} - P_x^{\mathrm{o}}e_z)\delta w' - (M_y^{\mathrm{o}})'\delta w\Big]_0^L\end{aligned} \tag{7.64}$$

results in the equilibrium equations of the beam at the initial state with corresponding boundary conditions

$$\begin{gathered}\frac{\mathrm{d}N^{\mathrm{o}}}{\mathrm{d}x} + p_x^{\mathrm{o}}(x) = 0, \quad \frac{\mathrm{d}^2 M_z^{\mathrm{o}}}{\mathrm{d}x^2} = 0, \quad \frac{\mathrm{d}^2 M_y^{\mathrm{o}}}{\mathrm{d}x^2} = 0, \\ \left.\begin{array}{lll} N^{\mathrm{o}} = P_x^{\mathrm{o}} & \text{or} & \delta u = 0 \\ M_z^{\mathrm{o}} = P_x^{\mathrm{o}} e_y & \text{or} & \delta v' = 0 \\ M_y^{\mathrm{o}} = P_x^{\mathrm{o}} e_z & \text{or} & \delta w' = 0 \\ (M_z^{\mathrm{o}})' = Q_y = 0 & \text{or} & \delta v = 0 \\ (M_y^{\mathrm{o}})' = Q_z = 0 & \text{or} & \delta w = 0 \end{array}\right\} \quad \text{at } x = 0 \text{ and } x = L\end{gathered} \tag{7.65}$$

Taking the variation of the term ΔU_{NL_2} gives

$$\delta(\Delta U_{NL_2}) = \int_0^L \left[M_z^o \phi' \delta w' - M_y^o \phi' \delta v' + \left(M_z^o w' - M_y^o v' + 2(\beta_y M_z^o + \beta_z M_y^o)\phi' \right) \delta\phi' \right] \mathrm{d}x \tag{7.66}$$

This will be reformulated after integrating by parts as

$$\begin{aligned}\delta(\Delta U_{NL_2}) = -\int_0^L \Big[&(M_z^o \phi')' \delta w - (M_y^o \phi')' \delta v + \\ &+ \left((M_z^o w')' - (M_y^o v')' + 2\beta_y (M_z^o \phi')' + 2\beta_z (M_y^o \phi')' \right) \delta\phi \Big] \mathrm{d}x \\ &+ \Big[M_z^o \phi' \delta w - M_y^o \phi' \delta v + 2\beta_y M_z^o \phi' \delta\phi + 2\beta_z M_y^o \phi' \delta\phi \Big]_{x=0}^{x=L}\end{aligned} \tag{7.67}$$

Combining this finally with (7.51) and (7.53) gives the homogeneous system of ordinary differential equations with boundary conditions describing the flexural-torsional buckling

$$\begin{aligned}&\left(EAu' \right)' = 0 \\ &(EI_z v'')'' - (N^o v')' - z_v (N^o \phi')' + (M_y^o \phi')' = 0 \\ &(EI_y w'')'' - (N^o w')' + y_v (N^o \phi')' - (M_z^o \phi')' = 0 \\ &\left(EI_\omega \phi'' \right)'' - (GI_t \phi')' + y_v (N^o w')' - z_v (N^o v')' - r^2 \left(N^o \phi' \right)' \\ &\qquad - (M_z^o w')' + (M_y^o v')' - 2\beta_y (M_z^o \phi')' - 2\beta_z (M_y^o \phi')' = 0\end{aligned} \tag{7.68}$$

and boundary conditions

$$\begin{aligned}&EAu' = N = 0 && \text{or} \quad \delta u = 0 \\ &-EI_z v'' = M_z = 0 && \text{or} \quad \delta v' = 0 \\ &-EI_z w'' = M_y = 0 && \text{or} \quad \delta w' = 0 \\ &-EI_\omega \phi'' = B = 0 && \text{or} \quad \delta\phi' = 0 \\ &-(EI_z v'')' + N^o (v' + z_v \phi') - M_y^o \phi' = Q_y = 0 && \text{or} \quad \delta v = 0 \\ &-(EI_y w'')' + N^o (w' - y_v \phi') + M_z^o \phi' = Q_z = 0 && \text{or} \quad \delta w = 0 \\ &-(EI_\omega \phi'')' + GI_t \phi' - N^o (y_v w' - z_v v' - r^2 \phi') + \\ &\qquad + 2\beta_y M_z^o \phi' + 2\beta_z M_y^o \phi' = M_\omega + M_t = 0 && \text{or} \quad \delta\phi = 0\end{aligned} \tag{7.69}$$

When keeping in the mind, that initial bending moment distributions M_z^o and M_y^o are constant with respect to the axial co-ordinate, the system (7.68) simplifies to the form

$$
\begin{aligned}
&\left(EAu'\right)' = 0 \\
&(EI_z v'')'' - (N^{o} v')' - z_v (N^{o} \phi')' + M_y^{o} \phi'' = 0 \\
&(EI_y w'')'' - (N^{o} w')' + y_v (N^{o} \phi')' - M_z^{o} \phi'' = 0 \\
&\left(EI_\omega \phi''\right)'' - (GI_t \phi')' + y_v (N^{o} w')' - z_v (N^{o} v')' - r^2 \left(N^{o} \phi'\right)' \\
&\qquad - M_z^{o} w'' + M_y^{o} v'' - 2\beta_y M_z^{o} \phi'' - 2\beta_z M_y^{o} \phi'' = 0
\end{aligned}
\tag{7.70}
$$

The energy method formulation will be obtained again by combining the equations (7.57) and (7.59). This results in the minimising problem: Find the minimum for the functional

$$
\begin{aligned}
\Pi = &\frac{1}{2}\int_0^L \left[EI_z (v'')^2 + EI_y (w'')^2 + EI_\omega (\phi'')^2 + GI_t (\phi')^2 \right] \mathrm{d}x \\
&+ \frac{1}{2}\int_0^L N^{o} \left[(w')^2 + (v')^2 - 2 y_v w' \phi' + 2 z_v v' \phi' + r^2 (\phi')^2 \right] \mathrm{d}x \\
&+ \int_0^L \left[M_z^{o} \left(w'\phi' + \beta_y (\phi')^2 \right) - M_y^{o} \left(v'\phi' - \beta_z (\phi')^2 \right) \right] \mathrm{d}x
\end{aligned}
\tag{7.71}
$$

among kinematically admissible functions $v(x), w(x), \phi(x)$. This formulation will yield an approximate solution for the problem of flexural-torsional buckling.

7.5. Lateral buckling

Lateral buckling is a phenomenon, in which a beam, loaded by an arbitrary bending including transverse distributed or concentrated loads along the beam axis, buckles by torsion and transverse deflection, Figure 4. In a more general case, also axial loads can be present. The problem of non-distortional lateral buckling of a straight beam with a thin-walled cross-section is investigated. The same principle as above is applied in deriving the equilibrium equations.

At the initial state, the external loading yields in the beam the normal stress distribution σ_x^{o} and shear stress distribution τ_{xs}^{o}. The consideration deviates from that of the flexural-torsional buckling problem in shear stresses and in corresponding shear force resultants Q_y^{o}, Q_z^{o}. The additional shear stresses bring to the analysis additional terms into the expressions of U_{NL_2}, and also in *V*. The initial shear stress distribution is of the form

$$
\tau_{xs}^{o} = -\frac{Q_y^{o} S_z(y)}{I_z t} - \frac{Q_z^{o} S_y(z)}{I_y t}
\tag{7.72}
$$

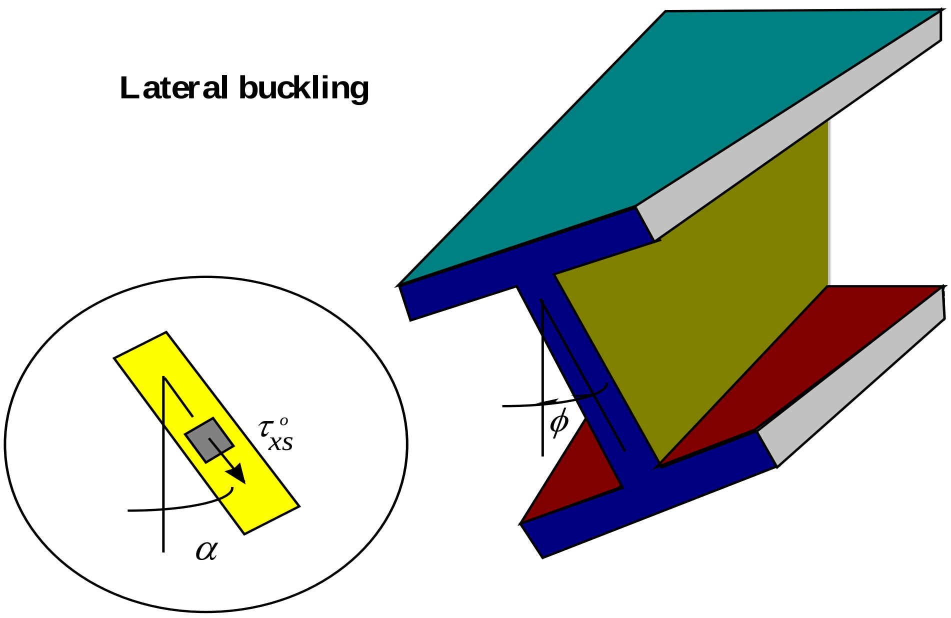


Figure 4. Lateral buckling of a beam

Here, $S_z(y)$ and $S_y(z)$ are the first moments of the cross-section, and are defined by

$$S_z(y) = \int_{A(y)} y(s)\mathrm{d}A, \qquad S_y(z) = \int_{A(z)} z(s)\mathrm{d}A, \tag{7.73}$$

At first, V takes the form

$$V = -\int_0^L \left[p_x^{\mathrm{o}}(x)u + p_y^{\mathrm{o}}(x)v + p_z^{\mathrm{o}}(x)w \right]\mathrm{d}x - \left[P_x^{\mathrm{o}}u + P_y^{\mathrm{o}}v + P_z^{\mathrm{o}}w + \bar{M}_z^{\mathrm{o}}v' + \bar{M}_y^{\mathrm{o}}w' \right]_0^L \tag{7.74}$$

Here to avoid confusion, the notations $\bar{M}_z^{\mathrm{o}}$ and $\bar{M}_y^{\mathrm{o}}$ are used for external bending moments at the ends of the beam. This as combined with U_{NL_1} in (7.43) and (7.59) takes the form

$$\delta(U_{NL_1} + V) = \int_0^L \left(N^{\mathrm{o}}\delta u' - M_z^{\mathrm{o}}\delta v'' - M_y^{\mathrm{o}}\delta w'' \right)\mathrm{d}x - \int_0^L \left[p_x^{\mathrm{o}}(x)\delta u + p_y^{\mathrm{o}}(x)\delta v + p_z^{\mathrm{o}}(x)\delta w \right]\mathrm{d}x$$
$$- \left[P_x^{\mathrm{o}}\delta u + P_y^{\mathrm{o}}\delta v + P_z^{\mathrm{o}}\delta w + \bar{M}_z^{\mathrm{o}}\delta v' + \bar{M}_y^{\mathrm{o}}\delta w' \right]_0^L \tag{7.75}$$

Performing the integration by parts results in

$$\delta(U_{NL_1}+V)=-\int_0^L\Big[\Big((N^{o})'+p_x^{o}(x)\Big)\delta u+\Big((M_z^{o})''+p_y^{o}(x)\Big)\delta v+\Big((M_y^{o})''+p_z^{o}(x)\Big)\delta w\Big]\mathrm{d}x$$
$$+\Big[(N^{o}-P_x^{o})\delta u+(M_z^{o}-\bar{M}_z^{o})\delta v'-((M_z^{o})'-P_y^{o})\delta v$$
$$+(M_y^{o}-\bar{M}_y^{o})\delta w'-((M_y^{o})'-P_z^{o})\delta w\Big]_0^L \tag{7.76}$$

Disappearance of this leads to the equilibrium equations and boundary conditions of the initial state

$$\frac{\mathrm{d}N^{o}}{\mathrm{d}x}+p_x^{o}(x)=0,\quad \frac{\mathrm{d}^2M_z^{o}}{\mathrm{d}x^2}+p_y^{o}(x)=0,\quad \frac{\mathrm{d}^2M_y^{o}}{\mathrm{d}x^2}+p_z^{o}(x)=0, \tag{7.77}$$

$$\left.\begin{array}{ll}
N^{o}=P_x^{o} & \text{or}\quad \delta u=0\\
M_z^{o}=\bar{M}_z^{o} & \text{or}\quad \delta v'=0\\
M_y^{o}=\bar{M}_y^{o} & \text{or}\quad \delta w'=0\\
(M_z^{o})'=Q_y=P_y^{o} & \text{or}\quad \delta v=0\\
(M_y^{o})'=Q_z=P_z^{o} & \text{or}\quad \delta w=0
\end{array}\right\}\quad \text{at } x=0 \text{ and } x=L$$

The stability analysis is based again on the term $U_L+U_{NL_2}$ of which U_L has no change as compared to the corresponding one derived in the case of torsional buckling in (7.50). Instead, the latter term U_{NL_2} will take an additional component due to the shear stresses at the initial configuration

$$\Delta U_{NL_2}=\int_V \tau_{xs}^{o}\gamma_{xs}^{NL}\mathrm{d}V=\int_V \tau_{xs}^{o}\left\{\left[w'+(y-y_v)\phi'\right]\phi\cos\alpha-\left[v'-(z-z_v)\phi'\right]\phi\sin\alpha\right\}\mathrm{d}V$$
$$=-\int_V\left[\frac{Q_y^{o}S_z(y)}{I_z t}+\frac{Q_z^{o}S_y(z)}{I_y t}\right]\left\{\left[w'+(y-y_v)\phi'\right]\phi\cos\alpha-\left[v'-(z-z_v)\phi'\right]\phi\sin\alpha\right\}\mathrm{d}V \tag{7.78}$$

in which γ_{xs}^{NL} is the non-linear part of strain component defined in (7.38). Inserting (7.72) into (7.78) requires certain integrals to be calculated over the cross-sectional area of the beam. These integrals are of the type

$$\int_A S_y(z)\underbrace{\sin\alpha}_{\mathrm{d}z/\mathrm{d}s}\,\underbrace{\mathrm{d}A}_{t\mathrm{d}s}=\left[S_y(z)tz\right]_z-\int_A z^2\mathrm{d}A=-I_y \tag{7.79}$$

and further,

$$
\begin{aligned}
&\int_A S_z(y)\cos\alpha \mathrm{d}A = -I_z \\
&\int_A S_z(y)\sin\alpha \mathrm{d}A = \int_A S_y(z)\cos\alpha \mathrm{d}A = I_{yz} = 0 \\
&\int_A S_y(z) z\sin\alpha \mathrm{d}A = -\int_A \tfrac{1}{2} z^3 \mathrm{d}A \\
&\int_A S_z(y) y\cos\alpha \mathrm{d}A = -\int_A \tfrac{1}{2} y^3 \mathrm{d}A \\
&\int_A S_y(z) y\cos\alpha \mathrm{d}A = -\int_A \tfrac{1}{2} zy^2 \mathrm{d}A \\
&\int_A S_z(y) z\sin\alpha \mathrm{d}A = -\int_A \tfrac{1}{2} yz^2 \mathrm{d}A
\end{aligned} \tag{7.80}
$$

Thus (7.78) takes the form

$$
\Delta U_{NL_2} = \int_0^L Q_y^{\mathrm{o}} \left(\beta_y \phi\phi' + w'\phi\right)\mathrm{d}x + \int_0^L Q_z^{\mathrm{o}} \left(\beta_z \phi\phi' - v'\phi\right)\mathrm{d}x \tag{7.81}
$$

with β_y and β_z defined in (7.61). Taking the variation of (7.81) gives

$$
\begin{aligned}
\delta(\Delta U_{NL_2}) = &\int_0^L Q_y^{\mathrm{o}} \left(\beta_y \phi'\delta\phi + \beta_y \phi\delta\phi' + w'\delta\phi + \phi\delta w'\right)\mathrm{d}x \\
&+ \int_0^L Q_z^{\mathrm{o}} \left(\beta_z \phi'\delta\phi + \beta_z \phi\delta\phi' - v'\delta\phi - \phi\delta v'\right)\mathrm{d}x
\end{aligned} \tag{7.82}
$$

and integration by parts yields further

$$
\begin{aligned}
\delta(\Delta U_{NL_2}) = &\int_0^L \left(\underline{\beta_y p_y^{\mathrm{o}} \phi\delta\phi} + Q_y^{\mathrm{o}} w'\delta\phi - (Q_y^{\mathrm{o}}\phi)'\delta w\right)\mathrm{d}x \\
&+ \int_0^L \left(\underline{\beta_z p_z^{\mathrm{o}} \phi\delta\phi} - Q_z^{\mathrm{o}} v'\delta\phi + (Q_z^{\mathrm{o}}\phi)'\delta v\right)\mathrm{d}x \\
&+ \left[\underline{(\beta_y Q_y^{\mathrm{o}} + \beta_z Q_z^{\mathrm{o}})\phi\delta\phi} + Q_y^{\mathrm{o}}\phi\delta w + Q_z^{\mathrm{o}}\phi\delta v\right]_{x=0}^{x=L}
\end{aligned} \tag{7.83}
$$

Combining this now with (7.51), (7.53) and (7.67) results in a homogeneous ordinary differential equation system

$$
\begin{aligned}
&(EAu')' = 0 \\
&(EI_z v'')'' - (N^{\mathrm{o}} v')' - z_v (N^{\mathrm{o}} \phi')' + (M_y^{\mathrm{o}} \phi)'' = 0 \\
&(EI_y w'')'' - (N^{\mathrm{o}} w')' + y_v (N^{\mathrm{o}} \phi')' - (M_z^{\mathrm{o}} \phi)'' = 0 \\
&(EI_\omega \phi'')'' - (GI_t \phi')' + y_v (N^{\mathrm{o}} w')' - z_v (N^{\mathrm{o}} v')' - r^2 (N^{\mathrm{o}} \phi')' \\
&\qquad - M_z^{\mathrm{o}} w'' + M_y^{\mathrm{o}} v'' - 2\beta_y (M_z^{\mathrm{o}} \phi')' - 2\beta_z (M_y^{\mathrm{o}} \phi')' + \\
&\qquad + \beta_y p_y^{\mathrm{o}} \phi + \beta_z p_z^{\mathrm{o}} \phi = 0
\end{aligned} \tag{7.84}
$$

with the boundary conditions

$$
\begin{aligned}
&EAu' = N = 0 && \text{or} \quad \delta u = 0 \\
&-EI_z v'' = M_z = 0 && \text{or} \quad \delta v' = 0 \\
&-EI_z w'' = M_y = 0 && \text{or} \quad \delta w' = 0 \\
&-EI_\omega \phi'' = B = 0 && \text{or} \quad \delta \phi' = 0 \\
&-(EI_z v'')' + N^{\mathrm{o}}(v' + z_v \phi') - (M_y^{\mathrm{o}} \phi)' = Q_y = 0 && \text{or} \quad \delta v = 0 \\
&-(EI_y w'')' + N^{\mathrm{o}}(w' - y_v \phi') + (M_z^{\mathrm{o}} \phi)' = Q_z = 0 && \text{or} \quad \delta w = 0 \\
&-(EI_\omega \phi'')' + GI_t \phi' - N^{\mathrm{o}}(y_v w' - z_v v' - r^2 \phi') + \\
&\qquad + M_z^{\mathrm{o}}(w' + 2\beta_y \phi') - M_y^{\mathrm{o}}(v' - 2\beta_z \phi') + \\
&\qquad + Q_y^{\mathrm{o}} \beta_y \phi + Q_z^{\mathrm{o}} \beta_z \phi = M_\omega + M_t = 0 && \text{or} \quad \delta \phi = 0
\end{aligned}
\tag{7.85}
$$

the first ones of equations (7.84) and (7.85) concern the initial state only, and are consequently meaningless in the stability analysis. When deriving system (7.84), the facts

$$
(M_z^{\mathrm{o}})' = Q_y^{\mathrm{o}} \qquad (M_y^{\mathrm{o}})' = Q_z^{\mathrm{o}}
$$

$$
(Q_y^{\mathrm{o}})' = -p_y^{\mathrm{o}} \qquad (Q_z^{\mathrm{o}})' = -p_z^{\mathrm{o}}
$$

have been utilised. The energy principle formulation can be built up by combining the integrals (7.57), (7.59) and (7.81) to give

$$
\begin{aligned}
\Pi = \frac{1}{2}\int_0^L &\left[EI_z (v'')^2 + EI_y (w'')^2 + EI_\omega (\phi'')^2 + GI_t (\phi')^2 \right] \mathrm{d}x \\
&+ \frac{1}{2}\int_0^L N^{\mathrm{o}} \left[(w')^2 + (v')^2 - 2 y_v w' \phi' + 2 z_v v' \phi' + r^2 (\phi')^2 \right] \mathrm{d}x \\
&+ \int_0^L \left[M_z^{\mathrm{o}} \left(w'\phi' + \beta_y (\phi')^2 \right) - M_y^{\mathrm{o}} \left(v'\phi' - \beta_z (\phi')^2 \right) \right] \mathrm{d}x \\
&+ \int_0^L Q_y^{\mathrm{o}} \left(\beta_y \phi \phi' + w' \phi \right) \mathrm{d}x + \int_0^L Q_z^{\mathrm{o}} \left(\beta_z \phi \phi' - v' \phi \right) \mathrm{d}x
\end{aligned}
\tag{7.86}
$$

The two last lines can still be combined, when the final expression for the total potential energy to be minimised is

$$
\begin{aligned}
\Pi = \frac{1}{2}\int_0^L & \left[EI_z(v'')^2 + EI_y(w'')^2 + EI_\omega(\phi'')^2 + GI_t(\phi')^2 \right]\mathrm{d}x \\
& + \frac{1}{2}\int_0^L N^{\mathrm{o}}\left[(w')^2 + (v')^2 - 2y_v w'\phi' + 2z_v v'\phi' + r^2(\phi')^2 \right]\mathrm{d}x \\
& + \int_0^L \left[\left((M_z^{\mathrm{o}}\phi)'w' + \beta_y (M_z^{\mathrm{o}}\phi)'\phi' \right) - \left((M_y^{\mathrm{o}}\phi)'v' - \beta_z (M_y^{\mathrm{o}}\phi)'\phi' \right) \right]\mathrm{d}x
\end{aligned} \tag{7.87}
$$

In pure lateral buckling without the compressive axial load, these equations will obviously be simpler.

In final equations (7.84) and (7.87), the position of the external load in *y*- and *z*-co-ordinate directions has no role. It is however obvious and easy to understand that this position plays an important role in the lateral buckling phenomenon. Taking this into account means actually a step outside the traditional one-dimensional beam theory, towards two- or three-dimensional analysis. It can be done by improving the kinematics due to the rigid body rotation, to include also the second order terms – to correspond to the second order theory used elsewhere. By considering Figure 5, we can deduce more generally the co-ordinates of a point A when undergoing rigid body rotation. A simple calculation gives

$$
\begin{aligned}
\bar{z} - z_v &= r\left(\cos(\theta-\phi) - \cos\theta\right) \\
&= r(\cos\theta\cos\phi + \sin\theta\sin\phi - \cos\theta) \\
&= r(\cos\theta(1 - \tfrac{1}{2}\phi^2 + \mathcal{O}(\phi^4) - 1) + \sin\theta(\phi + \mathcal{O}(\phi^3)) \\
&\approx r\cos\theta(-\tfrac{1}{2}\phi^2) + r\sin\theta(\phi) = -\tfrac{1}{2}(z - z_v)\phi^2 + (y - y_v)\phi
\end{aligned} \tag{7.88}
$$

and correspondingly

$$
\begin{aligned}
\bar{y} - y_v &= r\left(\sin(\theta-\phi) - \sin\theta\right) \\
&= r(\sin\theta\cos\phi - \cos\theta\sin\phi - \sin\theta \\
&= r(\sin\theta(1 - \tfrac{1}{2}\phi^2 + \mathcal{O}(\phi^4) - 1) - \cos\theta(\phi + \mathcal{O}(\phi^3)) \\
&\approx r\sin\theta(-\tfrac{1}{2}\phi^2) - r\cos(\phi) = -\tfrac{1}{2}(y - y_v)\phi^2 - (z - z_v)\phi
\end{aligned} \tag{7.89}
$$

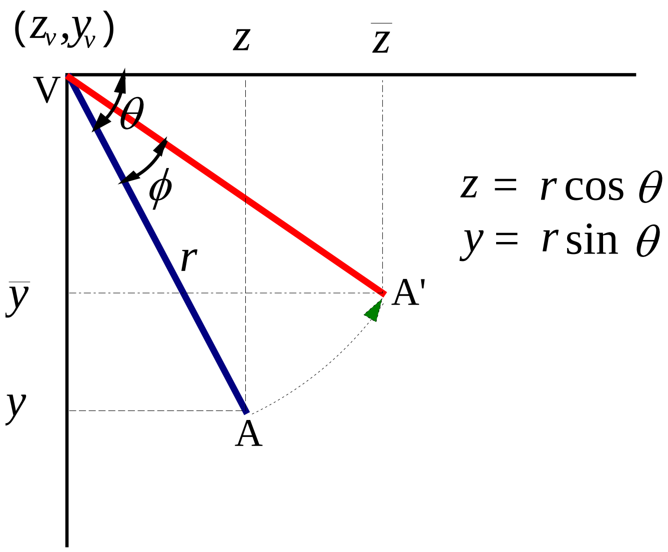


Figure 5. Rotation of a fiber

Adopting these in the definition of the kinematics applied yields the displacement vector

$$\mathbf{u}_o = (u - yv' - zw' - \omega\phi')\mathbf{i} + \left(v - (z - z_v)\phi \underline{- \tfrac{1}{2}(y - y_v)\phi^2}\right)\mathbf{j} + \left(w + (y - y_v)\phi \underline{- \tfrac{1}{2}(z - z_v)\phi^2}\right)\mathbf{k} \tag{7.90}$$

The additional underlined terms in (7.90) will produce terms of same order in the shear strains only. They are denoted by a superscript $(\cdot)^*$, and are

$$\begin{aligned} \gamma^*_{xy} &= 2e_{xy} = -2(y - y_v)\phi\phi' \\ \gamma^*_{zx} &= 2e_{zx} = -2(z - z_v)\phi\phi' \end{aligned} \tag{7.91}$$

These can be combined into the shear in the mid plane of each wall of the cross-section

$$\gamma^*_{xs} = -\left[(y - y_v)\cos\alpha + (z - z_v)\sin\alpha\right]\phi\phi' \tag{7.92}$$

This shear strain component results in an additional term in U_{NL_1} which includes exceptionally terms of second order in ▯.

$$\begin{aligned} \Delta U^*_{NL_1} &= \int_V \tau^o_{xs}\gamma^*_{xs}\mathrm{d}V = -\int_V \tau^o_{xs}\left\{\left[(y - y_v)\cos\alpha + (z - z_v)\sin\alpha\right]\phi\phi'\right\}\mathrm{d}V \\ &= \int_V \left[\frac{Q^o_y S_z(y)}{I_z t} + \frac{Q^o_z S_y(z)}{I_y t}\right]\left\{\left[(y - y_v)\cos\alpha + (z - z_v)\sin\alpha\right]\phi\phi'\right\}\mathrm{d}V \end{aligned} \tag{7.93}$$

Comparing this with the expression of ΔU_{NL_2} in (7.78) shows that (7.93) is a part of the former provided with an opposite sign and cancels it. This can be verified further by evaluating (7.93) to give

$$\Delta U^*_{NL_1} = -\int_0^L \left(Q^{\mathrm{o}}_y \beta_y + Q^{\mathrm{o}}_z \beta_z \right) \phi \phi' \mathrm{d}x \qquad (7.94)$$

Taking the first variation an integrating by parts yields

$$\begin{aligned}
\delta(\Delta U^*_{NL_1}) &= -\int_0^L \left(Q^{\mathrm{o}}_y \beta_y + Q^{\mathrm{o}}_z \beta_z \right) (\phi \delta\phi' + \phi' \delta\phi) \mathrm{d}x \\
&= \int_0^L \left(\beta_y \left((Q^{\mathrm{o}}_y \phi)' - Q^{\mathrm{o}}_y \phi' \right) + \beta_z \left((Q^{\mathrm{o}}_z \phi)' - Q^{\mathrm{o}}_z \phi' \right) \right) \delta\phi \mathrm{d}x - \left[(\beta_y Q^{\mathrm{o}}_y + \beta_z Q^{\mathrm{o}}_z) \phi \delta\phi \right]_0^L \\
&= \int_0^L \left(\beta_y (Q^{\mathrm{o}}_y)' + \beta_z (Q^{\mathrm{o}}_z)' \right) \phi \delta\phi \mathrm{d}x - \left[(\beta_y Q^{\mathrm{o}}_y + \beta_z Q^{\mathrm{o}}_z) \phi \delta\phi \right]_0^L \\
&= -\int_0^L \left(\beta_y p^{\mathrm{o}}_y + \beta_z p^{\mathrm{o}}_z \right) \phi \delta\phi \mathrm{d}x - \left[(\beta_y Q^{\mathrm{o}}_y + \beta_z Q^{\mathrm{o}}_z) \phi \delta\phi \right]_0^L \qquad (7.95)
\end{aligned}$$

Now, it is easy to see the cancelling terms both in the integral and boundary terms as compared to the underlined terms in (7.83). The potential of the external loads will include also additional terms due to the improved kinematics taking the expression

$$\begin{aligned}
V &= -\int_0^L \left[p^{\mathrm{o}}_y \left(v - (y_p - y_v)(1 - \cos\phi) \right) + p^{\mathrm{o}}_z \left(w - (z_p - z_v)(1 - \cos\phi) \right) \right] \mathrm{d}x \\
&= -\int_0^L \left[p^{\mathrm{o}}_y \left(v - \tfrac{1}{2}(y_p - y_v)\phi^2 \right) + p^{\mathrm{o}}_z \left(w - \tfrac{1}{2}(z_p - z_v)\phi^2 \right) \right] \mathrm{d}x
\end{aligned} \qquad (7.96)$$

The part complementing this, denoted again by star, is

$$V^* = \int_0^L (\tfrac{1}{2} p^{\mathrm{o}}_y e_y + \tfrac{1}{2} p^{\mathrm{o}}_z e_z) \phi^2 \mathrm{d}x \qquad (7.97)$$

where $e_y = y_p - y_v$ and $e_z = z_p - z_v$ are the distances of the loading points from the shear center. This will directly be included in the final differential equation system

$$\begin{aligned}
&(EI_z v'')'' - (N^{\mathrm{o}} v')' - z_v (N^{\mathrm{o}} \phi')' + (M^{\mathrm{o}}_y \phi)'' = 0 \\
&(EI_y w'')'' - (N^{\mathrm{o}} w')' + y_v (N^{\mathrm{o}} \phi')' - (M^{\mathrm{o}}_z \phi)'' = 0 \\
&(EI_\omega \phi'')'' - (GI_t \phi')' + y_v (N^{\mathrm{o}} w')' - z_v (N^{\mathrm{o}} v')' - r^2 (N^{\mathrm{o}} \phi')' \\
&\qquad - M^{\mathrm{o}}_z w'' + M^{\mathrm{o}}_y v'' - 2\beta_y (M^{\mathrm{o}}_z \phi')' - 2\beta_z (M^{\mathrm{o}}_y \phi')' + p^{\mathrm{o}}_y e_y \phi + p^{\mathrm{o}}_z e_z \phi = 0
\end{aligned} \qquad (7.98)$$

providing the system with the position factor of the external transverse load. The boundary conditions corresponding to these are

$$
\begin{aligned}
&EAu' = N = 0 && \text{or} \quad \delta u = 0 \\
&-EI_z v'' = M_z = 0 && \text{or} \quad \delta v' = 0 \\
&-EI_z w'' = M_y = 0 && \text{or} \quad \delta w' = 0 \\
&-EI_\omega \phi'' = B = 0 && \text{or} \quad \delta \phi' = 0 \\
&-(EI_z v'')' + N^{\mathrm{o}}(v' + z_v \phi') - (M_y^{\mathrm{o}} \phi)' = Q_y = 0 && \text{or} \quad \delta v = 0 \\
&-(EI_y w'')' + N^{\mathrm{o}}(w' - y_v \phi') + (M_z^{\mathrm{o}} \phi)' = Q_z = 0 && \text{or} \quad \delta w = 0 \\
&-(EI_\omega \phi'')' + GI_t \phi' - N^{\mathrm{o}}(y_v w' - z_v v' - r^2 \phi') + \\
&\quad + M_z^{\mathrm{o}}(w' + 2\beta_y \phi') - M_y^{\mathrm{o}}(v' - 2\beta_z \phi') = M_\omega + M_t = 0 && \text{or} \quad \delta \phi = 0
\end{aligned}
\tag{7.99}
$$

The corresponding modifications are also present in the expression of the total strain energy

$$
\begin{aligned}
\Pi = \frac{1}{2}\int_0^L \Big[EI_z (v'')^2 + EI_y (w'')^2 + EI_\omega (\phi'')^2 + GI_t (\phi')^2 \Big] \mathrm{d}x \\
+ \frac{1}{2}\int_0^L N^{\mathrm{o}} \Big[(w')^2 + (v')^2 - 2y_v w'\phi' + 2z_v v'\phi' + r^2 (\phi')^2 \Big] \mathrm{d}x \\
+ \int_0^L \Big[\Big((M_z^{\mathrm{o}}\phi)' w' + \beta_y M_z^{\mathrm{o}} (\phi')^2 \Big) - \Big((M_y^{\mathrm{o}}\phi)' v' - \beta_z M_y^{\mathrm{o}} (\phi')^2 \Big) \Big] \mathrm{d}x \\
+ \frac{1}{2}\int_0^L \Big(p_y^{\mathrm{o}} e_y \phi^2 + p_z^{\mathrm{o}} e_z \phi^2 \Big) \mathrm{d}x
\end{aligned}
\tag{7.100}
$$

The final expressions derived, both the differential equation system and the energy principle, form the basis to evaluate the critical load intensity for rather general stability problem of any one-dimensional beam. They are compatible with the equilibrium equations derived by using the fairly complicated tool of differential geometry in the general literature of structural stability. The procedure presented here is very systematic serving as an ideologically simple way to handle these problems.

7.6. Buckling of plates

Buckling of plates is as a problem very similar to the flexural buckling of beams, but extended to two dimensions only. The procedure applied here follows rather exactly the one used above in beam analyses. The structure considered is a two-dimensional rectangular plate located in three-dimensional Cartesian space so, that *x*- and *y*-co-ordinate axes coincide with the mid surface of the plate, and *z* is in the direction of the normal of plate. Linearly elastic material model is adopted, with the parameters E, ν, the Young’s modulus and Poisson’s ratio, respectively. The thickness of the plate is *h*.

The kinematics of the plate is given by applying the Love-Kirchhoff plate model, in which each normal is assumed to remain normal to the deformed mid surface of the deformed geometry. The displacement vector is thus

$$\mathbf{u} = (u - z\frac{\partial w}{\partial x})\mathbf{i} + (v - z\frac{\partial w}{\partial y})\mathbf{j} + w\mathbf{k} \tag{7.101}$$

Here, the displacement components are $u = u(x, y),\ v = v(x, y)$ and $w = w(x, y)$, and the unit vectors in the direction of co-ordinate axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The linear strain and rotation components are calculated again directly by applying the definitions (7.2) and (7.3) with $e_z = e_{yz} = e_{zx} = 0$

$$\begin{aligned}
e_x &= \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\
e_y &= \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\
e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \\
\omega_x &= \frac{\partial w}{\partial y} \\
\omega_y &= -\frac{\partial w}{\partial x} \\
\omega_z &= \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \approx 0
\end{aligned} \tag{7.102}$$

resulting in the nonzero strains $(\gamma_{yz} = \gamma_{zx} = 0)$ according to (7.7)

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}(\frac{\partial w}{\partial x})^2 \\
\varepsilon_y &= \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}(\frac{\partial w}{\partial y})^2 \\
\varepsilon_z &= \frac{1}{2}\left[(\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2\right] \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned} \tag{7.103}$$

The simplification performed is based on the assumption that the rotation in the plane of the plate is small as compared to the rotations out of the plate in buckling.

The expression of the strain energy

$$U = U^{\mathrm{o}} + U^{*} = \int_V \left[(\sigma_x^{\mathrm{o}} + \tfrac{1}{2}\sigma_x^{*})\varepsilon_x + (\sigma_y^{\mathrm{o}} + \tfrac{1}{2}\sigma_y^{*})\varepsilon_y + (\tau_{xy}^{\mathrm{o}} + \tfrac{1}{2}\tau_{xy}^{*})\gamma_{xy} + \tfrac{1}{2}\sigma_z^{*}\varepsilon_z\right]\mathrm{d}V \tag{7.104}$$

can further be split to give

$$U = \int_V \tfrac{1}{2}\left[\sigma_x^*(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}) + \sigma_y^*(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}) + \tau_{xy}^*(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}) \right] \mathrm{d}V$$
$$+ \int_V \left[\sigma_x^0(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}) + \sigma_y^0(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}) + \tau_{xy}^0(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}) \right] \mathrm{d}V \qquad (7.105)$$
$$+ \int_V \tfrac{1}{2}\left[\sigma_x^0(\frac{\partial w}{\partial x})^2 + \sigma_y^0(\frac{\partial w}{\partial y})^2 + \tau_{xy}^0 \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] \mathrm{d}V + \underbrace{\int_V \tfrac{1}{4}\sigma_z^* \left[(\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 \right] \mathrm{d}V}_{\approx 0}$$

This is composed of three parts following the practice above as

$$U = U_L + U_{NL_1} + U_{NL_2} \qquad (7.106)$$

in which U_L is the traditional strain energy expression due to linearised strains and the two following ones, U_{NL_1} and U_{NL_2}, take into account both the non-linear terms of strain components and the initial stresses, i.e.

$$U_L = \int_V \tfrac{1}{2}\left[\sigma_x^*(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}) + \sigma_y^*(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}) + \tau_{xy}^*(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}) \right] \mathrm{d}V$$
$$U_{NL_1} = \int_V \left[\sigma_x^0(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}) + \sigma_y^0(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}) + \tau_{xy}^0(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}) \right] \mathrm{d}V \qquad (7.107)$$
$$U_{NL_2} = \int_V \tfrac{1}{2}\left[\sigma_x^0(\frac{\partial w}{\partial x})^2 + \sigma_y^0(\frac{\partial w}{\partial y})^2 + \tau_{xy}^0 \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] \mathrm{d}V$$

When deriving the expression for U_L, the two dimensional linear elastic plane stress state as a material model is adopted. Then

$$\sigma_x^* = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$
$$\sigma_y^* = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad (7.108)$$
$$\tau_{xy}^* = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$

Inserting this into the expression of the linearised strain energy gives

$$U_L = \frac{1}{2}\int_V \left(\sigma_x^*\varepsilon_x + \sigma_y^*\varepsilon_y + \tau_{xy}^*\gamma_{xy} \right) \mathrm{d}V$$
$$= \frac{E}{2(1-\nu^2)}\int_V \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \tfrac{1}{2}(1-\nu)\gamma_{xy}^2 \right) \mathrm{d}V \qquad (7.109)$$

Incorporating still the linearised strains from (7.103) or (7.102), and denoting the bending stiffness of the plate by $D = Eh^3/12(1-\nu^2)$ we get

$$U_L = \frac{1}{2}\frac{E}{1-\nu^2}\underbrace{\int_{-h/2}^{h/2}\mathrm{d}z}_{=h}\int_A\left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1}{2}(1-\nu)(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2\right]\mathrm{d}x\mathrm{d}y$$
$$+\frac{1}{2}\underbrace{\frac{E}{1-\nu^2}\int_{-h/2}^{h/2}z^2\mathrm{d}z}_{=D}\int_A\left[(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)(\frac{\partial^2 w}{\partial x\partial y})^2\right]\mathrm{d}x\mathrm{d}y \quad (7.110)$$

Here, it is assumed that the integral $\int_{-h/2}^{h/2} z\mathrm{d}z = 0$. The non-linear parts of the potential energy expression are simplifying to the form

$$U_{NL_1} = \int_V\left[\sigma_x^{\mathrm{o}}\frac{\partial u}{\partial x} + \sigma_y^{\mathrm{o}}\frac{\partial v}{\partial y} + \tau_{xy}^{\mathrm{o}}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})\right]\mathrm{d}V$$
$$= \int_A\left[N_x^{\mathrm{o}}\frac{\partial u}{\partial x} + N_y^{\mathrm{o}}\frac{\partial v}{\partial y} + N_{xy}^{\mathrm{o}}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})\right]\mathrm{d}x\mathrm{d}y \quad (7.111)$$

where $N_x^{\mathrm{o}} = \sigma_x^{\mathrm{o}}h,\ N_y^{\mathrm{o}} = \sigma_y^{\mathrm{o}}h$ and $N_{xy}^{\mathrm{o}} = \tau_{xy}^{\mathrm{o}}h$, and consequently

$$U_{NL_2} = \frac{1}{2}\int_V\left[\sigma_x^{\mathrm{o}}(\frac{\partial w}{\partial x})^2 + \sigma_y^{\mathrm{o}}(\frac{\partial w}{\partial y})^2 + \tau_{xy}^{\mathrm{o}}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}V$$
$$= \frac{1}{2}\int_A\left[N_x^{\mathrm{o}}(\frac{\partial w}{\partial x})^2 + N_y^{\mathrm{o}}(\frac{\partial w}{\partial y})^2 + N_{xy}^{\mathrm{o}}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}x\mathrm{d}y \quad (7.112)$$

The potential of the external load V acting on the plane of the mid surface of plate and including the volume forces and the loads on the edges of plate is

$$V = -\int_V\left[p_x^{\mathrm{o}}u + p_y^{\mathrm{o}}v\right]\mathrm{d}V - \oint_S\left[t_x^{\mathrm{o}}u + t_y^{\mathrm{o}}v\right]\mathrm{d}S$$
$$= -\int_A\left[P_x^{\mathrm{o}}u + P_y^{\mathrm{o}}v\right]\mathrm{d}x\mathrm{d}y - \oint_s\left[T_x^{\mathrm{o}}u + T_y^{\mathrm{o}}v\right]\mathrm{d}s \quad (7.113)$$

Here, S covers the area of boundary surfaces, and s is the co-ordinate, following the boundary mid-line. The notations $P_x^{\mathrm{o}} = p_x^{\mathrm{o}}h,\ P_y^{\mathrm{o}} = p_y^{\mathrm{o}}h,\ T_x^{\mathrm{o}} = t_x^{\mathrm{o}}h,$ and $T_y^{\mathrm{o}} = t_y^{\mathrm{o}}h$ for the loading components are adopted. When considering the term

$$U_{NL_1} + V = \int_A \left[N_x^o \frac{\partial u}{\partial x} + N_y^o \frac{\partial v}{\partial y} + N_{xy}^o (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - P_x^o u + P_y^o v \right] \mathrm{d}x\mathrm{d}y$$
$$- \oint_s \left[T_x^o u + T_y^o v \right] \mathrm{d}s \tag{7.114}$$

taking the first variation

$$\delta(U_{NL_1} + V) = \int_A \left[N_x^o \frac{\partial \delta u}{\partial x} + N_y^o \frac{\partial \delta v}{\partial y} + N_{xy}^o (\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}) - P_x^o \delta u + P_y^o \delta v \right] \mathrm{d}x\mathrm{d}y$$
$$- \oint_s \left[T_x^o \delta u + T_y^o \delta v \right] \mathrm{d}s \tag{7.115}$$

and integrating by parts gives

$$\delta(U_{NL_1} + V) = -\int_A \left[(\frac{\partial N_x^o}{\partial x} + \frac{\partial N_{xy}^o}{\partial y} + P_x^o)\delta u + (\frac{\partial N_y^o}{\partial y} + \frac{\partial N_{xy}^o}{\partial x} + P_y^o)\delta v \right] \mathrm{d}x\mathrm{d}y$$
$$+ \int_{s_y} [(N_x^o - T_x^o)\delta u + (N_{xy}^o - T_y^o)\delta v]\,\mathrm{d}y + \int_{s_x} [(N_y^o - T_y^o)\delta v + (N_{xy}^o - T_x^o)\delta u]\,\mathrm{d}x \tag{7.116}$$

In the equilibrium this must vanish producing the equilibrium conditions of the initial state of the plate as

$$\frac{\partial N_x^o}{\partial x} + \frac{\partial N_{xy}^o}{\partial y} + P_x^o = 0$$
$$\frac{\partial N_y^o}{\partial y} + \frac{\partial N_{xy}^o}{\partial x} + P_y^o = 0 \tag{7.117}$$

with initial boundary conditions

$$\left.\begin{aligned} N_x^o &= T_x^o \quad \text{or} \quad \delta u = 0 \\ N_{xy}^o &= T_y^o \quad \text{or} \quad \delta v = 0 \end{aligned}\right\} \text{ on the boundaries parallell to } y\text{-axis}$$
$$\left.\begin{aligned} N_{xy}^o &= T_x^o \quad \text{or} \quad \delta u = 0 \\ N_y^o &= T_y^o \quad \text{or} \quad \delta v = 0 \end{aligned}\right\} \text{ on the boundaries parallell to } x\text{-axis} \tag{7.118}$$

The boundary conditions can be expressed on any boundary generally by

$$N_x^o n_x + N_{xy}^o n_y = T_x^o$$
$$N_y^o n_y + N_{xy}^o n_x = T_y^o \tag{7.119}$$

with n_x and n_y the direction cosines of the normal of the boundary surface. We have still the terms

$$U_L + U_{NL_2} = \frac{1}{2}\frac{Eh}{1-\nu^2}\int_A \left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\nu \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + 2(1-\nu)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 \right] \mathrm{d}x\mathrm{d}y$$

$$+ \frac{D}{2}\int_A \left[(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)(\frac{\partial^2 w}{\partial x \partial y})^2 \right. \tag{7.120}$$

$$\left. + \frac{N_x^{\mathrm{o}}}{D}(\frac{\partial w}{\partial x})^2 + \frac{N_y^{\mathrm{o}}}{D}(\frac{\partial w}{\partial y})^2 + 2\frac{N_{xy}^{\mathrm{o}}}{D}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] \mathrm{d}x\mathrm{d}y$$

The first term in this equation concerns the initial state of the plate and will be dropped. The rest of (7.120) takes after variation the form

$$\delta(U_L + U_{NL_2}) = D\int_A \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2} + \nu(\frac{\partial^2 \delta w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2}) \right.$$

$$+ 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y} \tag{7.121}$$

$$\left. + \frac{N_x^{\mathrm{o}}}{D}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{N_y^{\mathrm{o}}}{D}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + \frac{N_{xy}^{\mathrm{o}}}{D}(\frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y}) \right] \mathrm{d}x\mathrm{d}y$$

Integrating twice by parts simplifies the expression to

$$\delta(U_L + U_{NL_2}) = D\int_A \left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x^{\mathrm{o}}}{D}\frac{\partial^2 w}{\partial x^2} - \frac{N_y^{\mathrm{o}}}{D}\frac{\partial^2 w}{\partial y^2} - 2\frac{N_{xy}^{\mathrm{o}}}{D}\frac{\partial^2 w}{\partial x \partial y} \right] \delta w \mathrm{d}x\mathrm{d}y \tag{7.122}$$

without boundary conditions. The disappearance of the surface integral gives the final homogeneous partial differential equation, which is

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) = N_x^{\mathrm{o}}\frac{\partial^2 w}{\partial x^2} + N_y^{\mathrm{o}}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}^{\mathrm{o}}\frac{\partial^2 w}{\partial x \partial y} \tag{7.123}$$

The boundary conditions on all edges of the plate appearing in integration by parts follow the plate theory of Love-Kirchhoff and can be given in the form

$$\left. \begin{array}{l} \dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2} = \dfrac{M_x}{D} = 0, \quad \text{or} \quad \dfrac{\partial \delta w}{\partial x} = 0 \\ \dfrac{\partial^3 w}{\partial x^3} + (2-\nu)\dfrac{\partial^3 w}{\partial x \partial y^2} = \dfrac{1}{D}\underbrace{(Q_x + \dfrac{\partial M_{xy}}{\partial y})}_{V_x} = \dfrac{N_x^{\mathrm{o}}}{D}\dfrac{\partial w}{\partial x} + \dfrac{N_{xy}^{\mathrm{o}}}{D}\dfrac{\partial w}{\partial y}, \quad \text{or} \quad \delta w = 0 \end{array} \right\} \begin{array}{l} \text{on the} \\ \text{boundaries} \\ \text{parallell} \\ \text{to } y\text{-axis} \end{array} \tag{7.124}$$

$$\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}=\frac{M_y}{D}=0, \qquad \text{or} \qquad \frac{\partial \delta w}{\partial y}=0$$

$$\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}=\underbrace{\frac{1}{D}(Q_y+\frac{\partial M_{xy}}{\partial x})}_{V_y}=\frac{N_y^{\mathrm{o}}}{D}\frac{\partial w}{\partial y}+\frac{N_{xy}^{\mathrm{o}}}{D}\frac{\partial w}{\partial x}, \quad \text{or} \quad \delta w=0$$

on the boundaries parallell to x-axis

with V_x and V_y the Kirchhoff shear forces. These can be combined in boundary conditions in general form

$$M_x n_x + M_y n_y = 0 \quad \text{or} \quad \frac{\partial \delta w}{\partial x}n_x+\frac{\partial \delta w}{\partial y}n_y=0$$

$$V_x n_x + V_y n_y = (N_x^{\mathrm{o}}n_x+N_{xy}^{\mathrm{o}}n_y)\frac{\partial w}{\partial x}+(N_y^{\mathrm{o}}n_y+N_{xy}^{\mathrm{o}}n_x)\frac{\partial w}{\partial y} \quad \text{or} \quad \delta w=0 \tag{7.125}$$

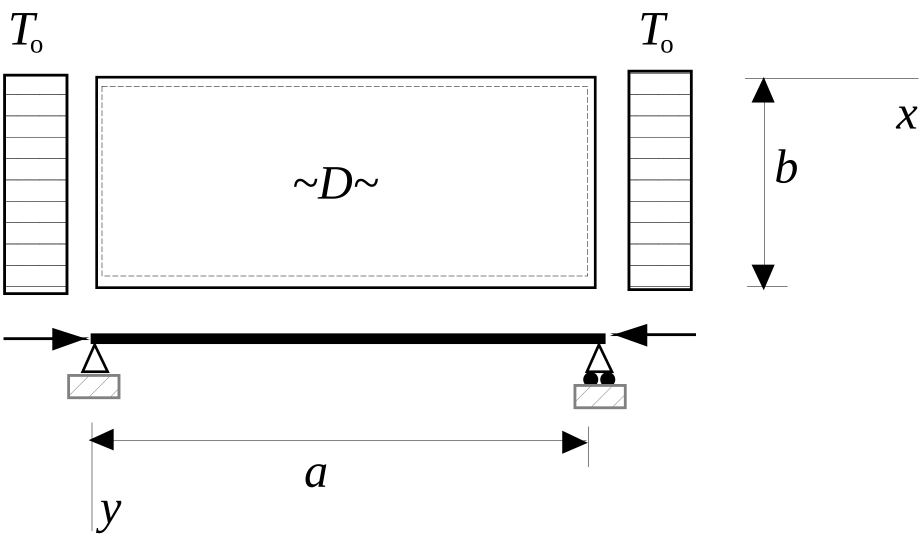
Solving equation (7.123) with boundary restrictions (7.124) determines finally the critical intensity for the loading of a plate due to lateral buckling.

The total potential energy formulation is actually in equation (7.120)

$$\Pi=\frac{D}{2}\int_A\left[(\frac{\partial^2 w}{\partial x^2})^2+(\frac{\partial^2 w}{\partial y^2})^2+2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2(1-\nu)(\frac{\partial^2 w}{\partial x \partial y})^2 \right.$$
$$\left. +\frac{N_x^{\mathrm{o}}}{D}(\frac{\partial w}{\partial x})^2+\frac{N_y^{\mathrm{o}}}{D}(\frac{\partial w}{\partial y})^2+2\frac{N_{xy}^{\mathrm{o}}}{D}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}x\mathrm{d}y \tag{7.126}$$

which has to be minimised in kinematically admissible deflection functions according to the energy principle.

Example: The problem is to find out the critical compressive load T_{o} for a plate shown with dimensions $(a \times b)$ and bending stiffness D. The plate is simply supported along the boundaries.



The differential equation for the problem takes the form

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{T_o}{D}\frac{\partial^2 w}{\partial x^2}$$

and the boundary conditions

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$$
$$\frac{\partial^2 w}{\partial x^2}(0,y) = \frac{\partial^2 w}{\partial x^2}(a,y) = \frac{\partial^2 w}{\partial y^2}(x,0) = \frac{\partial^2 w}{\partial y^2}(x,b) = 0$$

The Navier's solution for a plate with all edges simply supported fulfils the boundary conditions under consideration. It is

$$w(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})$$

Inserting this into the differential equation yields a condition

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\left[\frac{n^4\pi^4}{a^4} + 2\frac{n^2\pi^2}{a^2}\frac{m^2\pi^2}{b^2} + \frac{m^4\pi^4}{b^4} - \frac{T_o}{\mathrm{D}}\frac{n^2\pi^2}{a^2}\right]\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}) = 0$$

This condition will be fulfilled when the expression in brackets disappears. From this we can solve the value for the compressive load, which is

$$T_o = D\pi^2\frac{a^2}{n^2}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^2 = D\pi^2\left(\frac{n}{a} + \frac{m^2 a}{nb^2}\right)^2$$

We have to find out the minimum value for the load with respect to *m* and *n*. The minimum value for T_o is obtained with respect to *m*, when *m*=1, but for *n*, it must be calculated by differentiating. Thus it is obtained

$$\frac{\mathrm{d}T_o}{\mathrm{d}n} = 2D\pi^2\left(\frac{1}{a} - \frac{a}{n^2 b^2}\right)\left(\frac{n}{a} + \frac{a}{nb^2}\right) = 2D\pi^2\left(\frac{n}{a^2} - \frac{a^2}{n^3 b^4}\right) = 0 \quad \Rightarrow \quad n = \frac{a}{b}$$

Because *n* is an integer variable, the minimum value is obtained, when it will be taken as close to the value *a/b* as possible. If *a/b* is an integer, the minimum value is

$$T_o = \frac{4\pi^2 D}{b^2} \quad \square$$

The same problem can be handled by the potential energy formulation. Thus for example, a kinematically admissible basic set of functions to be used is

$$w(x, y) = w_o x(a - x) y(b - y)$$

This fulfils the boundary conditions $w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0$ for the deflection of the plate. By incorporating this into the energy integral gives at first, the derivatives

$$\begin{aligned}
&w_{,x}(x, y) = w_o(a - 2x) y(b - y) \\
&w_{,xx}(x, y) = -2w_o y(b - y) \\
&w_{,y}(x, y) = w_o x(a - x)(b - 2y) \\
&w_{,yy}(x, y) = -2w_o x(a - x) \\
&w_{,xy}(x, y) = w_o(a - 2x)(b - 2y)
\end{aligned}$$

and then

$$\begin{aligned}
\Pi &= \frac{D}{2}\int_A \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{N_x^o}{D}\left(\frac{\partial w}{\partial x}\right)^2 \right] \mathrm{d}x\mathrm{d}y \\
&= \frac{w_o^2 D}{2}\int_0^a\int_0^b \Big[4y^2(b-y)^2 + 4x^2(a-x)^2 + 8\nu xy(a-x)(b-y) + 2(1-\nu)(a-2x)^2(b-2y)^2 \\
&\qquad - \frac{T_o}{D}(a-2x)^2 y^2 (b-y)^2 \Big] \mathrm{d}x\mathrm{d}y
\end{aligned}$$

Taking the first variation and presuming it to disappear, gives

$$\begin{aligned}
\Pi &= \frac{Dw_o^2}{2}\left[4ab(3a^4 + 5a^2b^2 + 3b^4) - \frac{T_o a^3 b^5}{D} \right] \Rightarrow \\
\delta\Pi &= \frac{\partial \Pi}{\partial w_o}\delta w_o = Dw_o \left[4ab(3a^4 + 5a^2b^2 + 3b^4) - \frac{T_o a^3 b^5}{D} \right] \delta w_o = 0
\end{aligned}$$

This results in the critical value of the load parameter

$$T_o = \frac{4(3a^4 + 5a^2b^2 + 3b^4)D}{a^2 b^4} \quad \square$$

If $a = b$, the result is $T_o = 44D/b^2$, which is 11.45% higher than above.

If we still apply the set of functions

$$w(x, y) = w_o \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

and insert it into the energy equation, we will get

$$\Pi = \frac{D}{2}\int_A \left[(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)(\frac{\partial^2 w}{\partial x \partial y})^2 + \frac{N_x^{\mathrm{o}}}{D}(\frac{\partial w}{\partial x})^2 \right] \mathrm{d}x\mathrm{d}y$$

$$= \frac{w_{\mathrm{o}}^2 D}{2}\left[\left((\frac{\pi}{a})^4 + (\frac{\pi}{b})^4 + 2\nu(\frac{\pi}{a})^2(\frac{\pi}{b})^2 \right) \right] \int_0^a \int_0^b \sin^2(\frac{\pi x}{a}) \sin^2(\frac{\pi y}{b}) \mathrm{d}x\mathrm{d}y$$

$$+ (1-\nu) w_{\mathrm{o}}^2 D (\frac{\pi}{a})^2 (\frac{\pi}{b})^2 \int_0^a \int_0^b \cos^2(\frac{\pi x}{a}) \cos^2(\frac{\pi y}{b}) \mathrm{d}x\mathrm{d}y$$

$$+ \frac{N_x^{\mathrm{o}}}{2} (\frac{\pi}{a})^2 w_{\mathrm{o}}^2 \int_0^a \int_0^b \cos^2(\frac{\pi x}{a}) \sin^2(\frac{\pi y}{b}) \mathrm{d}x\mathrm{d}y$$

This will result in the equation

$$\Pi = w_{\mathrm{o}}^2 \left\{ \frac{D}{2}\left[\left((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2 \right)^2 \right] + \frac{N_x^{\mathrm{o}}}{2}(\frac{\pi}{a})^2 \right\}$$

By differentiating with respect to the displacement parameter, it is obtained

$$T_{\mathrm{o}} = -N_x^{\mathrm{o}} = \pi^2 D a^2 \left((\frac{1}{a})^2 + (\frac{1}{b})^2 \right)^2$$

which gives the exact solution to the problem.

7.7. Buckling of arches and rings

Buckling of beams curved in the plane, deviates from the previous ones through the analysis in curvilinear coordinates. The procedure applied here follows rather exactly the one used in beam analyses, but in the problem formulation, the derivatives of the unit vectors will be included. It will result in the equilibrium equations where the axial and bending analyses are coupled. Linearly elastic material model is adopted, with the parameters E, ν, the Young's modulus and Poisson's ratio, respectively. The height of the arch is *h*, and it is assumed to be small as compared to the radius of the arch.
The kinematics of the plate is given by applying the Euler-Bernoulli beam model, in which each normal is assumed to remain normal to the deformed mid surface of the deformed geometry. The displacement vector is thus

$$\mathbf{u} = (u - y\frac{\mathrm{d}v}{\mathrm{d}s})\mathbf{e}_s + v\mathbf{e}_y \tag{7.127}$$

Here, the displacement components $u = u(s)$ and $v = v(s)$, but also $\mathbf{e}_s = \mathbf{e}_s(s)$ and $\mathbf{e}_y = \mathbf{e}_y(s)$. In the continuation, the prime $(\bullet)'$ denotes differentiation with respect to the coordinate *s*. We have the rule for differentiation

$$\begin{aligned} \frac{\partial}{\partial \hat{x}} &= (1+\frac{y}{R})^{-1}\frac{\partial}{\partial s} \\ \frac{\partial}{\partial \hat{y}} &= \frac{\partial}{\partial y} \end{aligned} \tag{7.128}$$

Here, $(\hat{x}, \hat{y})$ are the coordinates of the global Cartesian coordinate system. The derivatives of the unit vectors are

$$\begin{aligned} \frac{\mathrm{d}\mathbf{e}_s}{\mathrm{d}s} &= -\frac{1}{R}\mathbf{e}_y \\ \frac{\mathrm{d}\mathbf{e}_y}{\mathrm{d}s} &= \frac{1}{R}\mathbf{e}_s \end{aligned} \tag{7.129}$$

The linear strain and rotation components are calculated again directly by applying the definitions (7.2) and (7.3) with $e_y = e_{ys} = e_z = e_{yz} = e_{zs} = 0.$ In addition, we restrict the consideration to circular arches and rings when the radius R is constant. The only nonzero strain component is ε_s

$$e_s = \frac{\mathrm{d}u}{\mathrm{d}s} + \left(1 + \frac{y}{R}\right)^{-1}\left[\frac{v}{R} - y\left(\frac{u}{R^2}\frac{\mathrm{d}R}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)\right)\right] \approx \frac{\mathrm{d}u}{\mathrm{d}s} + \frac{v}{R} - y\left(\frac{\mathrm{d}^2v}{\mathrm{d}s^2} + \frac{v}{R^2}\right) \tag{7.130}$$

The rotation components $\omega_s = \omega_y = 0,$ and

$$\omega_z = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial s}\cdot\mathbf{e}_y - \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{e}_s\right) = \frac{1}{2}\left(\frac{\partial v}{\partial s} - \frac{\partial u}{\partial y}\right) = \frac{\partial v}{\partial s} - \frac{u}{R} \tag{7.131}$$

So, we get for the strains

$$\begin{aligned} \varepsilon_s &= e_s + \tfrac{1}{2}(\omega_z^2 + \omega_y^2) = \frac{\mathrm{d}u}{\mathrm{d}s} + \frac{v}{R} - y\left(\frac{\mathrm{d}^2v}{\mathrm{d}s^2} + \frac{v}{R^2}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial s} - \frac{u}{R}\right)^2 \\ \varepsilon_y &= e_y + \tfrac{1}{2}(\omega_z^2 + \omega_s^2) = \frac{1}{2}\left(\frac{\partial v}{\partial s} - \frac{u}{R}\right)^2 \end{aligned} \tag{7.132}$$

At the initial loading state, consisting only of centric axial compression and bending moment, the normal stress is $\sigma_s^{\mathrm{o}} = \dfrac{N^{\mathrm{o}}}{A} + \dfrac{M_z^{\mathrm{o}}}{I_z}y$, while all the other stress components are zero. Following the same procedure as before, the incremental strain energy of the beam takes the form

$$\begin{aligned} U &= \int_V (\sigma_s^{\mathrm{o}}\varepsilon_s + \tfrac{1}{2}\sigma_s^*\varepsilon_s + \tfrac{1}{2}\sigma_y^*\varepsilon_y)\mathrm{d}V \\ &\approx \int_V (\sigma_s^{\mathrm{o}} + \tfrac{1}{2}\sigma_s^*)\left(u' + \tfrac{1}{R}v - y\left(v'' + \tfrac{1}{R^2}v\right) + \tfrac{1}{2}\left(v' - \tfrac{1}{R}u\right)^2\right)\mathrm{d}V \end{aligned} \tag{7.133}$$

When we take into account the linearly elastic material law

$$\begin{aligned} \sigma_s^* &= E\varepsilon_s = E\left[u' + \tfrac{1}{R}v - y\left(v'' + \tfrac{1}{R^2}v\right) + \tfrac{1}{2}\left(v' - \tfrac{1}{R}u\right)^2\right] \\ &\approx E\left(u' + \tfrac{1}{R}v - y\left(v'' + \tfrac{1}{R^2}v\right)\right) \end{aligned} \tag{7.134}$$

this will be

$$U=\frac{1}{2}\int_V E\left[u'+\tfrac{1}{R}v-y\left(v''+\tfrac{1}{R^2}v\right)\right]^2(1+\tfrac{y}{R})\mathrm{d}A\mathrm{d}s+$$
$$+\int_V \sigma_x^{o}\left[u'+\tfrac{1}{R}v-y\left(v''+\tfrac{1}{R^2}v\right)\right](1+\tfrac{y}{R})\mathrm{d}A\mathrm{d}s \qquad (7.135)$$
$$+\int_V \sigma_x^{o}\left(v'-\tfrac{1}{R}u\right)^2(1+\tfrac{y}{R})\mathrm{d}A\mathrm{d}s$$

This is a sum of three parts as

$$U=U_L+U_{NL_1}+U_{NL_2} \qquad (7.136)$$

with taking into account the initial stress distribution when

$$U_L=\frac{1}{2}\int_V E\left[u'+\frac{v}{R}-y\left(v''+\frac{v}{R^2}\right)\right]^2(1+\frac{y}{R})^{-1}\mathrm{d}A\mathrm{d}s$$
$$\approx\frac{1}{2}\int_0^L\left(EA(u'+\frac{v}{R})^2+EI_z(v''-\frac{u'}{R})^2\right)\mathrm{d}s$$
$$U_{NL_1}=\int_V \sigma_x^{o}\left[u'+\frac{v}{R}-y\left(v''+\frac{v}{R^2}\right)\right](1+\frac{y}{R})\mathrm{d}A\mathrm{d}s \qquad (7.137)$$
$$=\int_0^L\left[N^{o}(u'+\frac{v}{R})-M_z^{o}\left(v''-\frac{u'}{R}\right)\right]\mathrm{d}s$$
$$U_{NL_2}=\int_V \tfrac{1}{2}\sigma_x^{o}\left(v'-\frac{u}{R}\right)^2(1+\frac{y}{R})\mathrm{d}A\mathrm{d}s=\frac{1}{2}\int_0^L\left[N^{o}\left(v'-\frac{u}{R}\right)^2+\frac{M_z^{o}}{R}\left(v'-\frac{u}{R}\right)^2\right]\mathrm{d}s$$

The potential due to the external loads in the axial direction is

$$V=-\int_V\left[p_s^{o}(u-yv')+p_y^{o}v\right](1+\tfrac{y}{R})\mathrm{d}A\mathrm{d}s-\int_A\left[P_s^{o}(u-yv')+P_y^{o}v\right]_0^L\mathrm{d}A$$
$$=-\int_0^L\left[p_s^{o}u-m_zv'+\tfrac{m_z^{o}}{R}u+p_y^{o}v\right]\mathrm{d}s-\left[P_s^{o}u+P_y^{o}v+\hat{M}_zv'\right]_0^L \qquad (7.138)$$

where $m_z^{o}=\int_A p_s y\mathrm{d}A$ and $\hat{M}_z=[\int_A P_s^{o}y\mathrm{d}A]_0^L$

Now, the total potential energy is composed of terms

$$\Pi=U+V=U_L+U_{NL_1}+U_{NL_2}+V \qquad (7.139)$$

If we consider at first the part

$$U_{NL_1} + V = \int_0^L \left[N^{\mathrm{o}}(u' + \frac{v}{R}) - M_z^{\mathrm{o}} \left(v'' + \frac{u'}{R} \right) \right] \mathrm{d}x - \int_0^L \left[p_s^{\mathrm{o}} u - m_z^{\mathrm{o}} v' + \tfrac{m_z^{\mathrm{o}}}{R} u + p_y^{\mathrm{o}} v \right] \mathrm{d}s$$
$$- \left[P_s^{\mathrm{o}} u + P_y^{\mathrm{o}} v + \hat{M}_z v' \right]_0^L \qquad (7.140)$$

It is exactly the same as in the case of pure flexural buckling, and disappears when the initial structure is in equilibrium fulfilling the equation of equilibrium with boundary conditions

$$\delta(U_{NL_1} + V) = -\int_0^L \left[\left((N^{\mathrm{o}})' + \frac{(M_z^{\mathrm{o}})'}{R} + p_s^{\mathrm{o}} + \frac{m_z^{\mathrm{o}}}{R} \right) \delta u - \left(\frac{N^{\mathrm{o}}}{R} - (M_z^{\mathrm{o}})'' - (m_z^{\mathrm{o}})' - p_y^{\mathrm{o}} \right) \delta v \right] \mathrm{d}s$$
$$- \left[(N^{\mathrm{o}} - P_s^{\mathrm{o}}) \delta u + (M_z^{\mathrm{o}} - \hat{M}_z) \delta v' + ((M_z^{\mathrm{o}})' - P_y^{\mathrm{o}}) \delta v \right]_0^L \qquad (7.141)$$

The disappearance of this expression is equal to the equilibrium of the initial state of the arch yielding in the differential equations and boundary conditions

$$\frac{\mathrm{d}N^{\mathrm{o}}}{\mathrm{d}s} + \frac{1}{R} \frac{\mathrm{d}M_z^{\mathrm{o}}}{\mathrm{d}s} + p_s^{\mathrm{o}}(s) + \frac{m_z^{\mathrm{o}}}{R} = 0, \qquad \frac{\mathrm{d}^2 M_z^{\mathrm{o}}}{\mathrm{d}s^2} - \frac{N^{\mathrm{o}}}{R} + (m_z^{\mathrm{o}})' + p_y^{\mathrm{o}} = 0$$
$$\begin{cases} N^{\mathrm{o}} = P_s^{\mathrm{o}} \text{ or } \delta u = 0 \\ M_z^{\mathrm{o}} = \hat{M}_z \text{ or } \delta v' = 0 \quad \text{at } s = 0 \text{ and } s = L \\ (M_z^{\mathrm{o}})' = P_y^{\mathrm{o}} \text{ or } \delta v = 0 \end{cases} \qquad (7.142)$$

This part of equation disappears when the beam in the initial state is in equilibrium.

So, we have still left the equation

$$\Pi = U_L + U_{NL_2} = \frac{1}{2} \int_0^L \left(EA(u' + \frac{v}{R})^2 + EI_z (v'' - \frac{u'}{R})^2 \right) \mathrm{d}s$$
$$+ \frac{1}{2} \int_0^L \left[N^{\mathrm{o}} \left(v' - \frac{u}{R} \right)^2 + \frac{M_z^{\mathrm{o}}}{R} \left(v' - \frac{u}{R} \right)^2 \right] \mathrm{d}s \qquad (7.143)$$

isTaking at first the first variation of the results in

$$\delta\Pi = \int_0^L \left(EA(u' + \frac{v}{R})(\delta u' + \frac{\delta v}{R}) + EI_z (v'' - \frac{u'}{R})(\delta v'' + \frac{\delta u'}{R}) \right) \mathrm{d}s$$
$$+ \int_0^L \left[N^{\mathrm{o}} (v' - \frac{u}{R})(\delta v' - \frac{\delta u}{R}) + \frac{M_z^{\mathrm{o}}}{R} (v' - \frac{u}{R})(\delta v' - \frac{\delta u}{R}) \right] \mathrm{d}s \qquad (7.144)$$

By integrating by parts we get further

$$\delta\Pi = \int_0^L \left(-EA(u'' + \frac{v'}{R}) - EI_z(\frac{v'''}{R} - \frac{u''}{R^2}) - (N^o + \frac{M_z^o}{R})(\frac{v'}{R} - \frac{u}{R^2}) \right)\delta u \mathrm{d}s$$
$$+ \int_0^L \left[EA(\frac{u'}{R} + \frac{v}{R^2}) + EI_z(v'''' - \frac{u'''}{R}) - (N^o + \frac{M_z^o}{R})(v'' - \frac{u'}{R}) \right]\delta v \mathrm{d}s \qquad (7.145)$$
$$+ \left[\left(EAu' + EI_z(\frac{v''}{R} - \frac{u'}{R^2}) \right)\delta u + EI_z v''\delta v' - \left(EI_z v''' - (N^o + \frac{M_z^o}{R})(v' - \frac{u}{R}) \right)\delta v \right]_0^L$$

The differential equations obtained are thus

$$\begin{aligned} -EA(\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + \frac{1}{R}\frac{\mathrm{d}v}{\mathrm{d}s}) - EI_z(\frac{1}{R}\frac{\mathrm{d}^3v}{\mathrm{d}s^3} - \frac{1}{R^2}\frac{\mathrm{d}^2u}{\mathrm{d}s^2}) - (N^o + \frac{M_z^o}{R})(\frac{1}{R}\frac{\mathrm{d}v}{\mathrm{d}s} - \frac{u}{R^2}) = 0 \\ EA(\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{v}{R^2}) + EI_z(\frac{\mathrm{d}^4v}{\mathrm{d}s^4} - \frac{1}{R}\frac{\mathrm{d}^3u}{\mathrm{d}s^3}) - (N^o + \frac{M_z^o}{R})(\frac{\mathrm{d}^2v}{\mathrm{d}s^2} - \frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}s}) = 0 \end{aligned} \qquad (7.146)$$

and corresponding boundary conditions at $x = 0, L$

$$\begin{aligned} & EAu' + EI_z(\frac{v''}{R} - \frac{u'}{R^2}) = 0 \quad \text{or} \quad \delta u = 0 \\ & EI_z v''' - (N^o + \frac{M_z^o}{R})(v' - \frac{u}{R}) = 0 \quad \text{or} \quad \delta v = 0 \\ & EI_z v'' = 0 \quad \text{or} \quad \delta v' = 0 \end{aligned} \qquad (7.147)$$

This system of equations can be solved for the critical compressive load generally. If we multiply the equation by the factor R^3 / EI, introduce the notations $\lambda = N^o R^3 / EI_z$ and $\rho = R^2 A / I_z$ change the variable $R\mathrm{d}\theta = \mathrm{d}s$ and assume still that $M_z^o = 0$ we get

$$\begin{aligned} -\rho(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \frac{\mathrm{d}v}{\mathrm{d}\theta}) + (\frac{\mathrm{d}^3v}{\mathrm{d}\theta^3} + \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}) - \lambda(\frac{\mathrm{d}v}{\mathrm{d}\theta} - u) = 0 \\ -\rho(\frac{\mathrm{d}u}{\mathrm{d}\theta} + v) - (\frac{\mathrm{d}^4v}{\mathrm{d}\theta^4} - \frac{\mathrm{d}^3u}{\mathrm{d}\theta^3}) - \lambda(\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} - \frac{\mathrm{d}u}{\mathrm{d}\theta}) = 0 \end{aligned} \qquad (7.148)$$

By assuming now the displacements $v = A_n \cos n\theta$ and $u = B_n \sin n\theta$, the system of equations take the form

$$\begin{aligned} \left[A_n \left(\rho n + n^3 + \lambda n \right) + B_n \left(\rho n^2 - n^2 + \lambda \right) \right] \sin n\theta = 0 \\ \left[A_n \left(-\rho - n^4 + \lambda n^2 \right) - B_n \left(\rho n + n^3 - \lambda n \right) \right] \cos n\theta = 0 \end{aligned} \qquad (7.149)$$

This gives for the loading parameter the value 4, corresponding to $n = 2$ when the critical load is $N^o = 4EI / R^3$.

Another way to solve this system of differential equations is to use a symbolic notation for the derivative operator $D = \mathrm{d}/\mathrm{d}x$ when the system of equations can be written in the form

$$\begin{aligned} L_1 v + L_2 u &= 0 \\ L_3 v + L_4 u &= 0 \end{aligned} \tag{7.150}$$

in which

$$\begin{aligned} L_1 &= \rho D - D^3 + \lambda D & \qquad L_2 &= \rho D^2 - D^2 + \lambda \\ L_3 &= -\rho - D^4 - \lambda D^2 & \qquad L_4 &= -\rho D + D^3 + \lambda D \end{aligned} \tag{7.151}$$

Then, we can utilize the commutativity property of linear differential operators

$$\begin{aligned} (L_1 L_4 - L_2 L_3) v &= 0 \\ (L_1 L_4 - L_2 L_3) u &= 0 \end{aligned} \tag{7.152}$$

Thus we get a sixth order ordinary differential equation

$$(2-\rho) v^{(6)} - (2\rho + \lambda) v'''' + \rho v'' + \lambda \rho v = 0 \tag{7.153}$$

It includes only even order of derivatives and results thus in a polynomial of third order as the characteristic equation.

7.8. Buckling of shells of revolution

Buckling of shells of revolution is actually a one-dimensional problem. The procedure applied here follows rather exactly the one used in beam analyses, but in the problem formulation, the derivatives of the unit vectors will be included. It will result in the equilibrium equations where the membrane and bending analyses are coupled. Linearly elastic material model is adopted, with the parameters E, ν, the Young's modulus and Poisson's ratio, respectively. The thickness of the shell is h, and it is assumed to be small as compared to the radii of the arch.

The kinematics of the shell is given by applying the Love-Kirchhoff shell or plate model, in which each normal is assumed to remain normal to the deformed mid surface of the deformed geometry. The displacement vector is thus

$$\mathbf{u}(\vartheta, z) = \left[u(\vartheta) - z\theta_\phi(\vartheta) \right] \mathbf{e}_\vartheta(\vartheta) + w(\vartheta)\mathbf{e}_z(\vartheta) \tag{7.154}$$

Here, the displacement components $u = u(s)$ and $v = v(s)$, but also $\mathbf{e}_s = \mathbf{e}_s(s)$ and $\mathbf{e}_\phi = \mathbf{e}_\phi(s)$. In the continuation, the prime $(\bullet)'$ denotes differentiation with respect to the coordinate s.

The linear nonzero strain components in a shell of revolution exposed to axial symmetric loading are

$$e_\vartheta = (1+\frac{z}{R_\vartheta})^{-1}\left[\frac{\mathrm{d}u}{R_\vartheta \mathrm{d}\vartheta} - \frac{z}{R_\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}(\frac{\mathrm{d}w}{R_\vartheta \mathrm{d}\vartheta}) + \frac{z}{R_\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}(\frac{u}{R_\vartheta}) + \frac{w}{R_\vartheta}\right]$$

$$= \frac{\mathrm{d}u}{R_\vartheta \mathrm{d}\vartheta} + \frac{w}{R_\vartheta} - \frac{z}{R_\vartheta}\left[\frac{u}{R_\vartheta^2}\frac{\mathrm{d}R_\vartheta}{\mathrm{d}\vartheta} + \frac{w}{R_\vartheta} + \frac{\mathrm{d}}{\mathrm{d}\vartheta}(\frac{\mathrm{d}w}{R_\vartheta \mathrm{d}\vartheta})\right] + \mathcal{O}(z^2)$$

$$e_\phi = (1+\frac{z\sin\vartheta}{r})^{-1}\left[\frac{u}{r}\cos\vartheta + \frac{w}{r}\sin\vartheta - \frac{z}{r}(\frac{\mathrm{d}w}{R_\vartheta \mathrm{d}\vartheta} - \frac{u}{R_\vartheta})\cos\vartheta\right]$$

$$= \frac{u}{r}\cos\vartheta + \frac{w}{r}\sin\vartheta - \frac{z\cos\vartheta}{r}\left[\frac{\mathrm{d}w}{R_\vartheta \mathrm{d}\vartheta} - \frac{u}{R_\vartheta}(1-\sin\vartheta) + \frac{w}{R_\vartheta}\sin\vartheta\tan\vartheta\right] + \mathcal{O}(z^2)$$

These will be simplified in the case of cylindrical shell to the form with the relations $R_\vartheta \mathrm{d}\vartheta = \mathrm{d}s,\ 1/R_\vartheta = 0,\ \cos\vartheta = 0,\ \sin\vartheta = 1$

$$e_s = \frac{\mathrm{d}u}{\mathrm{d}s} - z(\frac{\mathrm{d}^2 w}{\mathrm{d}s^2})$$

$$e_\phi = \frac{w}{r}$$

In addition, the rotation term ω_z is the only nonzero, while $\omega_\phi = \omega_s = 0$

$$\omega_z = \frac{\mathrm{d}w}{\mathrm{d}s}$$

The total expressions for the strains are

$$\varepsilon_s = \frac{\mathrm{d}u}{\mathrm{d}s} - z(\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}) + \frac{1}{2}(\frac{\mathrm{d}w}{\mathrm{d}s})^2 = u' - zw'' + \tfrac{1}{2}(w')^2$$

$$\varepsilon_\phi = \frac{w}{r} + \frac{1}{2}(\frac{\mathrm{d}w}{\mathrm{d}s})^2 = \frac{w}{r} + \tfrac{1}{2}(w')^2$$

The expression of the strain energy

$$U = U^{\mathrm{o}} + U^* = \int_V \left[(\sigma_s^{\mathrm{o}} + \tfrac{1}{2}\sigma_s^*)\varepsilon_s + (\tfrac{1}{2}\sigma_\phi^*)\varepsilon_\phi\right]\mathrm{d}V \qquad (7.155)$$

can further be split to give

$$U = \int_V \tfrac{1}{2}\left[\sigma_s^*(\frac{\mathrm{d}u}{\mathrm{d}s} - z\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}) + \sigma_\phi^*(\frac{w}{r})\right]\mathrm{d}V + \int_V \left[\sigma_s^{\mathrm{o}}(\frac{\mathrm{d}u}{\mathrm{d}s} - z\frac{\mathrm{d}^2 w}{\mathrm{d}s^2})\right]\mathrm{d}V + \int_V \tfrac{1}{2}\left[\sigma_s^{\mathrm{o}}(\frac{\mathrm{d}w}{\mathrm{d}s})^2\right]\mathrm{d}V \qquad (7.156)$$

This is composed of three parts following the practice above as

$$U = U_L + U_{NL_1} + U_{NL_2} \qquad (7.157)$$

in which U_L is the traditional strain energy expression due to linearised strains and the two following ones, U_{NL_1} and U_{NL_2}, take into account both the non-linear terms of strain components and the initial stresses.

When we take into account the linearly elastic material law

$$\begin{aligned} \sigma_s^* &= \frac{E}{1-\nu^2}\left(\varepsilon_s + \nu\varepsilon_\phi\right) \\ \sigma_\phi^* &= \frac{E}{1-\nu^2}\left(\varepsilon_\phi + \nu\varepsilon_s\right) \end{aligned} \tag{7.158}$$

Inserting this into the expression of the linearised strain energy gives

$$U_L = \frac{1}{2}\int_V \left(\sigma_s^*\varepsilon_s + \sigma_\phi^*\varepsilon_\phi\right)\mathrm{d}V = \frac{E}{2(1-\nu^2)}\int_V \left(\varepsilon_s^2 + \varepsilon_\phi^2 + 2\nu\varepsilon_s\varepsilon_\phi\right)\mathrm{d}V \tag{7.159}$$

Incorporating still the linearized strains from (7.103) or (7.102), and denoting the bending stiffness of the plate by $D = Eh^3/12(1-\nu^2)$ we get

$$\begin{aligned} U_L &= \frac{1}{2}\frac{E}{1-\nu^2}\underbrace{\int_{-h/2}^{h/2}\mathrm{d}z}_{=h}\int_A\left[(\frac{\mathrm{d}u}{\mathrm{d}s})^2 + (\frac{w}{r})^2 + 2\nu\frac{\mathrm{d}u}{\mathrm{d}s}\frac{w}{r}\right]r\mathrm{d}s\mathrm{d}\phi \\ &+ \underbrace{\frac{1}{2}\frac{E}{1-\nu^2}\int_{-h/2}^{h/2} z^2\mathrm{d}z}_{=D}\int_A\left[(\frac{\mathrm{d}^2w}{\mathrm{d}s^2})^2\right]r\mathrm{d}s\mathrm{d}\phi \end{aligned} \tag{7.160}$$

The non-linear parts of the potential energy expression are simplifying to the form

$$U_{NL_1} = \int_V\left[\sigma_s^{\mathrm{o}}\frac{\mathrm{d}u}{\mathrm{d}s}\right]\mathrm{d}V = \int_A\left[N_s^{\mathrm{o}}\frac{\mathrm{d}u}{\mathrm{d}s}\right]r\mathrm{d}s\mathrm{d}\phi \tag{7.161}$$

where $N_x^{\mathrm{o}} = \sigma_x^{\mathrm{o}}h,\ N_y^{\mathrm{o}} = \sigma_y^{\mathrm{o}}h$ and $N_{xy}^{\mathrm{o}} = \tau_{xy}^{\mathrm{o}}h$, and consequently

$$U_{NL_2} = \frac{1}{2}\int_V\left[\sigma_s^{\mathrm{o}}(\frac{\mathrm{d}w}{\mathrm{d}s})^2\right]\mathrm{d}V = \frac{1}{2}\int_A\left[N_s^{\mathrm{o}}(\frac{\mathrm{d}w}{\mathrm{d}s})^2\right]r\mathrm{d}s\mathrm{d}\phi \tag{7.162}$$

The potential of the external load V acting on the plane of the mid surface of plate and including the volume forces and the loads on the edges of plate is

$$V = -\int_V\left[p_s^{\mathrm{o}}u\right]\mathrm{d}V - \oint_S\left[t_s^{\mathrm{o}}u\right]\mathrm{d}S = -\int_A\left[P_s^{\mathrm{o}}u\right]\mathrm{d}x\mathrm{d}y - \oint_s\left[T_s^{\mathrm{o}}u\right]\mathrm{d}s \tag{7.163}$$

Here, S covers the area of boundary surfaces. The notations $P_x^{\mathrm{o}} = p_x^{\mathrm{o}}h,$ and $T_x^{\mathrm{o}} = t_x^{\mathrm{o}}h$ for the loading components are adopted. When considering the term

$$U_{NL_1} + V = \int_A \left[N_s^o \frac{du}{ds} + P_s^o u \right] r \mathrm{d}s\mathrm{d}\phi - \oint_s \left[T_s^o u \right] \mathrm{d}s \tag{7.164}$$

taking the first variation

$$\delta(U_{NL_1} + V) = \int_A \left[N_s^o \frac{d\delta u}{ds} + P_s^o \delta u \right] r \mathrm{d}s\mathrm{d}\phi - \oint_s \left[T_s^o \delta u \right] \mathrm{d}S \tag{7.165}$$

and integrating by parts gives

$$\delta(U_{NL_1} + V) = -\int_A \left[(\frac{dN_x^o}{dx} + P_x^o)\delta u \right] r \mathrm{d}s\mathrm{d}\phi + \int_{s_y} [(N_s^o - T_s^o)\delta u\,] r \mathrm{d}\phi \tag{7.166}$$

In the equilibrium this must vanish producing the equilibrium conditions of the initial state of the plate as

$$\frac{dN_s^o}{ds} + P_s^o = 0 \tag{7.167}$$

with initial boundary conditions

$$N_s^o = T_s^o \quad \text{or} \quad \delta u = 0 \tag{7.168}$$

We have still the terms

$$\begin{aligned} U_L + U_{NL_2} &= \frac{1}{2}\frac{Eh}{1-\nu^2}\int_A \left[(\frac{du}{ds})^2 + (\frac{w}{r})^2 + 2\nu \frac{du}{ds}\frac{w}{r} \right] r \mathrm{d}s\mathrm{d}\phi \\ &+ \frac{D}{2}\int_A \left[(\frac{d^2 w}{ds^2})^2 + \frac{N_x^o}{D}(\frac{dw}{ds})^2 \right] r \mathrm{d}s\mathrm{d}\phi \end{aligned} \tag{7.169}$$

The first term in this equation concerns the initial state of the plate and will be dropped. The rest of (7.120) takes after variation the form

$$\begin{aligned} \delta(U_L + U_{NL_2}) &= \frac{Eh}{1-\nu^2}\int_A \left[\frac{du}{ds}\frac{d\delta u}{ds} + \frac{w}{r}\frac{\delta w}{r} + \nu(\frac{d\delta u}{ds}\frac{w}{r} + \frac{du}{ds}\frac{\delta w}{r}) \right] r \mathrm{d}s\mathrm{d}\phi \\ &+ D\int_A \left[\frac{d^2 w}{ds^2}\frac{d^2 \delta w}{ds^2} + \frac{N_s^o}{D}\frac{dw}{ds}\frac{d\delta w}{ds} \right] r \mathrm{d}s\mathrm{d}\phi \end{aligned} \tag{7.170}$$

Integrating twice by parts simplifies the expression to

$$\delta(U_L + U_{NL_2}) = -\frac{Eh}{1-\nu^2}\int_A \left[\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + \nu(\frac{\mathrm{d}w}{r\mathrm{d}s}) \right] \delta u \, r\mathrm{d}s\mathrm{d}\phi$$

$$+ \int_A \left[D\frac{\mathrm{d}^4 w}{\mathrm{d}s^4} - N_s^{\mathrm{o}} \frac{\mathrm{d}^2 w}{\mathrm{d}s^2} + \frac{Eh}{r(1-\nu)}(\frac{w}{r} + \frac{\mathrm{d}u}{\mathrm{d}s}) \right] \delta w \, r\mathrm{d}s\mathrm{d}\phi \qquad (7.171)$$

$$+ \frac{Eh}{1-\nu^2}\int_{s_u} (\frac{\mathrm{d}u}{\mathrm{d}s} + \nu\frac{w}{r})\delta u \, r\mathrm{d}\phi + \int_{s_u} \left[D\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}\delta w' - (D\frac{\mathrm{d}^3 w}{\mathrm{d}s^3} - N_s^{\mathrm{o}}\frac{\mathrm{d}w}{\mathrm{d}s})\delta w \right] r\mathrm{d}\phi$$

The disappearance of the surface integral gives the final system of two homogeneous ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + \nu(\frac{\mathrm{d}w}{r\mathrm{d}s}) = 0 \\ &D\frac{\mathrm{d}^4 w}{\mathrm{d}s^4} - N_s^{\mathrm{o}}\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} + \frac{Eh}{r(1-\nu)}(\frac{w}{r} + \frac{\mathrm{d}u}{\mathrm{d}s}) = 0 \end{aligned} \qquad (7.172)$$

By utilizing the first equation, the final formulation for the buckling of a cylindrical shell will be obtained

$$D\frac{\mathrm{d}^4 w}{\mathrm{d}s^4} + T_x^{\mathrm{o}}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{Eh}{r^2} w = 0 \qquad (7.173)$$

Here, the negative sign of the compressive load is taken into account.
The principle of minimum potential energy is obtained from the expression

$$\begin{aligned} U_L + U_{NL_2} &= \frac{1}{2}\frac{Eh}{1-\nu^2}\int_A \left[(\frac{\mathrm{d}u}{\mathrm{d}s})^2 + (\frac{w}{r})^2 + 2\nu\frac{\mathrm{d}u}{\mathrm{d}s}\frac{w}{r} \right] r\mathrm{d}s\mathrm{d}\phi \\ &+ \frac{D}{2}\int_A \left[(\frac{\mathrm{d}^2 w}{\mathrm{d}s^2})^2 + \frac{N_x^{\mathrm{o}}}{D}(\frac{\mathrm{d}w}{\mathrm{d}s})^2 \right] r\mathrm{d}s\mathrm{d}\phi \end{aligned} \qquad (7.174)$$

$$\left(EAu'\right)' = 0$$

$$(EI_z v'')'' - (N^{\text{o}} v')' - z_v (N^{\text{o}} \phi')' + (M_y^{\text{o}} \phi')' = 0$$

$$(EI_y w'')'' - (N^{\text{o}} w')' + y_v (N^{\text{o}} \phi')' - (M_z^{\text{o}} \phi')' = 0$$

$$\left(EI_\omega \phi''\right)'' - (GI_t \phi')' + y_v (N^{\text{o}} w')' - z_v (N^{\text{o}} v')' - r^2 \left(N^{\text{o}} \phi'\right)'$$

$$- (M_z^{\text{o}} w')' + (M_y^{\text{o}} v')' - 2\beta_y (M_z^{\text{o}} \phi')' - 2\beta_z (M_y^{\text{o}} \phi')' = 0$$