# CIV-E4100 - Stability of Structures D, 01.03.2021-18.04.2021

A? MyCourses

These are the two lectures of the first week

## Content

First week

0. **Basic concepts**
   **Equilibrium, Stability**
   **The energy criterion of stability**

following weeks

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

In short: In this course, we study the
Elastic stability of slender structures – Rakenteiden stabiilius

The content is conceptually very concise with only three fundamental and general concepts to study:

1) Equilibrium - tasapaino
2) The stability properties (stable, neutral, unstable)– tasapainon laatu
3) Sensitivity of an equilibrium to imperfections –herkkyys häiriöille

Many applications of structural stability of typical structural elements commonly used in civil engineering will be studied.

The method:
two consecutive lectures & two cessions of guided exercises for doing the weekly compulsory homework

One topic per week

| Mo | Tu | We | Th | Fr | Sa | Su |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |

March
April

**Lecturer**
Djebar Baroudi, Dr.
Civil Engineering Department
Aalto University

version 2 Mars 2021

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Many applications of structural stability of typical structural elements commonly used in civil engineering will be studied.

HW-1 will introduce these genral consepts

## Sensitivity of an equilibrium to imperfections

- *stable symmetric* - always imperfection insensitive
- *unstable symmetric* - imperfection sensitive
- *unstable asymmetric* - imperfection sensitive (more than in the symmetric unstable case)

# Content of this 1st week two lectures:

These are the two lectures of the first week

## Content

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   **Equilibrium, Stability**
   **The energy criterion of stability**
1. Flexural buckling (nurjahdus)
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4. Buckling of thin plates
5. Buckling of shells (lommahdus)

following weeks

Main topics

- Content of this course
- Literature & additional course material
- Practicalities

- **Introduction**
- What equilibrium and stability mean?
- The key questions
- How stability is investigated?

- Stability loss as a phenomenon
  - Examples of loss of stability
- Basic concepts of stability
  - Static & dynamic stability
- Structural design and stability
- Methods of stability study

- **Energy criteria of stability**
- Lagrange-Dirichlet Stability theorem

- **Equilibrium paths**
- Critical equilibrium points, bifurcation, limit points
- Stability of an equilibrium
- Linear Buckling Analysis
- Non-Linear Buckling Analysis (GNA)

- **Types of bifurcation instabilities**
- **Effect of imperfections on the post-buckling behavior**

- **Illustration examples**

# Literature

- **[1]** CHAI H. YOO & SUNG C. LE . ***STABILITY OF STRUCTURES - Principles and Applications,***2011 Elsevier **e-textbook** (our course main textbook)
- **[2]** Lecturer (D. Baroudi) additional material: lectures-notes: (may be updated weekly or weakly)

## Content

Topics of the lectures and homework

First week

0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

**Lectures slides**: Each week new material

**Week 1-2 topics** **Fundaments of Stability:**

[The PDF:] https://mycourses.aalto.fi/pluginfile.php/664042/course/sect

(I provide the pdf I have written. This **stability key topics** are missing in the textbook!)

**My notes on stability:**

**5) Version 27 FEB 2021: https://mycourses.aalto.fi/pluginfile.php/1260824/course/section/173429 /Main_book_Structural_Mechanics_2020_STAB._home_Optimized_27FEV2021_PM.pdf**

Not compulsory

**Additinal reading:**

Not compulsory

- [3] S.P. Timoshenko & J.M. Gere. *Theory of Elastic Stability. 2nd Ed., 1985.* (Classical textbook)
- [4] Juha Paavola. *Structural Stability.* Lecture note - 2018 (pdf in MyCourses) . https://mycourses.aalto.fi/pluginfile.php /1260824/course/section/158920/Fundaments_in_Elastic_Stability_Energy_method_by_JPaavola_2018.pdf This is a must-read for those interrested in general and systematic energetic approach for elastic stability.
- [5] *Structural Stability* (Lecture notes by Prof Markku Tuomala, in Finish). This is a complete textbook with plenty of solved exercises https://mycourses.aalto.fi/pluginfile.php/1260824/course/section/158920

# Practicalities

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CIV-E4100 - Stability of Structures

litterature

» Reading assignments and Lecture notes [2021]

» Homwork assignments : Topics/Schedule & remote-EXAM [2021]

» HW & HW-solution for lecturer 2020]

» Homework and course points [2020]

» Exams, grades & solutions [2020]

» For the Lecturer only

» Material [old, 2018]

» Lecture Slides [old, 2018]

- Reading assignments
- Lecture augmented slides
- Detailed content
- Reading assignments

## First week - content

BEGIN WEEK 1-2

==========================================================

**Week 1-2: [2] Chapter 4, from 4.1 till section 4.11.1:** ***The fundaments of Elastic Stability***

(I provide the pdf I have written. This **stability key topics** are missing in the textbook!)

**Week 1-2 topics**

[PDF Slides] : https://mycourses.aalto.fi/pluginfile.php/952895/course/section/133502/Week_1_Lecture_Slides_DB.pdf

- **Introduction**
  - What is stability? Phenomenon, elastic stability
  - Structural design and stability – some standards
- **Basic concepts**
  - Equilibrium, equilibrium paths
  - Critical points: bifurcation, limit point
  - Stability of equilibrium (branches)
- **Energy criteria of stability**
- About post-buckling analysis
- **Types of bifurcational instabilities**
- **Sensitivity to imperfections**
- **Illustrative examples**

==========================================================

From e-textbook ---- you can read also,

**Week 1-2: [1] Chapter 1:** *Buckling of columns* ...

& especially, section 1.6: *Introduction to calculus of variation* (variaatiolaskenta)

# Practicalities

## Homework

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**Assignment week 1 [2021] - Fundaments of Elas Stability [Deadline 10 March < 23:55]**

» Homwork assignments : Topics/Schedule & remote-EXAM [2021]

HW_1_2021.pdf

**Fundaments of Elastiic Stability**:

- Buckling load,
- energy criteria of stability
- equilibrium paths,
- post-critical analysis,
- effect of imperfections
- asymptotic post-buckling analysis

The next home work, we wil

The key consepts and method are invariant structure.

An introduction to the calculus of variations

1): https://www.open.edu/openlearn/ocw/pl /3/Introduction%20to%20the%20calculus%20of%20variations_ms327.pdf

2): http://courses.theophys.kth.se/5A

### Homework #1 (1st Week) Fundamental concepts

February 28, 2021

**Topics**: *Buckling load, equilibrium paths, post-critical analysis, effect of imperfections and asymptotic post-buckling analysis.*

Contents



**Why these simple exercises?** Before tackling various modes of stability loss of different type of structures it is wise to get a bigger picture about

**Schedule for guided exercises and Homework 2021**

**Please notice this schedule:** has been sent to you through mycourses

# CIV-E4100 - Stability of Structures D, 01.03.2021-18.04.2021


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CIV-E4100 - Stability of Structures

litterature

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- For the Lecturer only
- Material [old, 2018]
- Lecture Slides [old, 2018]

## Passing the course

- Having obtained from HW-assignements >= 40% of compulsory points togather with passing successfully the written exam.
- when the **written** exam is successfully passed, then the homework points rise the examination grade (arvosana) at most by 1 grade if **homework points >= 2/3 of homework compulsory maximum points**.
- There will be organised only two examinations

## Assignments

- readings from textbook and the additional lecturer's pdf-material
- doing weekly homework (probably five topics) *student delivery.* solutions
- one computer analysis: linear buckling and post-buckling analysis: *student delivery.* solutions and report

*The purpose of assignments is to train and deepen active learning. All the cession of exercises are guided.*

Axial buckling failure in service.

# Some videos on stability of structures

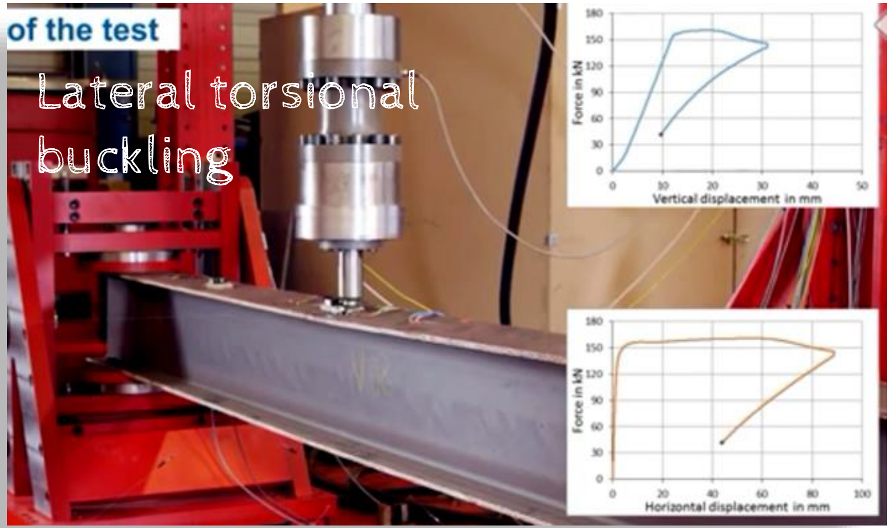
https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop

1: Lateral torsional buckling of I-beam (kiepahdus)

**Comment**: Good experiment with load-displacement curves

The student can clearly see the transition from bending in the vertical plane to bending in the horizontal plane and torsion

(24.02.2010)




https://www.youtube.com/watch?feature=youtu.be&v=cYRicTk-Q08&app=desktop

(24.02.2010)

2: Torsional buckling of L-shape cross-section (angle) column (vääntönurjahdus)

**Comment**: Good experiment with a funny professor.

Note that, the apparent (torsional) rigidity gets dramatically reduced close to the buckling load

(24.02.2010)


https://www.youtube.com/watch?v=0WN8RP7Bz6Q

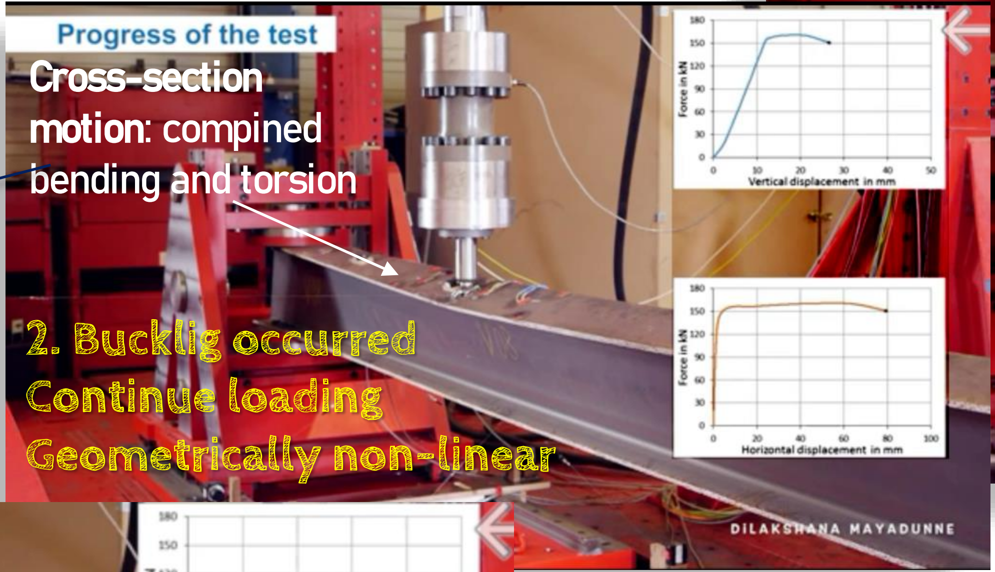

3: buckling of upper cord of a truss

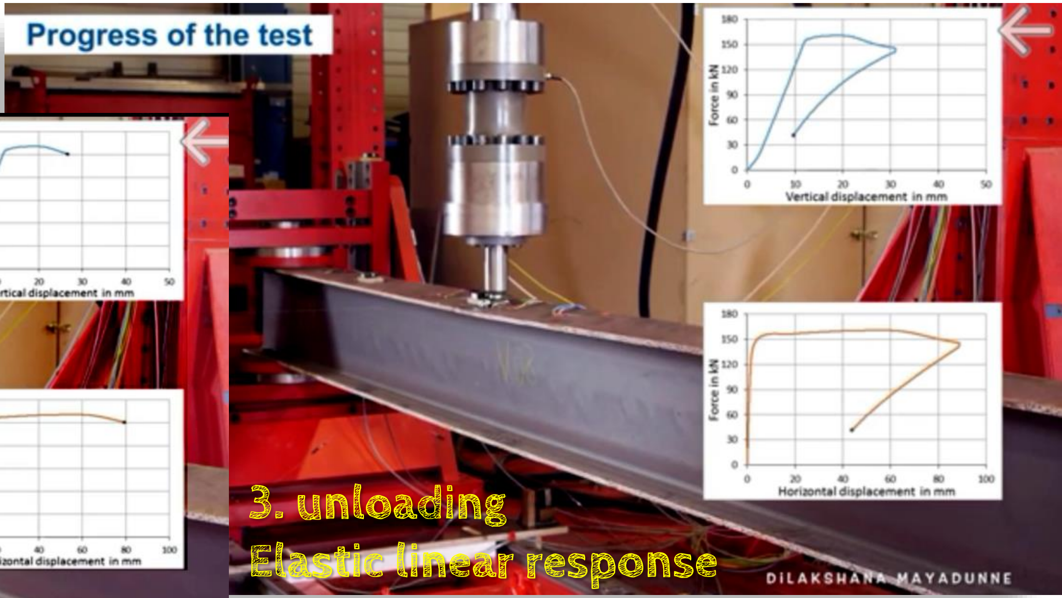
**Comment**: Note the FAST and DYNAMICAL transition from the primary equilibrium (unbuckled) to the secondary equilibrium state (buckled)

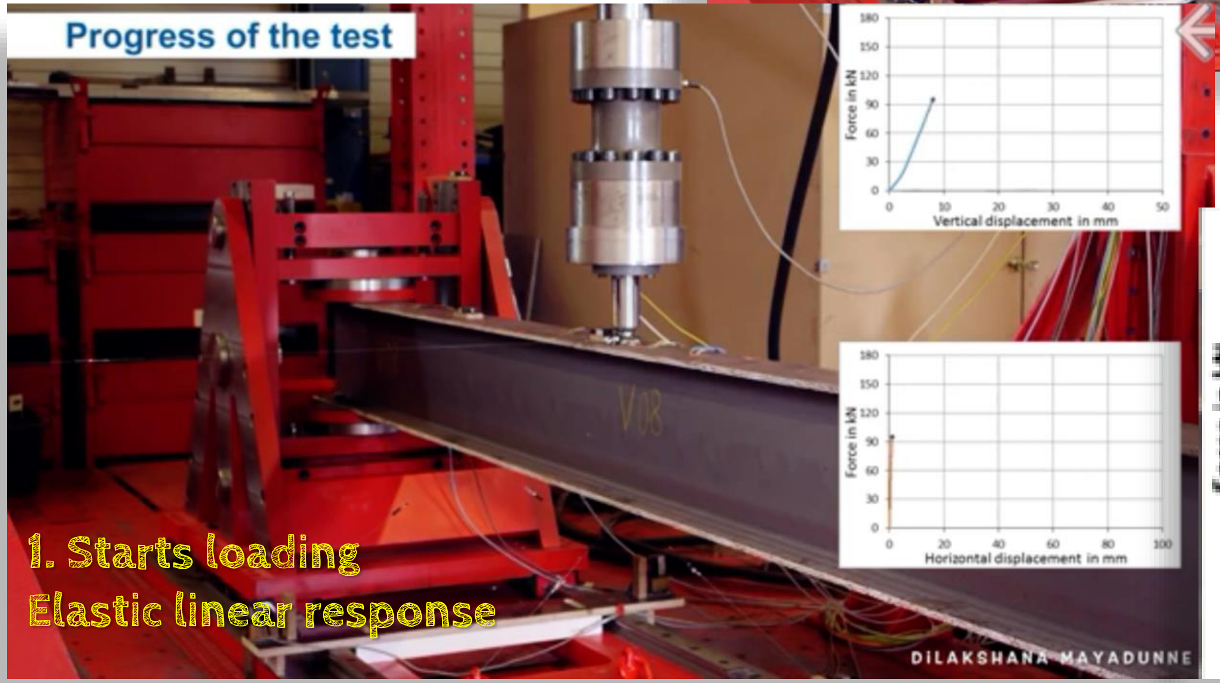
# Some videos on stability of structures

https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop
(24.02.2010)

Lateral torsional buckling of I-beam
(kiepahdus)

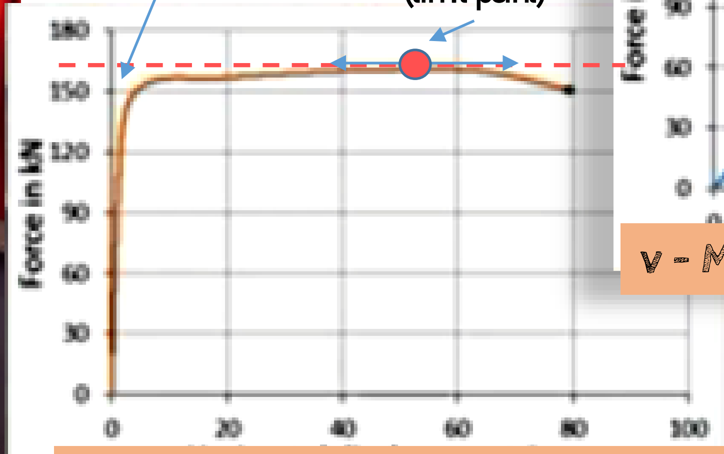


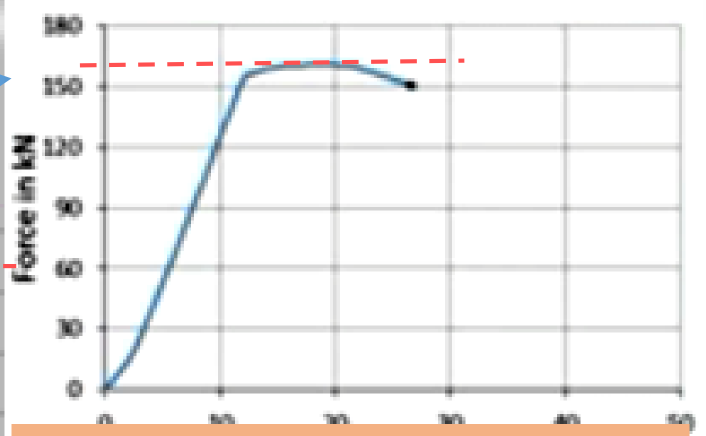




Buckling load

Limit load
(limit point)




w – Measured horizontal displ. (mm)




v – Measured vertical displ. (mm)

Experimental load-displacement curves
= equilibrium paths

# Systems with multiple equilibrium states

**Open**: equilibrium position #1

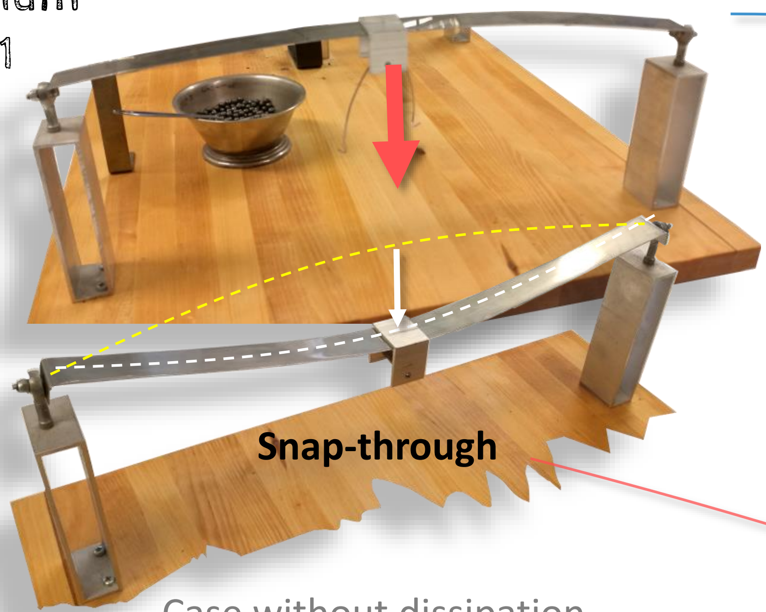
**Closed**: equilibrium position #2

Simple bi-stable system

Shallow arch

Model in our coffe-room

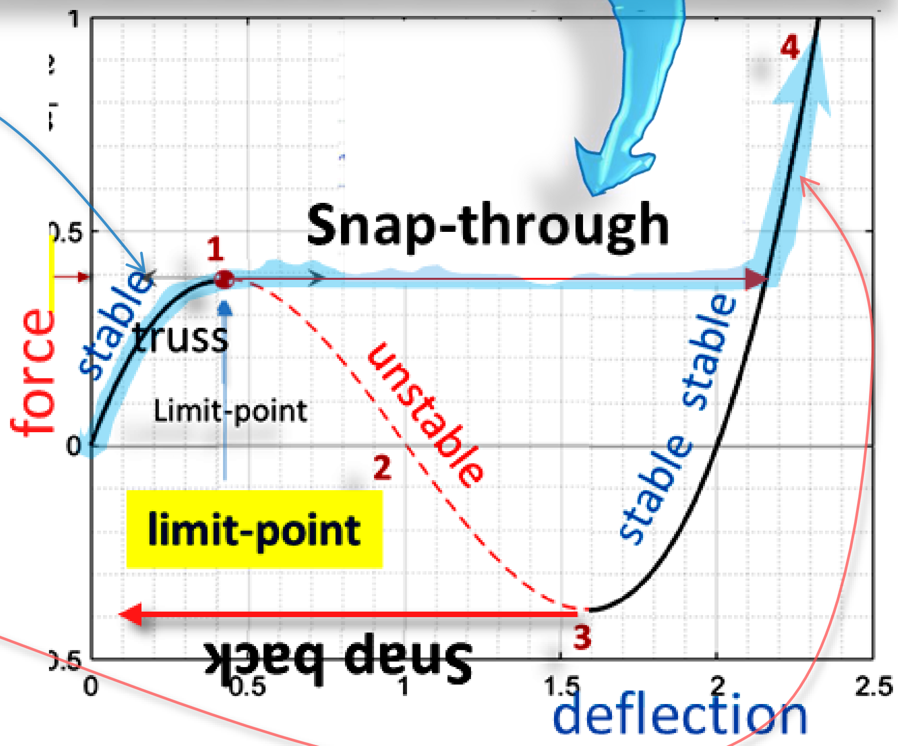
Equilibrium state #1




Equilibrium state #2

Case without dissipation

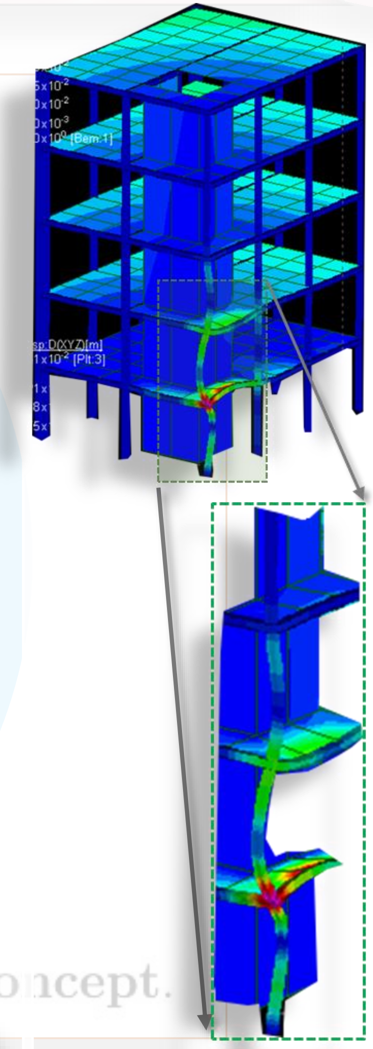
- In such systems, the structure can move from one equilibrium state to another if enough perturbed
- For structural engineers such motion is not desired and is called loss of stability




Snap-through of shallow arch

*Stability* is a concept related to systems having *more than one equilibrium states*. Simplest examples are the *bistable* systems, with two equilibrium states where the system can rest in either one two states

# The key stability question in structural design



Soil material instability



D. Baroudi, Dr.



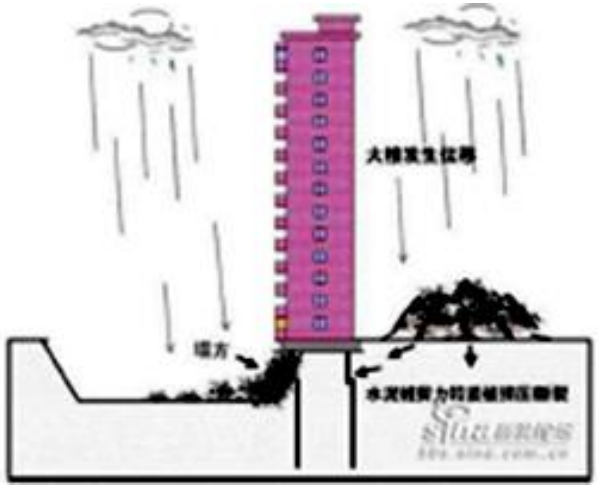
Version: 16.2.2020

# The key stability question in structural design

*BASIC CONCEPTS*

**Equilibrium?** Yes.
But, is it **stable**? No.

Figure 3.32: *Equilibrium* and *Stability* - **the key concept**.



**Soil material instability**

Lecture slides for internal use only
D. Baroudi, PhD


Version: 25.2.2019

# What are the key question in stability investigation?

Here the content of this course in four points through questions that will be addressed:

or limit load

1. can we predict the buckling (critical) load?

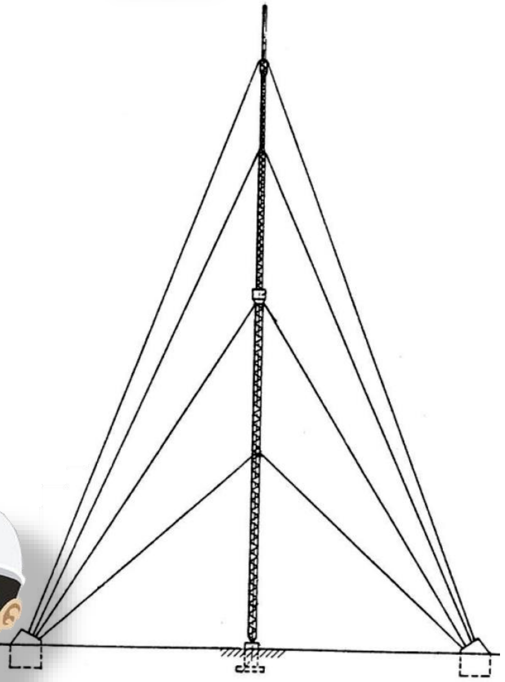
2. what happens at the bifurcation (or limit) point? (*i.e.*, after the buckling) Post-buckling behavior

3. can we determine the post-critical branches? What would be their shape? Nature of stability? Stable, instable?

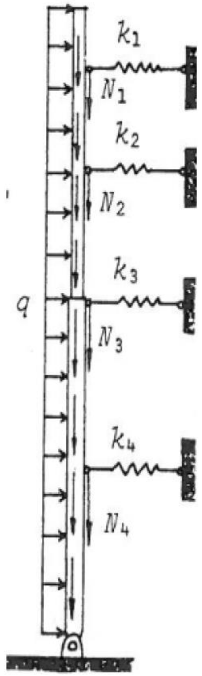
How much ?

4. what imperfection-sensitive is the structure under study?

Cable-stayed tower



A model




Stability?

# Stability of an equilibrium

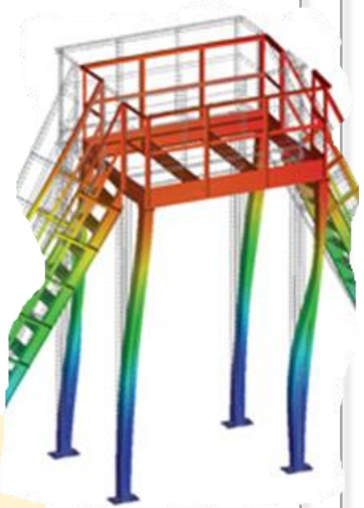
*Equilibrium* and *Stability*

- **Static stability:**
  - ➢ Consider a structure which is initially in equilibrium
  - ➢ Introduce an arbitrary tiny perturbation ;
  - ➢ I
    - ▪ Then what happens?

**DOES** The structure sustains or comes back to its initial equilibrium configuration or very close to it?

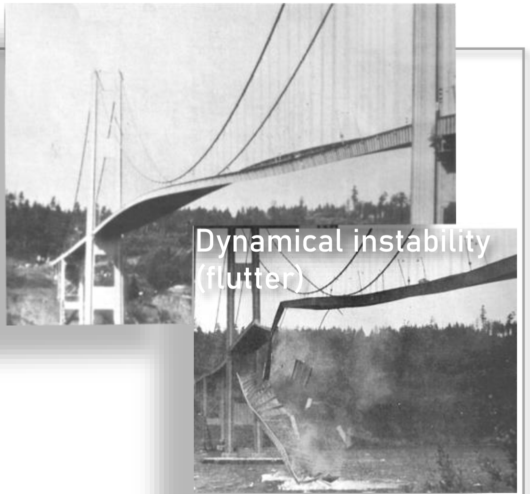
YES: The equilibrium stable

NO: The equilibrium unstable




- **Dynamic stability:**
  - ➢ **Dynamic stability** considers time history of the motion after an arbitrary tiny perturbation
    - ▪ Then what happens?

**DOES** the change in amplitude keep not enhanced in time, after such perturbation?




**YES**: dynamically stable

# Methods of stability study

In our course, we will systematically the energy approach.

In our course, we will systematically the energy approach.

**There exits three (analytical) methods** for studying the stability of an equilibrium:

1) **Bifurcation approach** – write the equilibrium equations in a deformed configuration and determine the onset of buckling

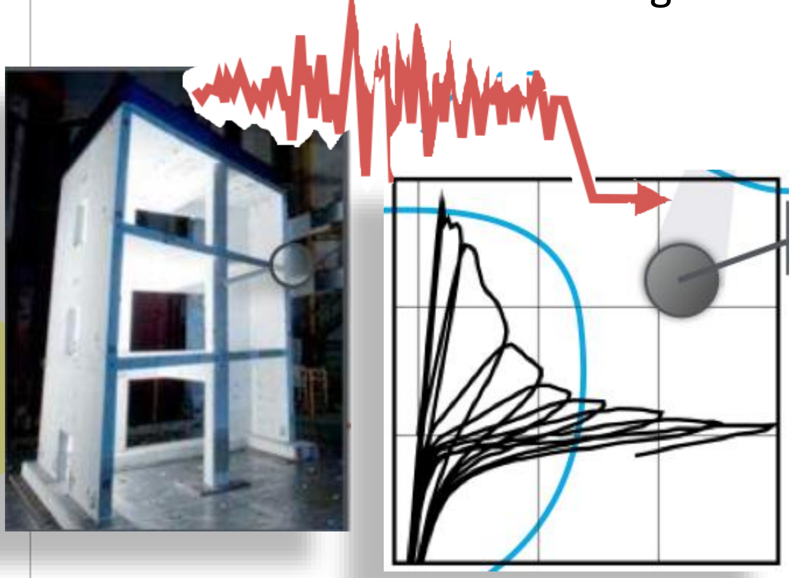
2) Energy approach – the change of total potential energy of the system between two neighboring equilibrium states is used to derive the equations of equilibrium and to study its stability

3) Dynamic approach – the equations of motion of the system are established.
   i) natural frequencies decreasing to zero, correspond to the onset of instability or
   ii) investigate how an initial perturbation develops with time (full dynamics)

1) **For design purposes**, detailed stability behaviour of structures can be analysed **numerically** by performing a **geometrical and material non-linear (GNMA)** analysis on the real structure with the inherent real and possible imperfections using capability of the Finite Element technology. Such analysis provides full load-displacement curves which are used to identify bifurcations or/and limit points for determining the limit loads

5) **Experimental approach** is needed since models are only approximations and very often, they are a very incomplete approximations. For some structures, experiments are of primary importance

**Experimental**: stability of concrete structures under seismic loading



**Experiment: NASA NESC Shell Buckling**


Buckling of thin-walled cylinders

# Basic Concepts

'Two types' of stability

*N.B.* the **stability** loss is nothing else than the motion during transition of the structure from one equilibrium state to another one and therefore stability loss is dynamical by nature. Despite, that, under certain conditions one can treats the problem of stability loss in statics framework. Naturally, many other problems of stability must be set as dynamical problems to be correctly solved.

## Static Stability (SS):

Treated in this course

**Stability of static equilibrium** configurations of a mechanical system (Euler 1707-1783)

*ex.* structures: columns, beam-columns, frames, plates, shells …

## Dynamic Stability (DS):

Not treated in this course

**Stability of the motion of dynamic systems** (Lyapunov 1857-1918)

(*e.g.*, a train trajectory, system with following forces: reaction propulsion rackets, korkeushyppääjän sauva, aerodynamic forces on structures: **flutter** bridges, …)

Conservative systems

**Assumptions for static stability criteria:**

- **Elastic* material** (strain energy exists) displacements and rotations, not necessarily small
- **Loads are conservative**** (= derivable from a potential)

**Examples of conservative forces**: *gravity* and *hydrostatic loads, elastic force, …*

**Examples of non-conservative forces**: friction, (drag) hydro- & aerodynamic and jet-propulsion loads, gyroscopic forces, following forces…

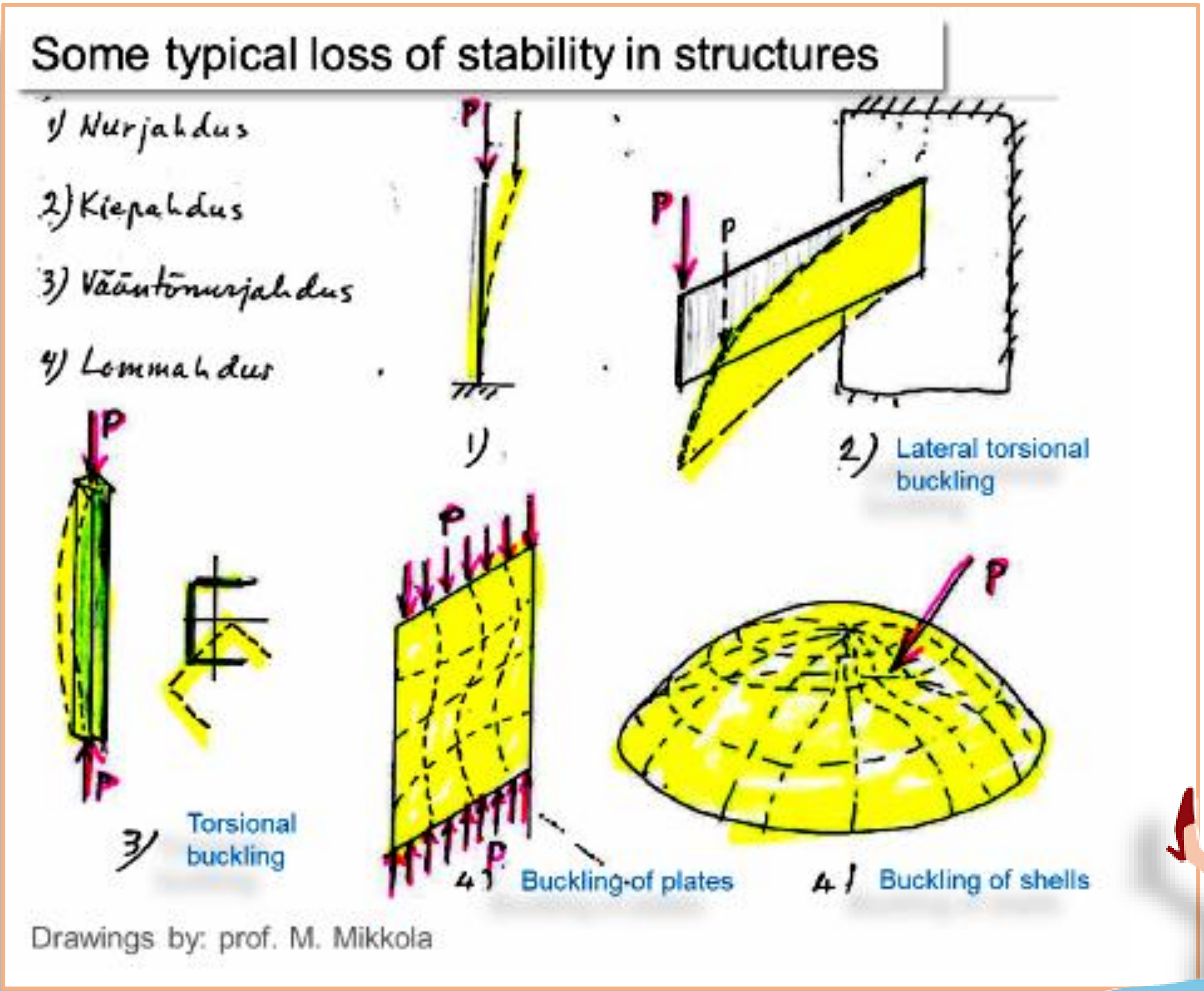
*Ex.* non-conservative forces:

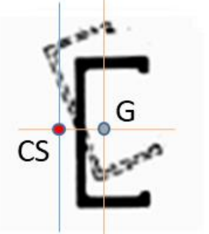

Conservative and non-conservative systems

Jet propulsion force

## **Stability loss** as we see its … consequences


*Loss of stability* of a column. Original temporary renforcement method.



## What is stability as a phenomenon?

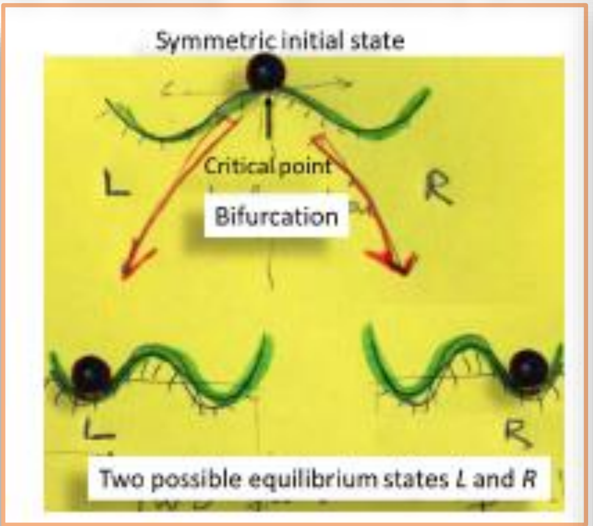
Stability is a fundamental property of (dynamical systems) systems having *more than one equilibrium states* where the system can rest in one or in the other states

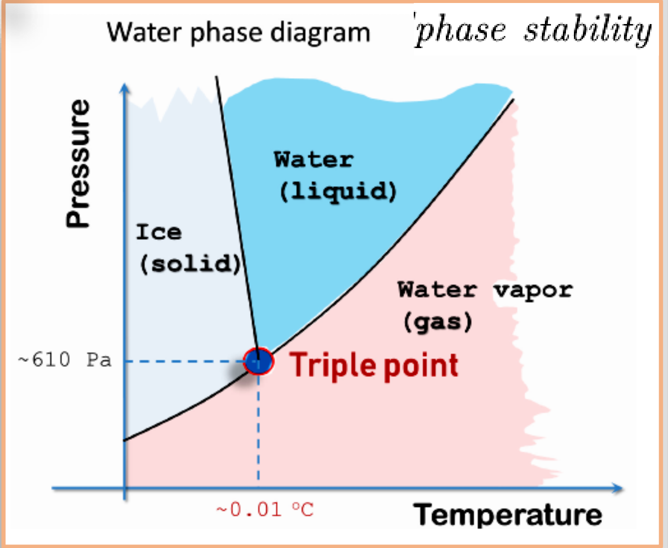
- These **equilibrium** states correspond to **local minima** in **potential energy** of the system
- Between two local minima a local maximum should exist. The state at this critical point is *unstable*. This local maxima is termed as *potential barrier*.
- A tiny external perturbation can make the system switch to another equilibrium state if enough energy input is given to jump the barrier separating the local minima

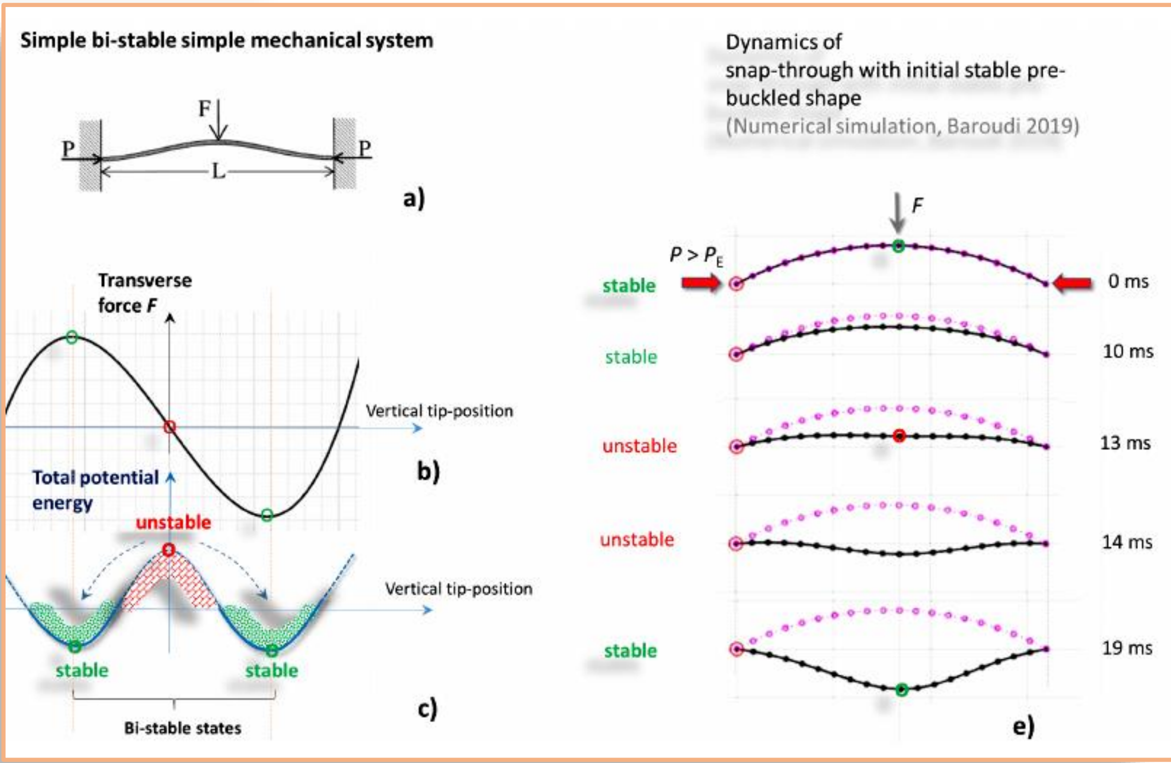



*Loss of stability* of a column. Original temporary renforcement method.

### Loss of Stability = symmetry breaking



# What is stability as a phenomenon?

**Stability loss** as a **symmetry breaking** phenomena

- In **physics, loss of stability** belongs to the class of ***symmetry breaking*** phenomena …
  - where action of **infinitely small perturbations** (fluctuations) on the ***system*** *being* ***close to a critical point*** …
  - leads to **sudden branching** *via* ***bifurcation*** (or limit point) to some other **neighboring state**

- **Loss of stability is dynamic by nature**

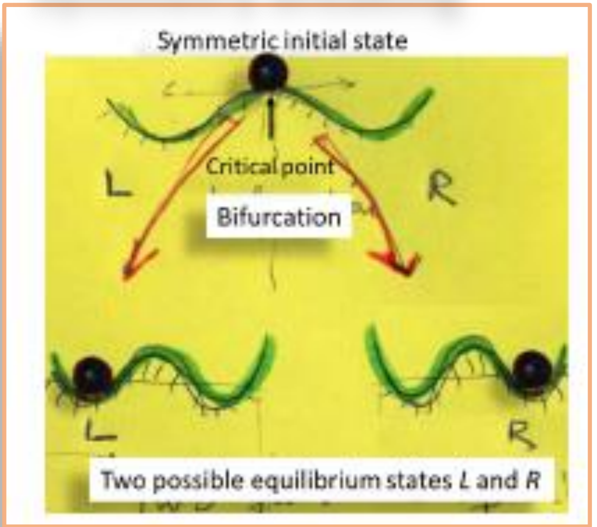
Examples

- ✓ *snap-through* of a shallow arch
- ✓ *resonance* in parametric excitation of stay-cables of a bridge or cable of guyed tower,
- ✓ Flutter (a dynamic instability of an elastic structure interacting with a fluid flow … Tacoma Bridge)
- ✓ … ..

A tiny perturbation


Straight primary equilibrium
Is this position stable?

## Symmetry breaking



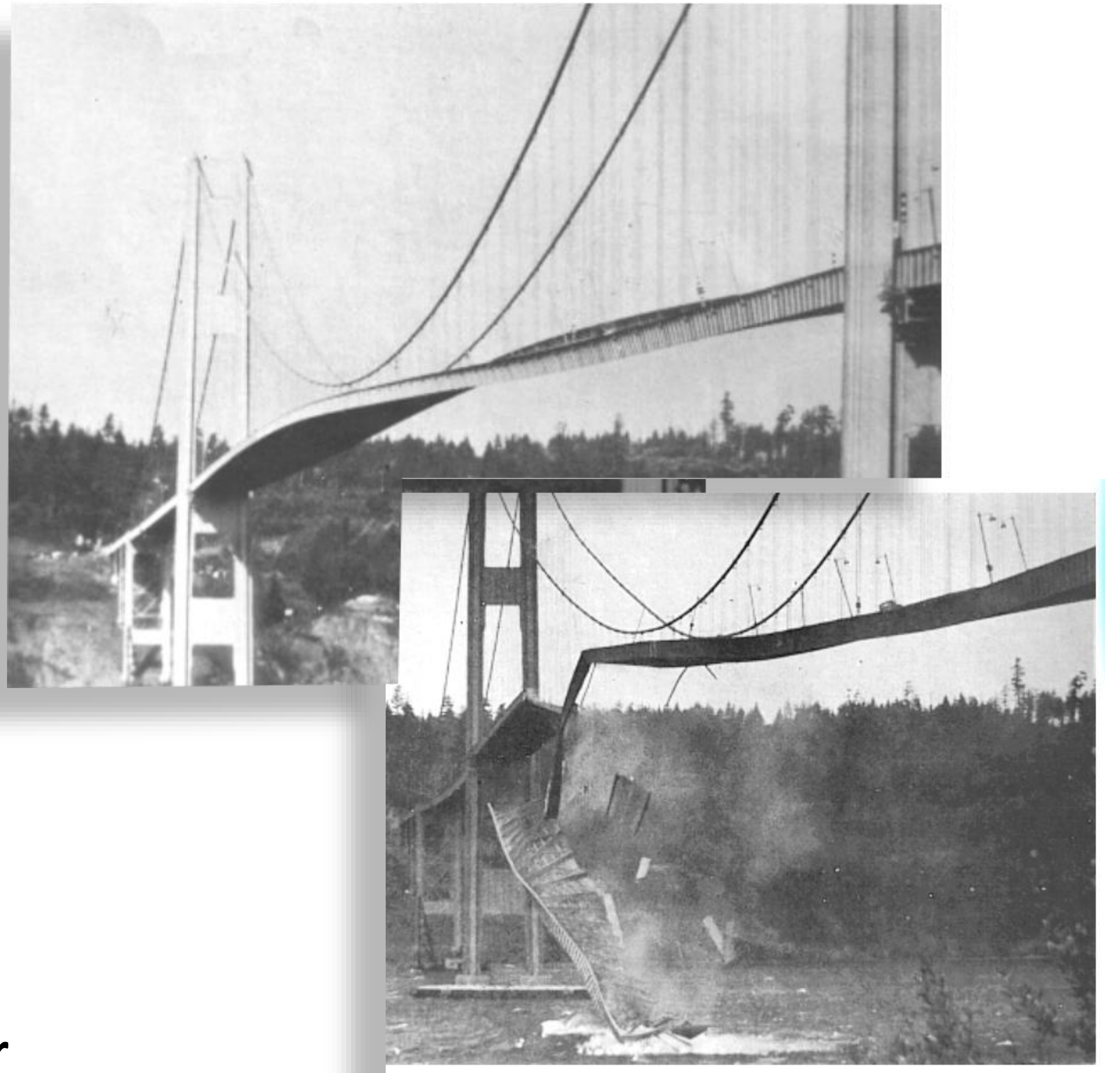

**Theory of supersymmetry:**
*Physicists believe that just after the* ***big bang,*** *all of the forces of nature were identical and all elementary particles were the same.* But within an 'instant', symmetry was broken . . . and then . . . we and the universe are here …

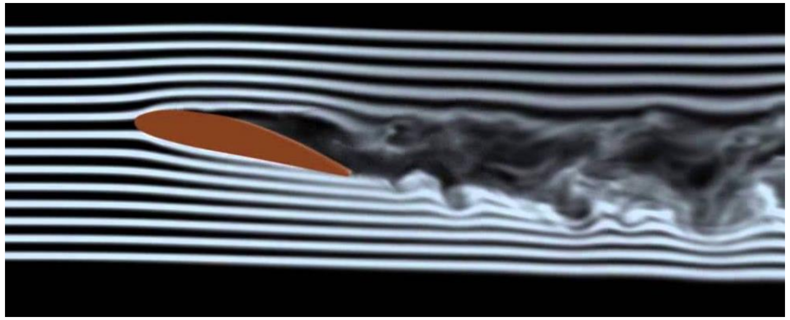
It seems that we are the consequence of a stability loss of the primary universe!

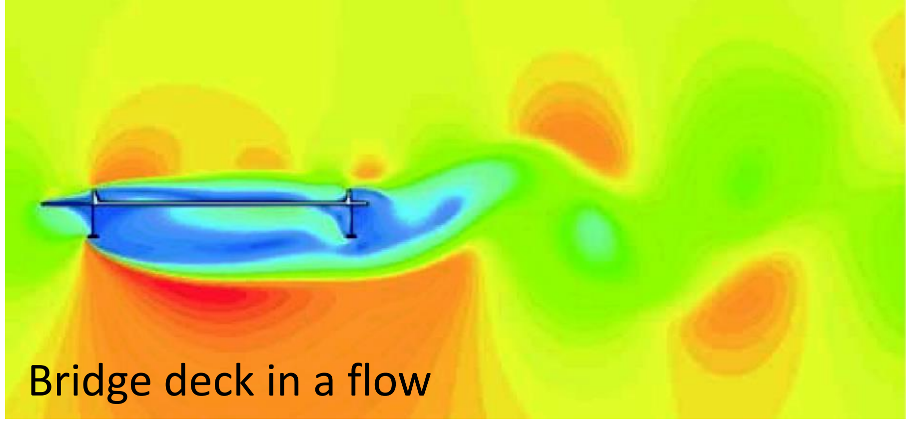
I ask physicists what does '*instant* means before our time even existed!

# What is stability as a phenomenon?

Oscillation and subsequent collapse of the Tacoma Narrows Bridge.








**Flutter**

Coupling structure-fluid motion

Bending and torsional modes … have same frequency

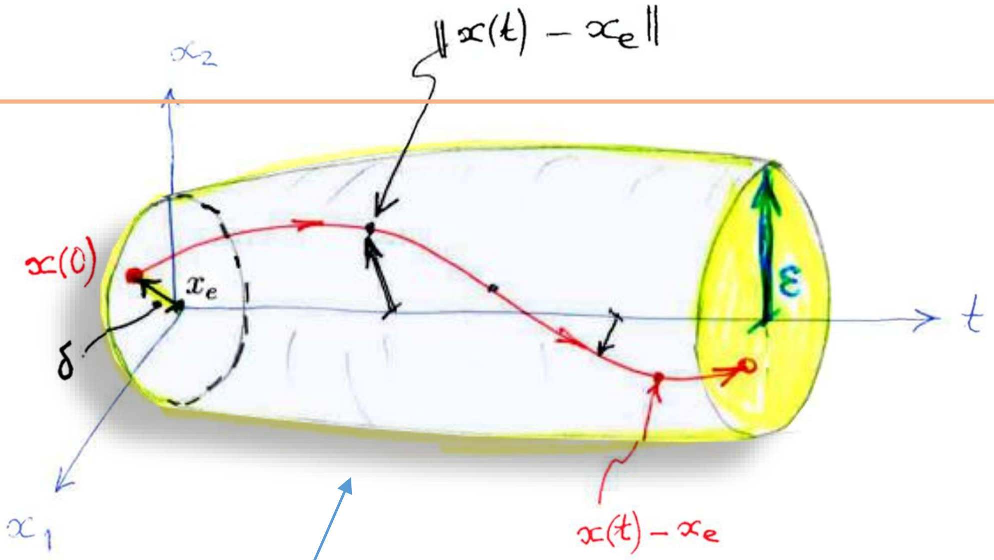
***Lyapunov dynamic stability*** criteria is naturally in use in structural dynamics

# What is stability as a phenomenon?

## **Stability loss** as a dynamical process

- **Dynamical systems** are generally described by non-linear differential equation set

$$\begin{cases} \dot{x}(t) = f(x(t)),\ t > 0 \\ x(0) = x_0. \end{cases}$$




### Equilibrium point

- The system posses equilibrium points (states) $x_e$ defined by $f(x_e) = 0$

$x_e$ is an equilibrium point, if $f(x_e) = 0$

$$\Downarrow$$

$\dot{x}_e = 0$ so, with zero velocity

$\Longrightarrow$ the point is at rest

**Lyaponov stability**: *If at an equilibrium point $x_e$, two solutions (time series) having initial conditions close to each other remains close to each another for ever then the equilibrium point $x_e$ is Lyaponov stable.*

## Lyapunov dynamic stability:

equilibrium point, $x_e$ is *Lyapunov stable*,

$$\forall \epsilon > 0,\ \exists \delta \text{ such that } \|x(0) - x_e\| \le \delta \implies \|x(t) - x_e\| \le \epsilon,\ \forall t \ge 0.$$

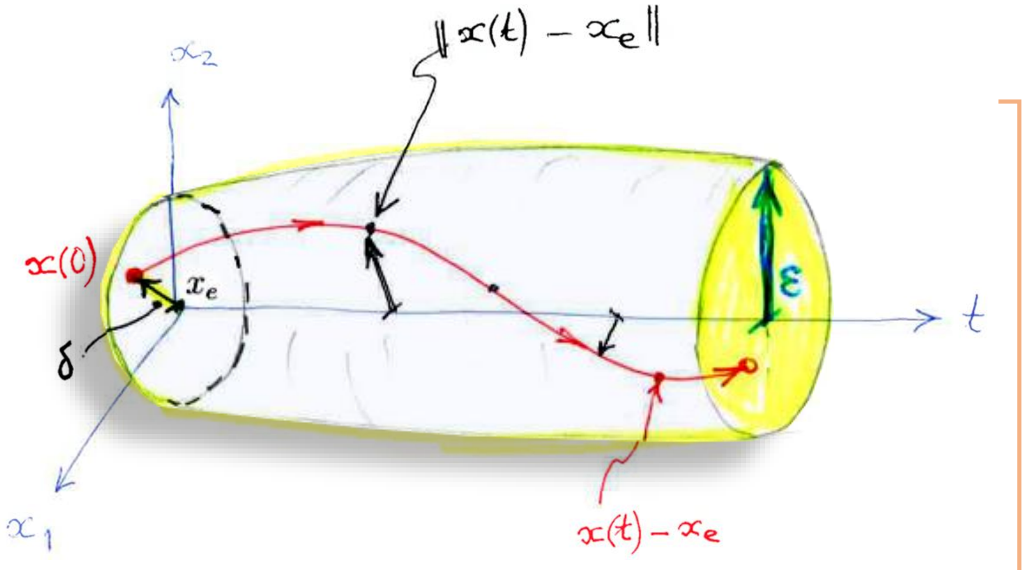
# What is stability as a phenomenon?

- **Dynamical systems** are generally described by **non-linear differential equation set**

$$\begin{cases} \dot{x}(t) = f(x(t)),\ t > 0 \\ x(0) = x_0. \end{cases}$$

- The system posses equilibrium points (states)

$$x_e \ \text{defined by} \ f(x_e) = 0$$




- The discrete **equation of motion** of a mechanical system, can be *recast* in terms of a canonical **non-linear dynamical problem**

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}, & \text{linear case} \\ \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}(\mathbf{u}(t), t), & \text{non-linear case.} \end{cases}$$

velocity $\mathbf{v} = \dot{\mathbf{u}}$ as a change of variable

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix} \implies \underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$

For the linear case:

$$\mathbf{f} := \mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u}$$

Lyapunov dynamic stability:

**Lyaponov stability**: *If at an equilibrium point $x_e$, two solutions (time series) having initial conditions close to each other remains close to each another for ever then the equilibrium point $x_e$ is Lyaponov stable.*

Lyapunov dynamic stability criteria is naturally in use in structural dynamics

# What is stability as a phenomenon?

- The discrete **equation of motion** of a mechanical system, can be *recast* in terms of a canonical **non-linear dynamical problem**

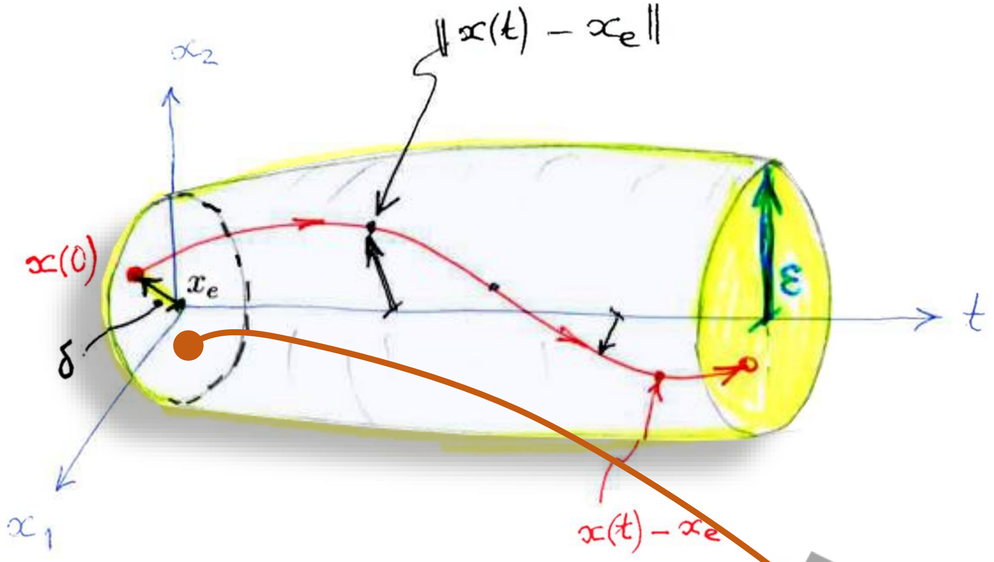
$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}, & \text{linear case} \\ \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}(\mathbf{u}(t), t), & \text{non-linear case.} \end{cases}$$

velocity $\mathbf{v} = \dot{\mathbf{u}}$ as a change of variable

For the linear case:

$$\mathbf{f} := \mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix} \implies \underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$




Lyapunov dynamic stability criteria is naturally in use in structural dynamics

Includes also **aero-elastic forces from the interaction of the structure with flow**

Flutter leads to collapse

✓ **Flutter** - a dynamic instability of an elastic structure interacting with a fluid flow … Tacoma Bridge

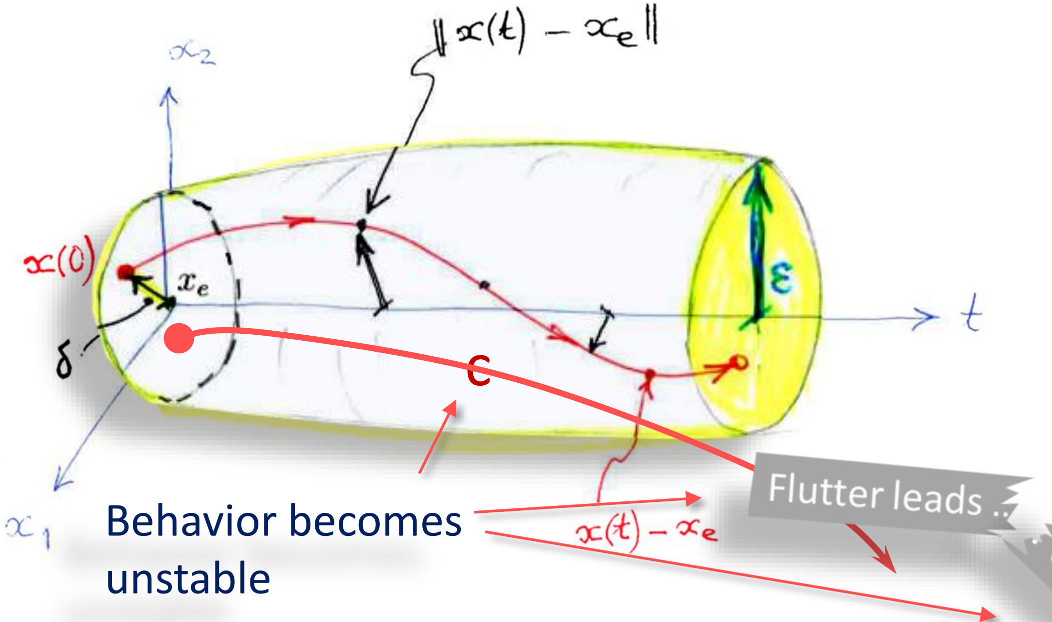
Discrete **equation of motion** of mechanical systems

$$\mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}(\mathbf{u}(t), t), \qquad \text{non-linear}$$

$$\underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$

**Includes in forces *f***
**aero-elastic forces from the interaction of the structure with flow**

## **Lyapunov** *dynamic stability criteria*




Tacoma Bridge

**Example of dynamic instability: Flutter** : *instability* of an elastic structure resulting from structure-fluid interaction

# Structural design and stability

## Standards: design of steel structures

- Local buckling ............................. EN 1993-1-5
- Flexural buckling ........................ EN 1993-1-1 hot rolled columns
- Lateral torsional buckling .......... EN 1993-1-1 beams
- Lateral

- Flexural torsional buckling ........
- Local-global .................................
- Distortional .................................
- Shear buckling

EN 1993-1-3
EN 1993-1-5

- Shell buckling ............................ EN 1993-1-6
  - Linear elastic Bifurcation Analysis (LBA) (= linear buckling analysis)
  - Geometrically Non-linear Analysis (GNA)
  - Geometrically Non-linear Analysis with Imperfections
  - ... LA , LBA , GNA , GNIA, ...

(= post-buckling analysis for perfect structure and structure with imperfections)

* * *

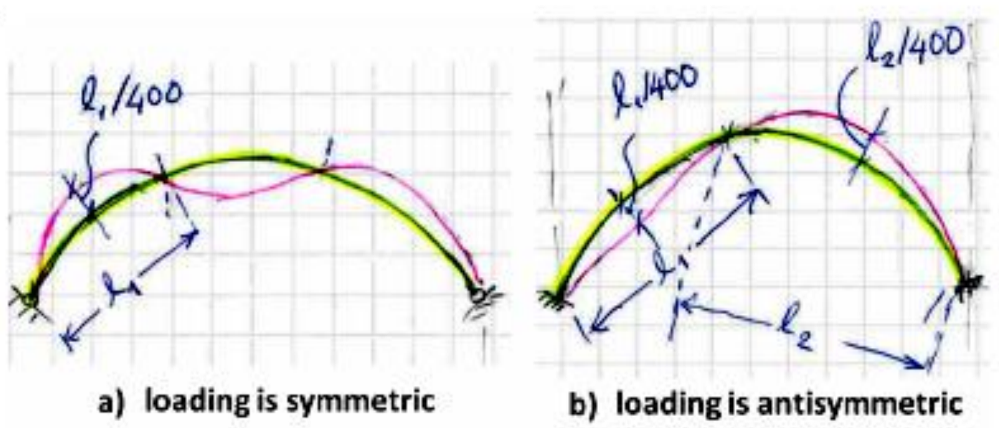
## Standards: design of wood structures

- Stability issues & imperfections ......... EN 1995-1-1

## Standards: design of concrete structures

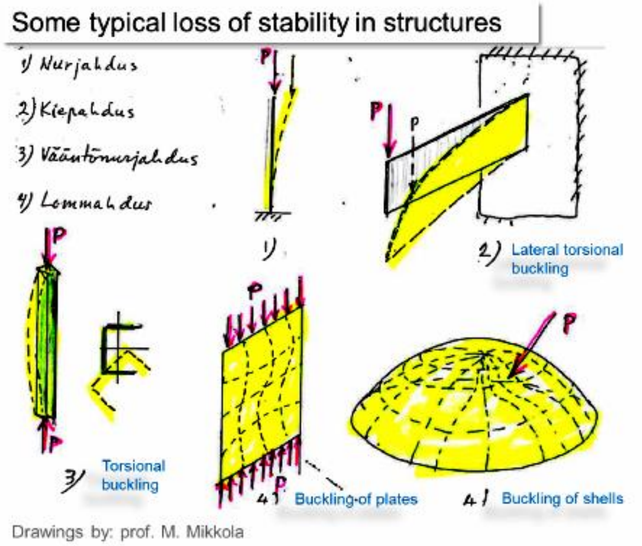
- Sect. 5.8 Second order effects with axial load..... EN 1992-1-1

Some standards related to stability issues in structural design.




**Example of initial shape imperfections in an arch** (Standards: design of wood structures - EN 1995-1-1)

Example of initial shape imperfections in wooden arches to be accounted in the structural analysis.

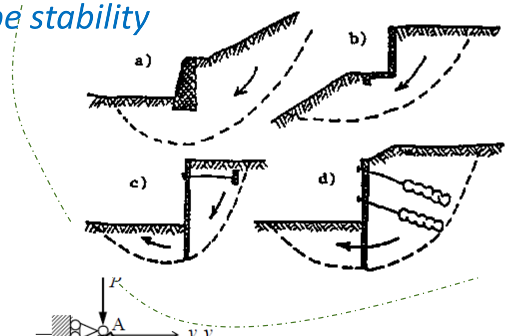


# Examples of loss of stability

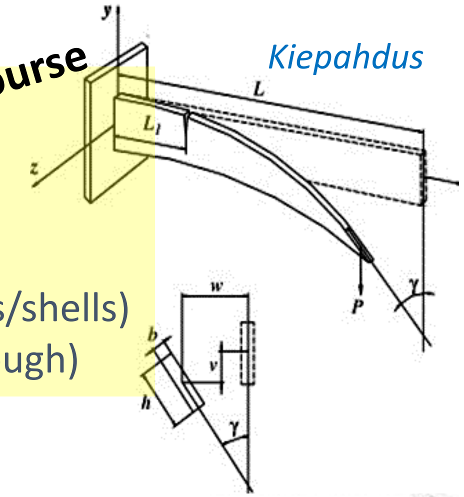
**Rakenteiden epästabiiliusilmiöitä:**

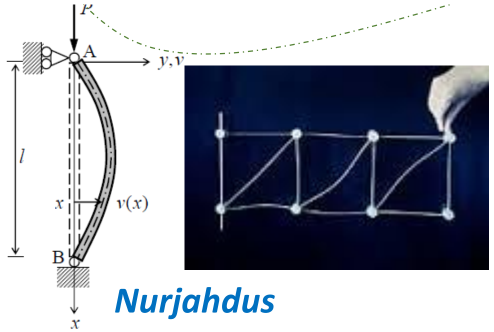
- sauvojen **nurjahdus** (flexural buckling) ← This course
- **vääntönurjahdus** (torsional buckling)
- **kiepahdus** (lateral torsional buckling)
- levyjen ja kuorien **lommahdus** (buckling of plates/shells)
- laakeiden kaarien ja kuorien **läpilyönti** (snap through)

*Slope stability*

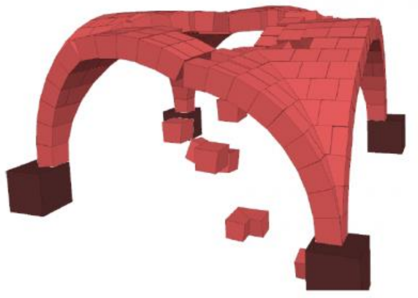



*Kiepahdus*

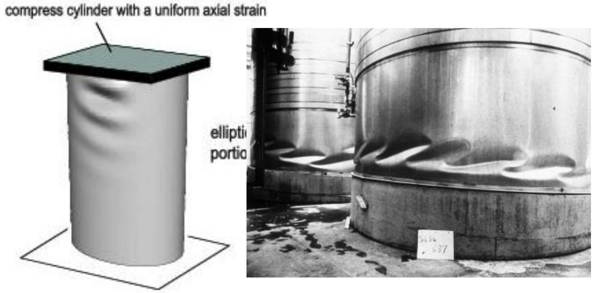


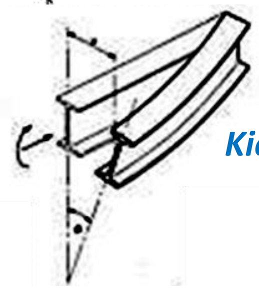




***Nurjahdus***



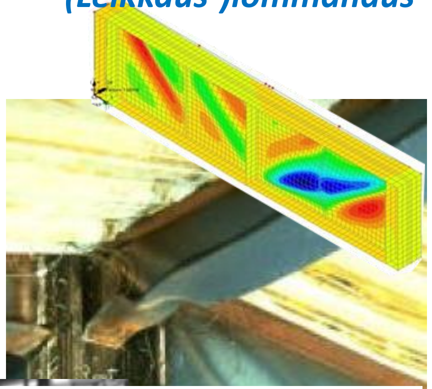
***Lommahdus***

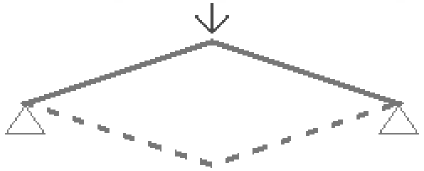


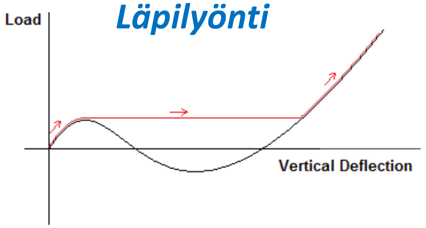



***Kiepahdus***

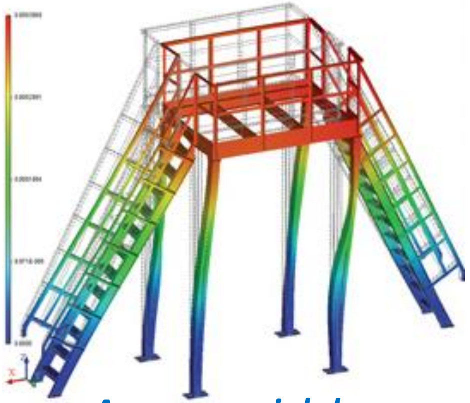
***(Leikkaus-)lommahdus***



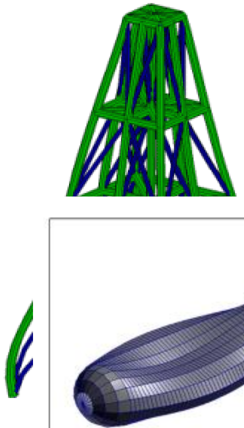




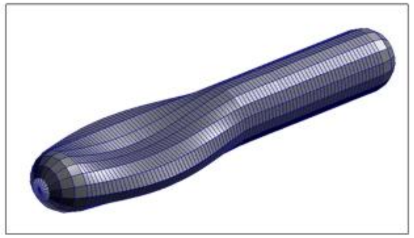

***Läpilyönti***

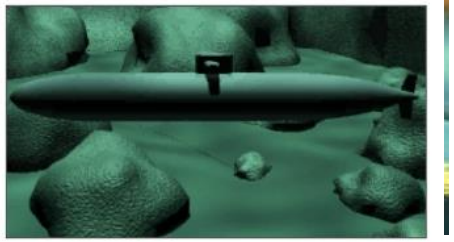


***Avaruusnurjahdus***



***Lommahdus***

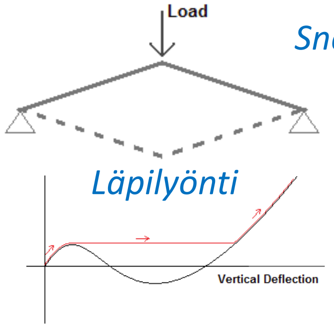




# Examples of loss of stability

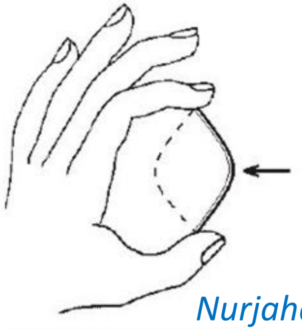
**Rakenteiden stabiiliusilmiöitä** - sekalaista

*Snap-through*

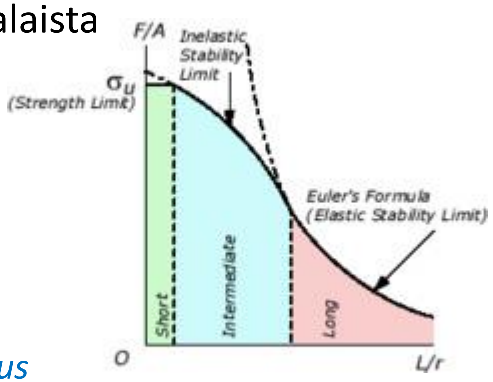


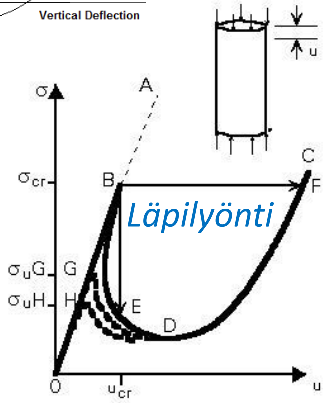

*Läpilyönti*

*Kimmoton nurjahdus*
*Inelastic buckling*



*Nurjahdus*

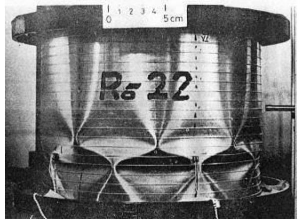


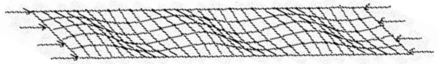




*Läpilyönti*

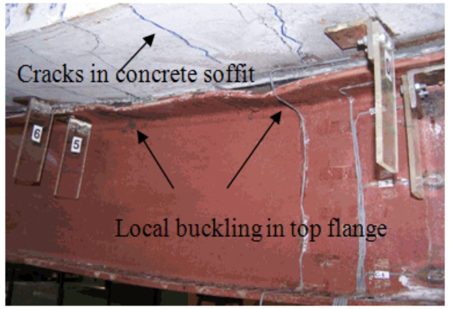
Figure 6 Post-buckling behaviour of an imperfect axially compressed cylindrical shell

*Lommahdus*





*Lommahdus*



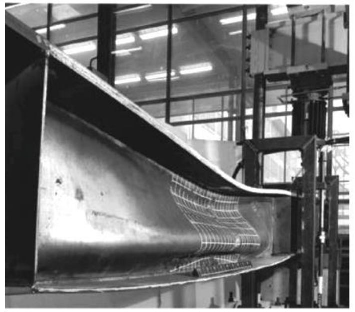
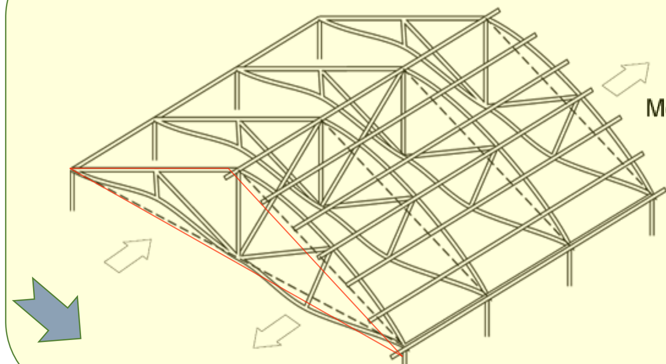
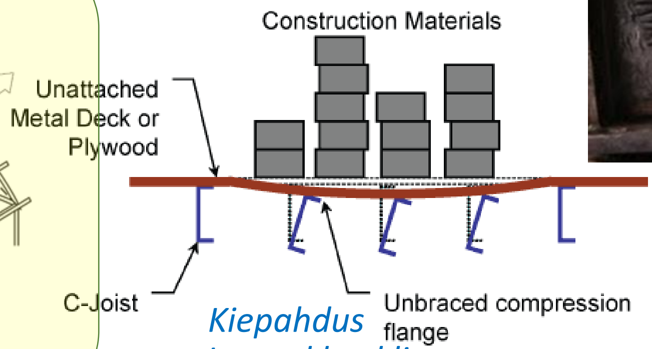



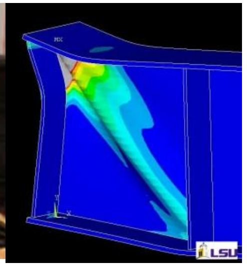
Figure 2. Vertical web buckling.

*Lommahdus*






*Kiepahdus*
*Lateral buckling*



*Lommahdus*

*vrt. kiepahdukseen – vaikkakin alapaarre on vedossa*

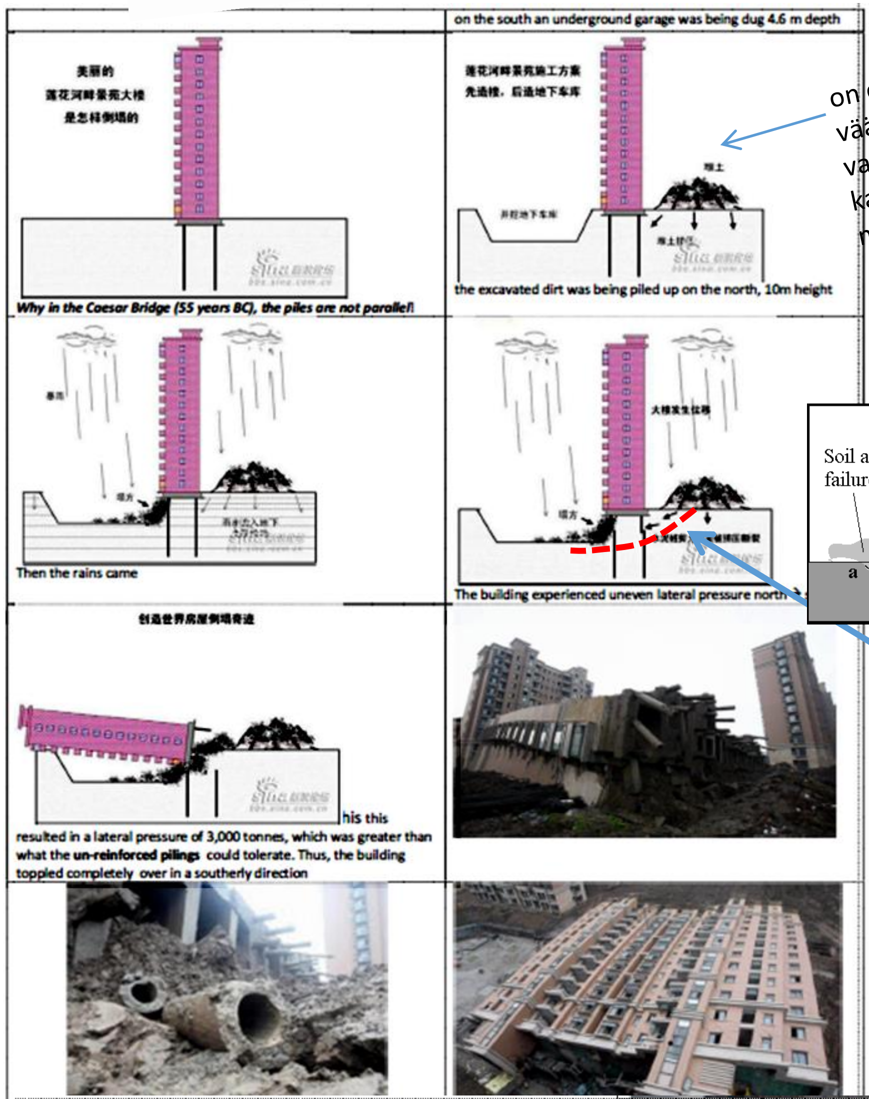
# Maapohja petti ... maan leikkausmurto

## Soil Foundation Bearing capacity Failure – China

You want to know more? join the courses of **Geotechnics** and **Soil and rock- mechanics**

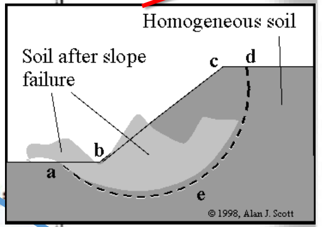
An anthology of errors accumulation

Material instabilities



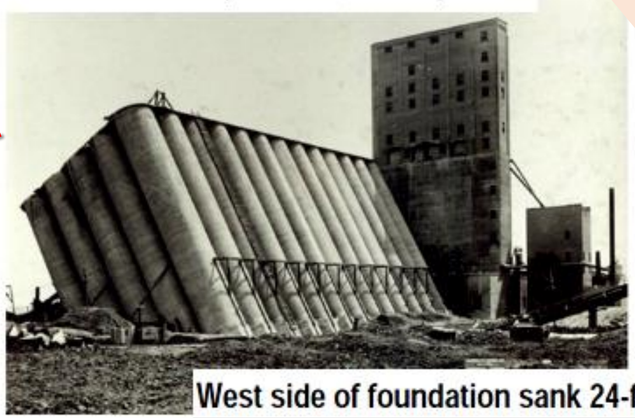

on ehdottomasti väärä paikka varastoida kaivauksen maa-massa...

**Slope stability**



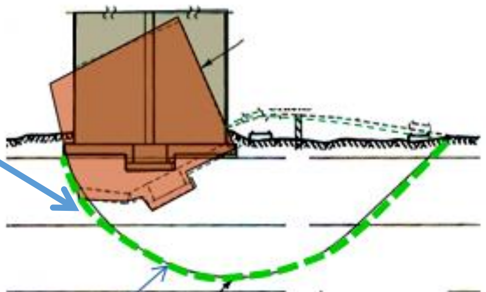

SUMMARY: 13 stories built on the surface of the ground . . . **with no basement** and attached **to hollow concrete pilings with NO steel reinforcing**. (*Where have all the true engineers gone?*)

Static: Transcosna Grain Elevator Canada (Oct. 18, 1913)



West side of foundation sank 24-ft

Soil Foundation Bearing Capacity Failure



Todennäköinen leikkauspinnat
Shear-slip surface

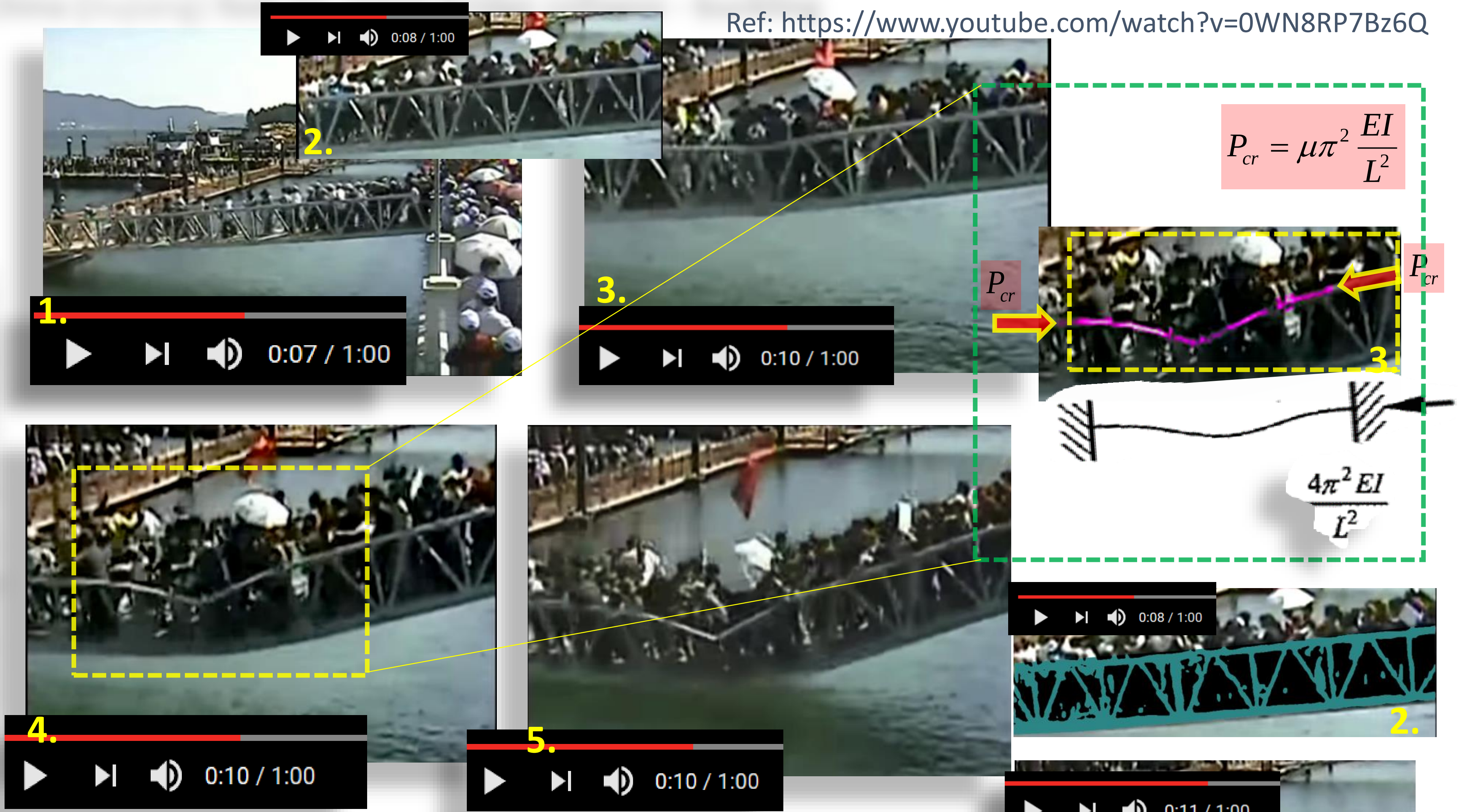
Material instabilities

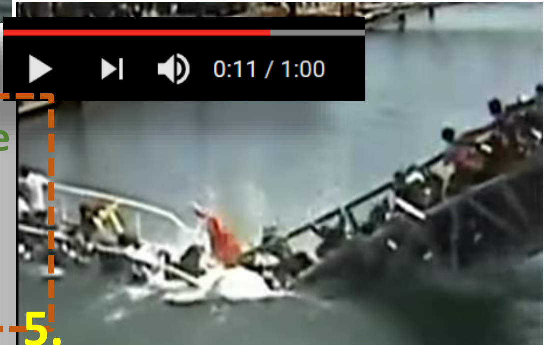
# China (Jiujiang) foot (or ramp) bridge collapse - buckling

Ref: https://www.youtube.com/watch?v=0WN8RP7Bz6Q

## Examples of stability failure

## Why instability is an unwanted event in desing of structures?



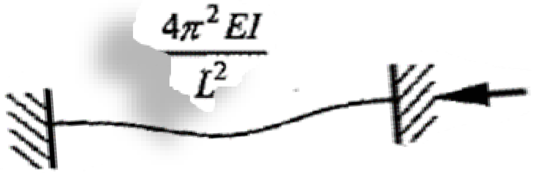

**Buckling failure happens –** watch it, you'll see how fast & unpredictable

- **At once and global (in the whole structure or its member)**
- **Unpredictable (you do not have time to see the signs that is too late already) … happens between 7th s and 9th s!**

# Examples of stability failure

Foot bridge (ramp) collapse in Jiujiang City (China's Jiangxi)






**Railway bridge collapse, Russia ~1890**

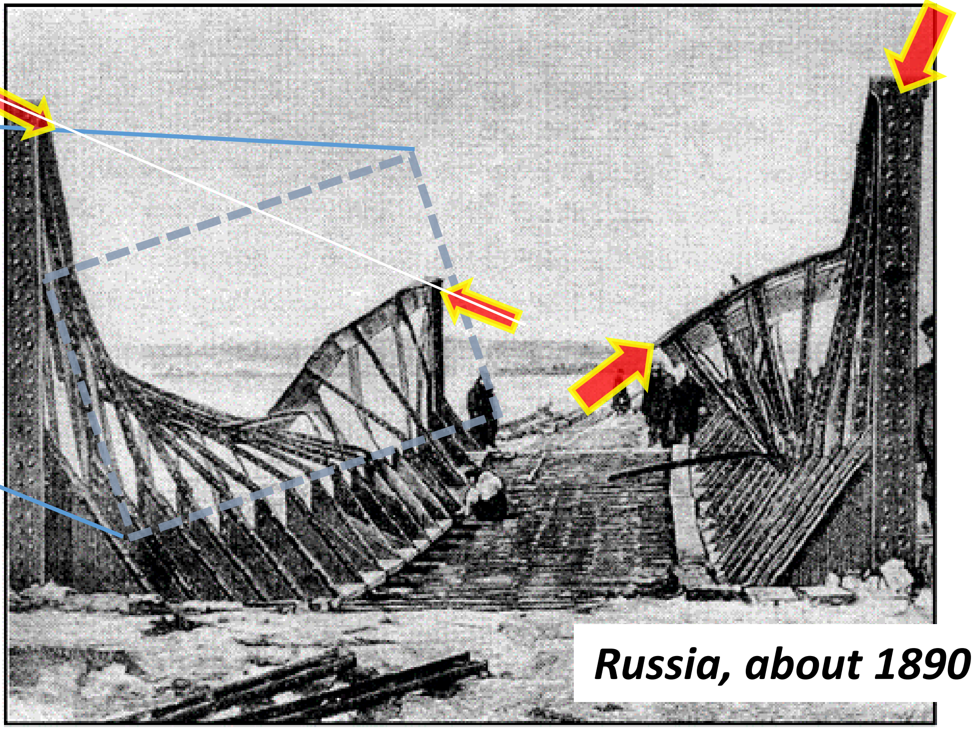



*fig. 8.3. - Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).*

***buckling***

***Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).***

Why instability is an unwanted event in desing of structures?

$$P_{cr} = \mu \pi^2 \frac{EI}{L^2}$$

**The mechanical cause of the collapse is of the same type:**
flexural buckling of compressed upper chord of the truss
(yläpaarteen nurjahdus)

Examples of stability failure

# Why instability is an unwanted event in desing of structures?

## Consequences of stability loss

Good engineering final result should always provide the product itself together with *a quantified safety margin* of it operation.

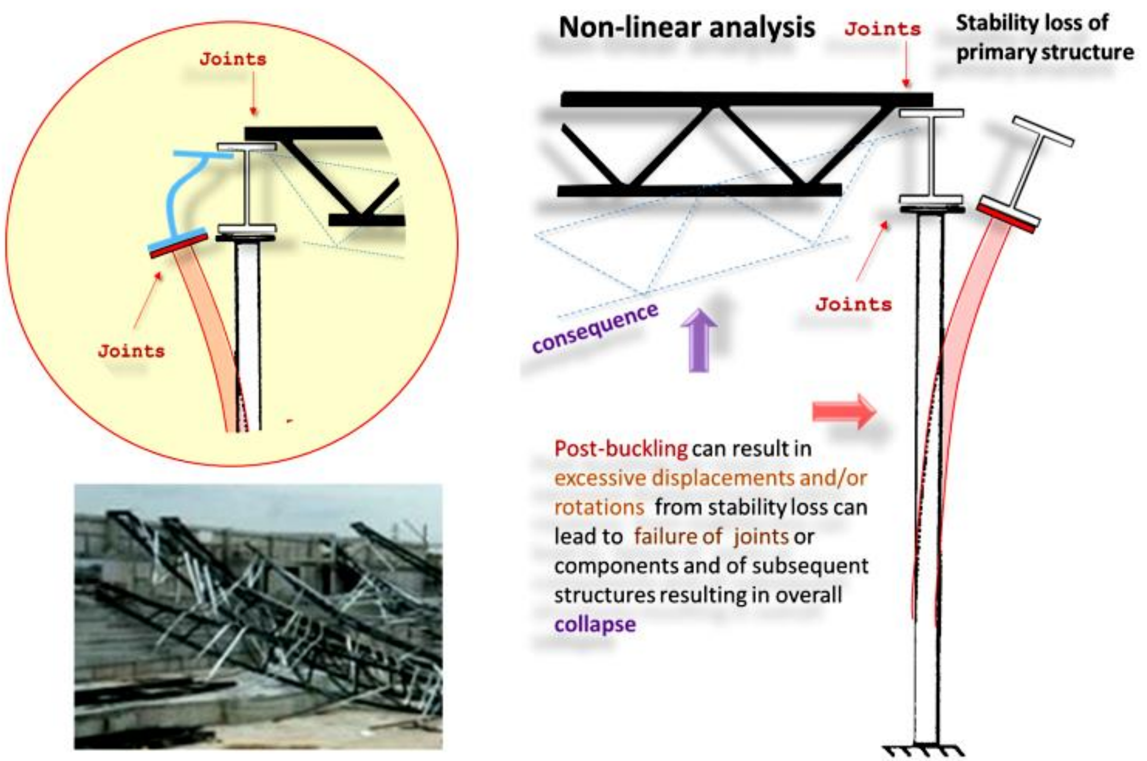
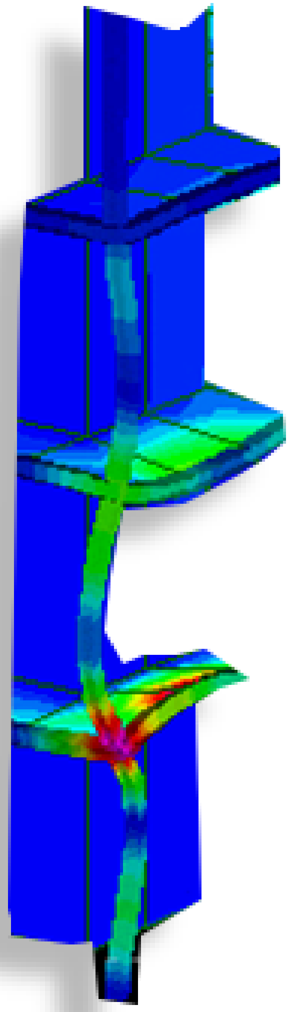



Figure 3.26: Illustration of possible consequence of too large rotations or displacements after stability loss.

# Structural design and stability


Figure 3.23: Examples of various types of loss of stability in simple structures. From left to right: lateral-torsional buckling, buckling of spherical and cylindrical shells, buckling of slender columns, buckling of a rail-road rail bonded to a support and plate buckling represented by shear buckling of the flanges and compressive buckling of web.



Flexural buckling

# A bit of history

Pieter van Musschenbroek
(1692 – 1761)

A Dutch scientist

Physics, mathematics, philosophy, medicine, astronomy

He did pioneering studies on the buckling of compressed struts

$$P_{failure} \quad \alpha \quad 1/\ell^2$$

- Performed experiments on column buckling (1729)
- Observed that the maximum compressive load a column can sustain prior to failure is proportional to $1/\ell^2$

experiment

theory

Compare with Euler's buckling load: (obtained theoretically, 1744)

$$P_{cr} = \frac{\pi^2 EI}{l^2} \sim 1/\ell^2$$

# STATIC STABILITY OF STRUCTURES

Exp: Djebar Baroudi, PhD

Euler, the Great

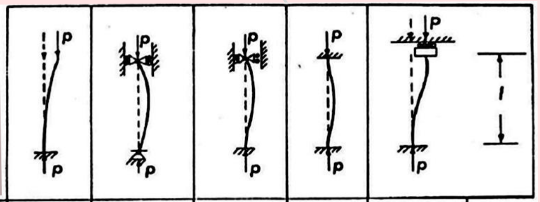
Djebar BAROUDI, PhD.
Short-version, state: 7.10.2016


7.10.2016

Leonhard Euler

## To: Structural Mechanics:

## Beams and frame structures



| Тапоus | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| μ | 0.25 | 1 | 2.046 | 4 | 1 |

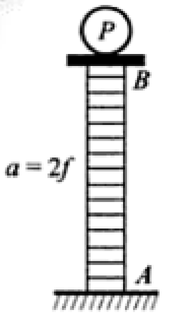
$$\sigma = \frac{F}{A} = \frac{\pi^2 E}{(\ell/r)^2}$$

Writing by Euler himself; can you identify **the formula?**

Modern writing $\frac{\pi^2 EI}{L^2}$ ↔ $eſt = \pi\pi.\frac{Ekk}{aa}$,

SUR
LA FORCE DES COLONNES.
PAR M. EULER.

le poids que cette colonne eſt capable de ſoutenir ſans ſe plier, eſt $= \pi\pi.\frac{Ekk}{aa}$,




Leonard Euler

He derived the theoretical critical load for buckling of a column already in **1774**! At that time no one understood the importance of such result.

2..718281828459045235360287471352662497757247093699995...

[101] **Pieter van Musschenbroek** (1692 – 1761: A Dutch scientist Physics, mathematics, philosophy, medicine, astronomy) did, about 30 years before *Euler*, pioneering experimental studies on the buckling of compressed struts. He Performed experiments on column buckling (1729) and he observed that the maximum compressive load a column can sustain prior to failure is proportional to $1/\ell^2$ Compare with Euler's buckling load formula $P_{cr} = \pi^2 EI/\ell^2$ obtained theoretically and about 30 years later in 1774. At that time nobody has understood the importance of such result. Even Coulomb was saying that these results,including experimental ones are wrong because many experiments show that the compressive strength of columns was proportional to the cross-section area and not to the square of its length. These last experiments were done with short iron and wooden columns where the failure mode was the crashing or material failure and not buckling. At that time the concept of slenderness was not understood yet. At the end, they were all right but each one on the opposite side of the slenderness axis. This critical slenderness point divides the failure mode into material failure and elastic buckling, for axially compressed members.

Leonard Euler

He derived the theoretical critical load for buckling of a column already in **1774**! At that time no one understood the importance of such result.

(1692 – 1761)

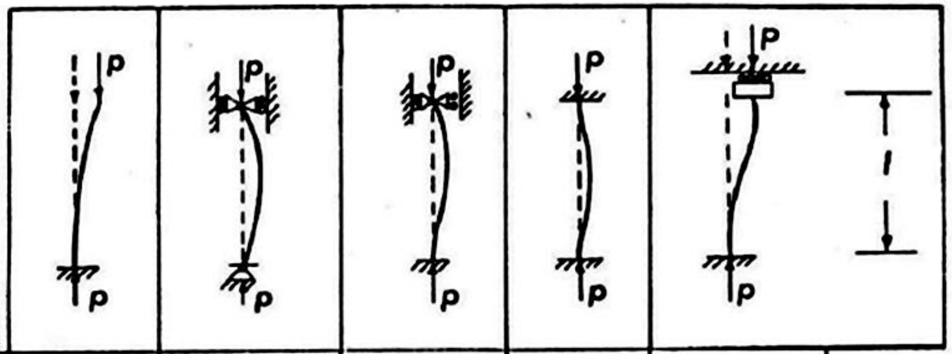
A Dutch scientist

Physics, mathematics, philosophy, medicine, astronomy

theory

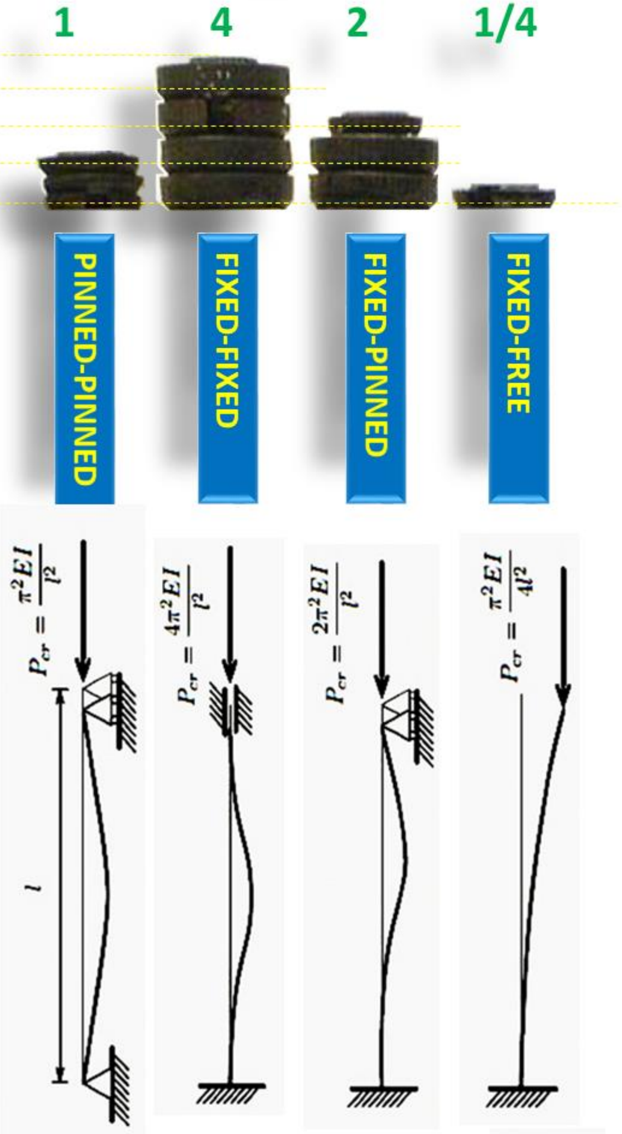
$$P_{failure} \quad \alpha \quad 1/\ell^2$$

- Performed experiments on column buckling (1729)
- Observed that the maximum compressive load a column can sustain prior to failure is proportional to $1/\ell^2$




| Τύπος | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| μ | 0.25 | 1 | 2.046 | 4 | 1 |

# Effects of boundary conditions – experimental evidence for Euler's buckling formulas


$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad \frac{4\pi^2 EI}{l^2} \quad \frac{2\pi^2 EI}{l^2} \quad \frac{\pi^2 EI}{4l^2}$$

1 4 2 1/4

# Stability analysis investigates

- Equilibrium configurations existence of multiples equilibriums or limit points
- Stability of these equilibriums with respect to small perturbations
- Sensitivity with respect to imperfections
  - shape
  - geometry
  - Loads (eccentricity)
  - Material imperfections

In addition to the above points, following questions will be answered:

- Can we predict the critical load?
- What happens at the bifurcation (or limit) point?
- Can we describe or determine the post-critical branches? What would be their shape? Their nature?

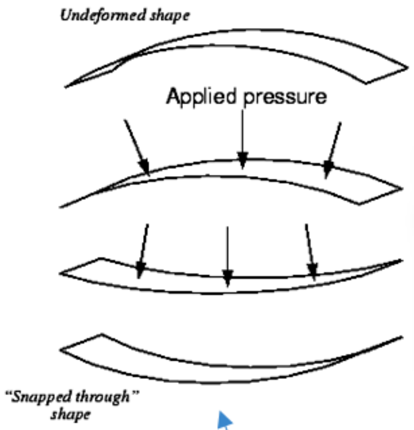

Equilibrium? **Yes.** $\Leftarrow \delta\Pi[\mathbf{u}] = 0$

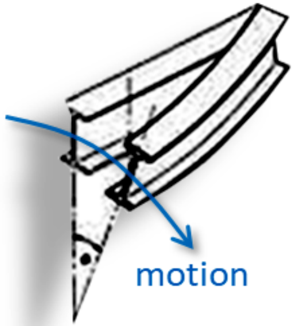
But, is it stable? **No.** $\Leftarrow \delta^2\Pi[\mathbf{u}] < 0$

# Fundamental Concepts

- **Stable equilibrium state**
  - Let's perturb a bit a structure Initially resting at an *initial equilibrium state* away from this equilibrium state. If after removal of all disturbing factors the structure *returns to its initial equilibrium state* then the initial equilibrium state is **stable** (for elastic structures)
  - Even when the structure *tends only to return to its initial equilibrium state* one can assume that this initial equilibrium state is stable.

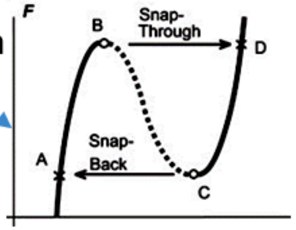
    *Such behaviour occurs when material behaviour is plastic or elasto-plastic. In such cases the structure returns only incompletely to its initial configuration because of residual deformations.*
  - In case of rigid bodies the **stability of position** is considered
- When the above listed behaviour is not fulfilled we say that the equilibrium is unstable

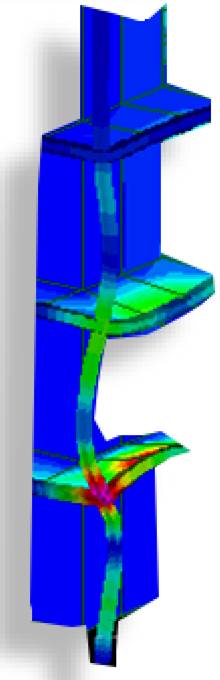




Lateral torsional buckling

Snap-through



Flexural buckling

# Method for Study of Stability

$$\Pi' = 0$$
$$\delta\Pi = 0$$

**The stability theorem**

**Lagrange-Dirichlet Theorem:** *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$$\Pi'' > 0$$
$$\delta^2\Pi(u) > 0$$

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

$$\delta(\Delta\Pi) = 0$$

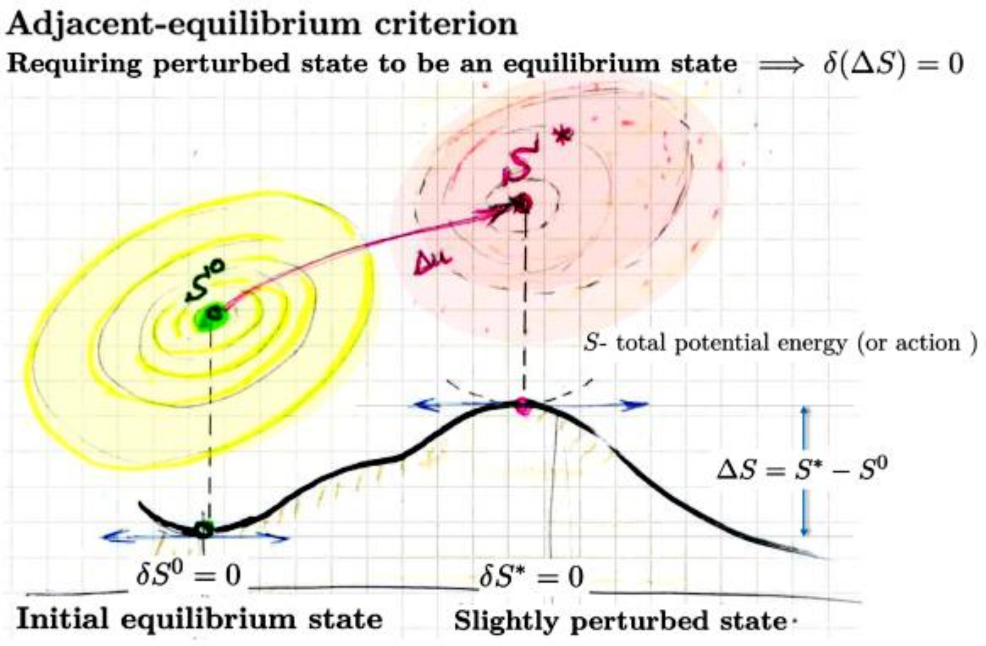
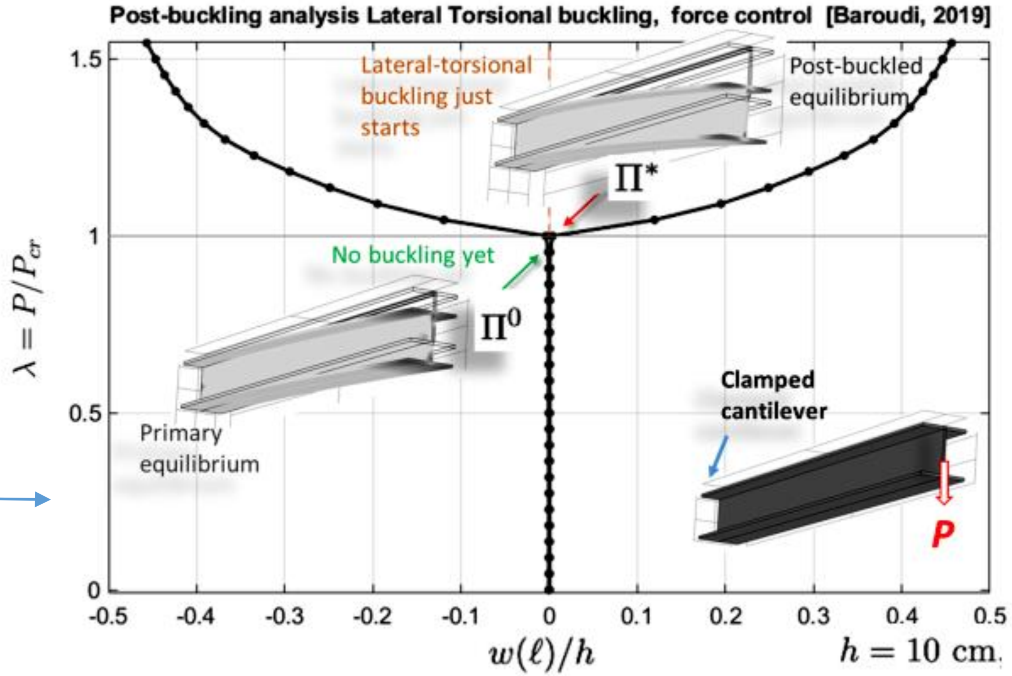



Figure 3.36: Generic illustration of the adjacent-equilibrium criterion in its variational version (up). (Δu is an infinitesimal perturbation corresponds to $w$ on the lowest plot). Finite element post-buckling analysis (Lower figure). Note the initiation of combined lateral and torsional motion of the




For conservative systems we consider stability of static equilibrium

# Stability theorem of **Lagrange-Dirichlet**

$$\Pi' = 0$$
$$\delta\Pi = 0$$

**Lagrange-Dirichlet Theorem:** *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$$\Pi'' > 0 \qquad \delta^2\Pi(u) > 0$$

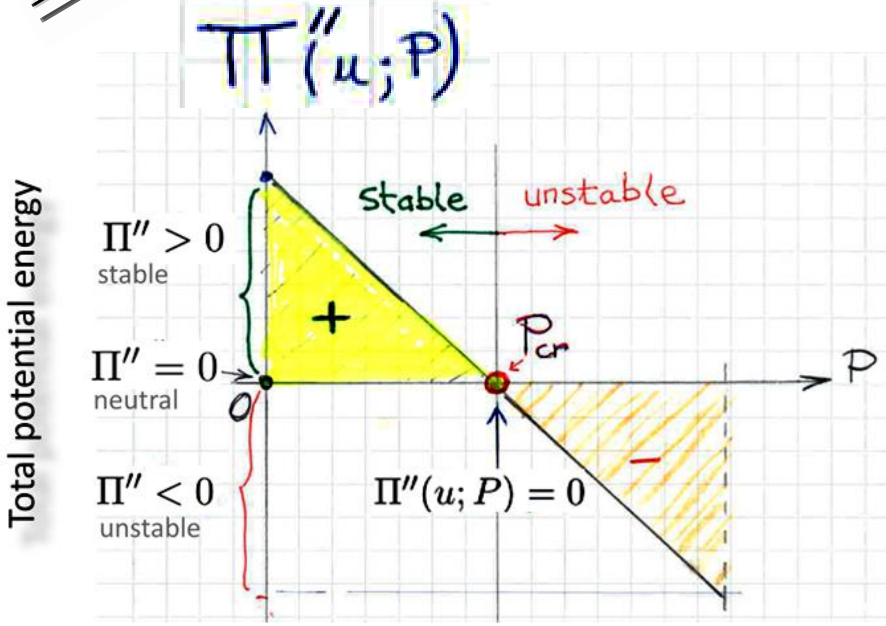
**Trefftz** condition

for stability of an equilibrium:

$$\begin{cases} \delta^2\Pi(u) > 0, & \text{stable,} \\ \delta^2\Pi(u) = 0, & \text{neutral,} \\ \delta^2\Pi(u) < 0, & \text{unstable.} \end{cases}$$

- Is a *global energy criterion for stability*
- will be used systematically to derive the all the equations of stability (loss) we need for all elastic structure s

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$





Lagrange


Dirichlet

*Lagrange-Dirichlet theorem* and investigate the sign of the

increment $\Delta\Pi = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \delta^4\Pi + \ldots$

(More general than Trefftz)

**Trefftz** is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

# Stability theorem of **Lagrange-Dirichlet**

$$\Pi' = 0$$
$$\delta\Pi = 0$$

**Lagrange-Dirichlet Theorem:** *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$$\Pi'' > 0 \qquad \delta^2\Pi(u) > 0$$

$\Longrightarrow$ $\Downarrow$ $\Nearrow$

**Trefftz** condition

for stability of an equilibrium:

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## Buckling or instability or neutral equilibrium condition:

$$\delta\Pi = \underbrace{\delta\Pi_0}_{=0,\text{ initial equilibrium}} + \delta(\Delta\Pi) = 0, \qquad \forall \text{ perturbation } \delta v$$

$$\Longrightarrow \delta(\Delta\Pi) = 0 \quad \text{at buckling,} \qquad \forall\delta v$$

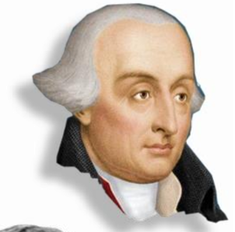
Consequently, the condition

$$\boxed{\text{At buckling} \quad \delta(\Delta\Pi) = 0, \quad \forall\delta v}$$

(More general than Trefftz)

The 3nergy approach permits a geometrical visualization of equilibrium and its stability as topographical maps where it is easy to recognize equilibrium paths and their stability properties

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$



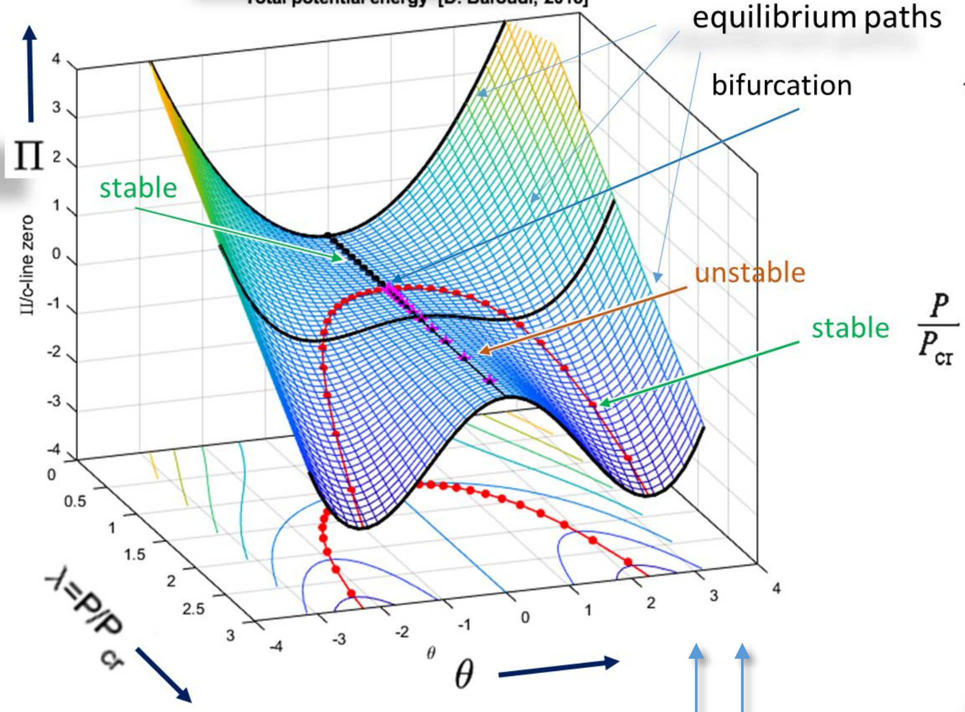
Lagrange

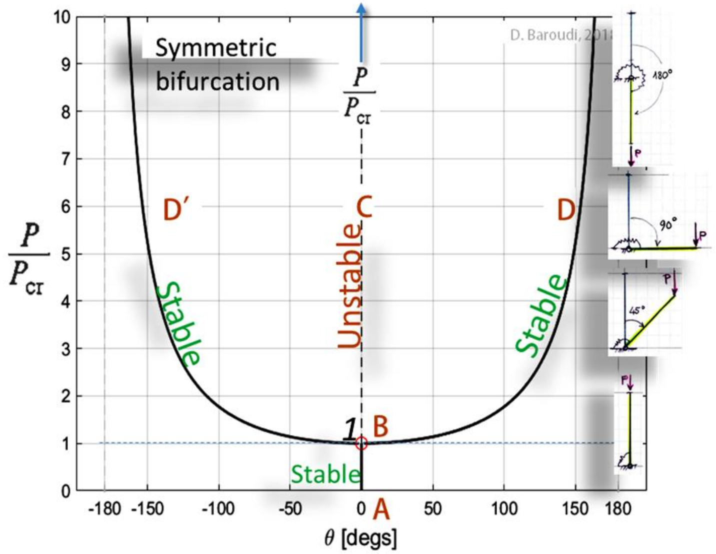
Dirichlet

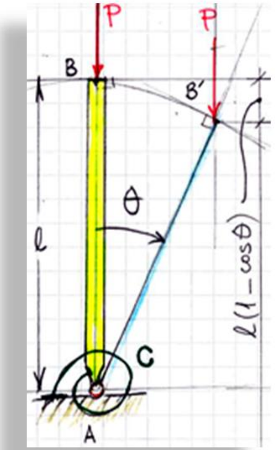
# The geometry of equilibrium and stability

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta)$$

Total potential energy [D. Baroudi, 2018]



$$
\begin{cases}
\Pi'' > 0, & \text{stable,} \\
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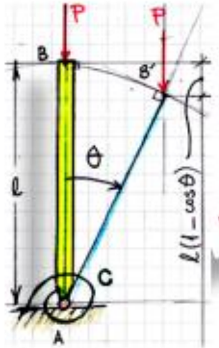
\end{cases}
$$



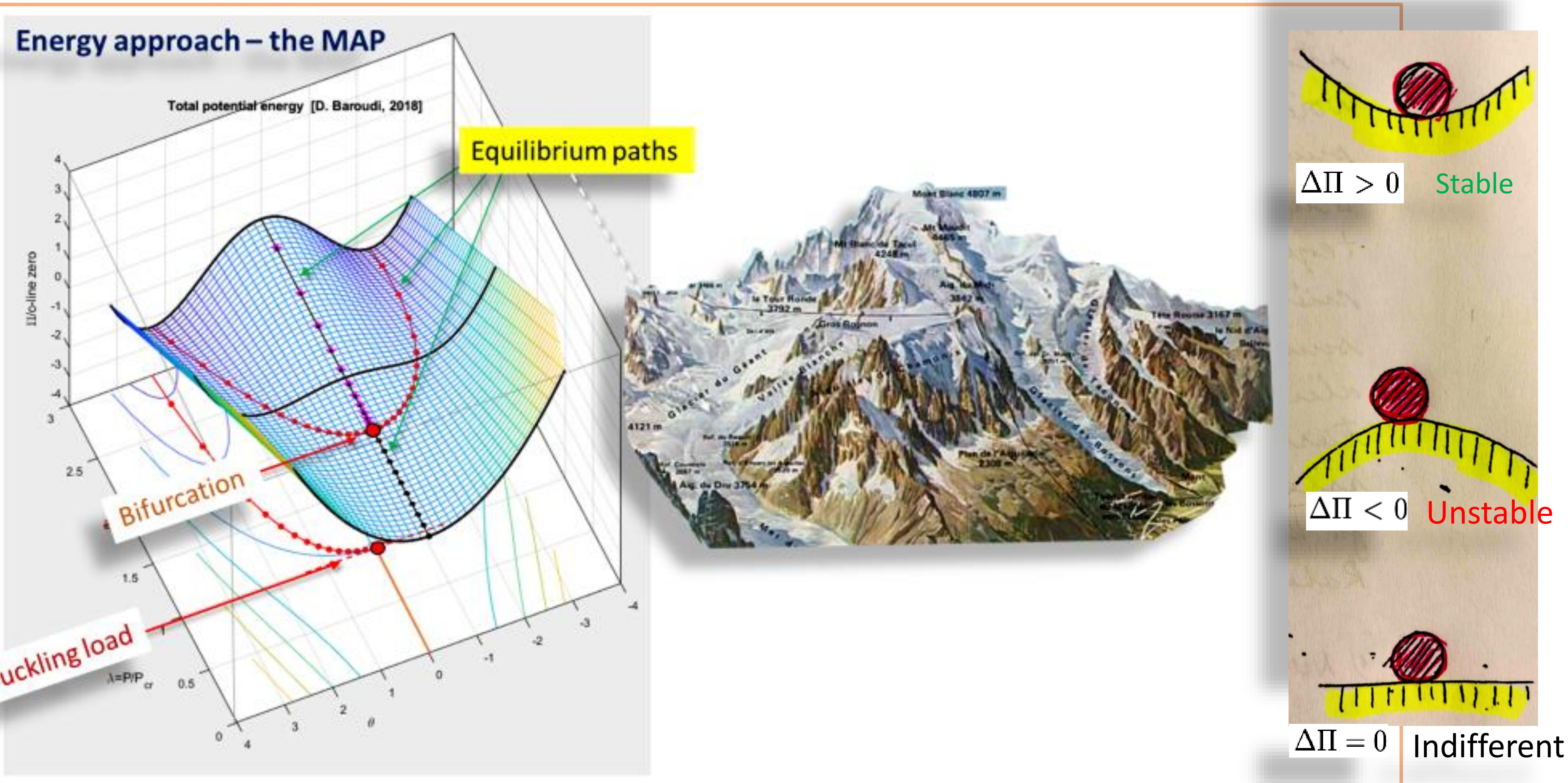




Figure 3.24: *Map* of total potential energy $\Delta\Pi$ of an elastic structure. Equilibrium paths corresponds to locations where $\delta(\Delta\Pi) = 0$ while keeping $P$ constant. Stable equilibrium is achieved there where $\delta^2(\Delta\Pi) > 0$ . Note the analogy with the topographical map of piece of Chamonix (Alpes).

Application of **Lagrange-Dirichlet** Stability theorem

## Testing stability of an equilibrium:

- a static **equilibrium configuration**

+ *plus*

- a **small perturbation**

Perturb & release

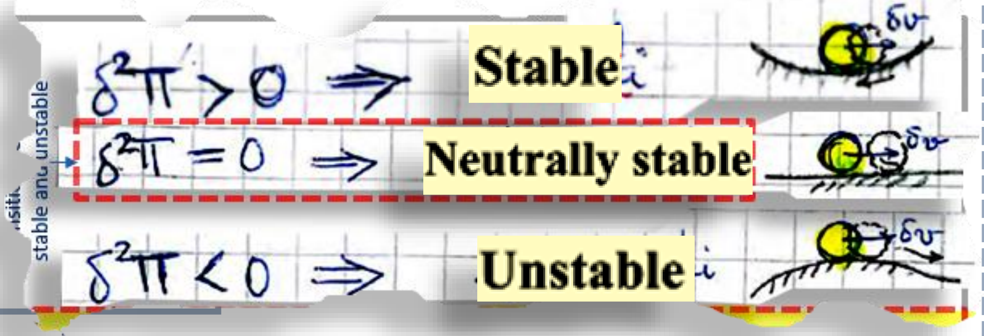
$\delta u$

$\delta u$ -infinitesimal perturbation keeping loading parameter unchanged

$\delta^2\Pi < 0$

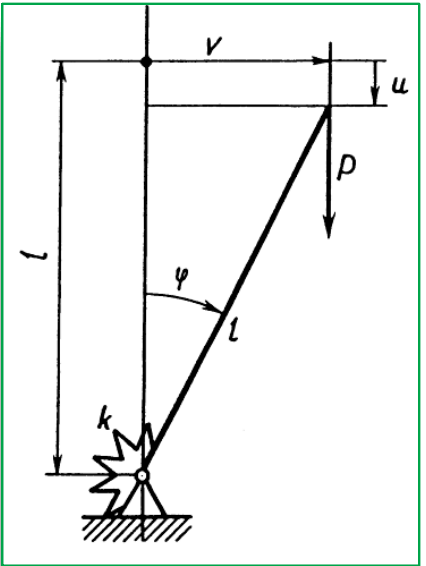
**Equilibrium**? Yes.

But, is it **stable**? No.



# Equilibrium Path for Deformed Systems

Illustration example: the simplest systems with one DOF




Equilibrium equation: $$P l \sin\varphi = k\varphi ,$$

Two independent solutions:

$$\varphi = 0 \quad \text{for any } P;$$ ← primary path (primary equilibrium)

$$P = \frac{k}{l}\,\frac{\varphi}{\sin\varphi}.$$ ← secondary path

Equilibrium paths: are graphs of equilibrium solutions (= **load-displacement curves)**

Primary branches = pre-buckled behaviour

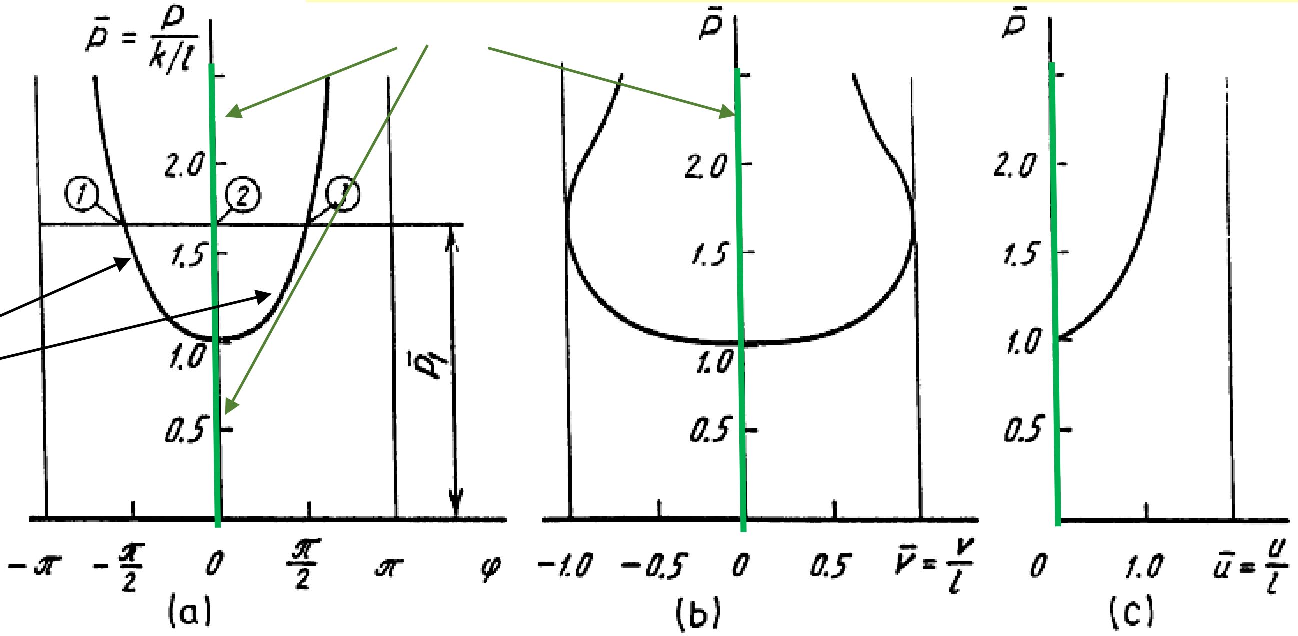
A key behavior of a loaded structure can be studied through:

- measuring or deriving or the relation between the *load* and the *displacement* at some characteristic point(s)
- Graphical representations of such relations

are called equilibrium paths (or equilibrium curves)

Secondary branches

Post-buckled behaviour



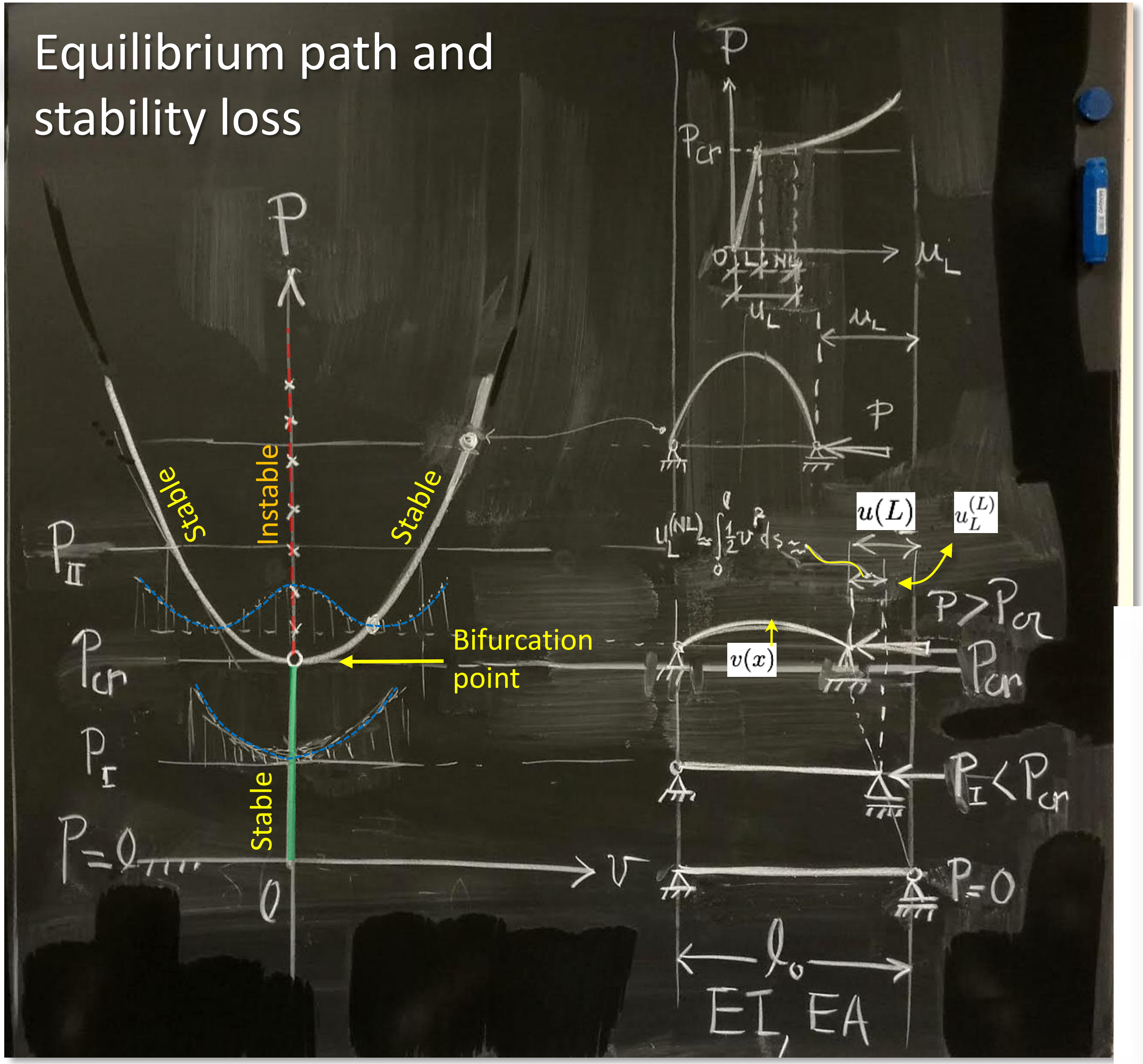

Rotation | Horizontal displacement | **Vertical displacement**

Ref: this example is from:

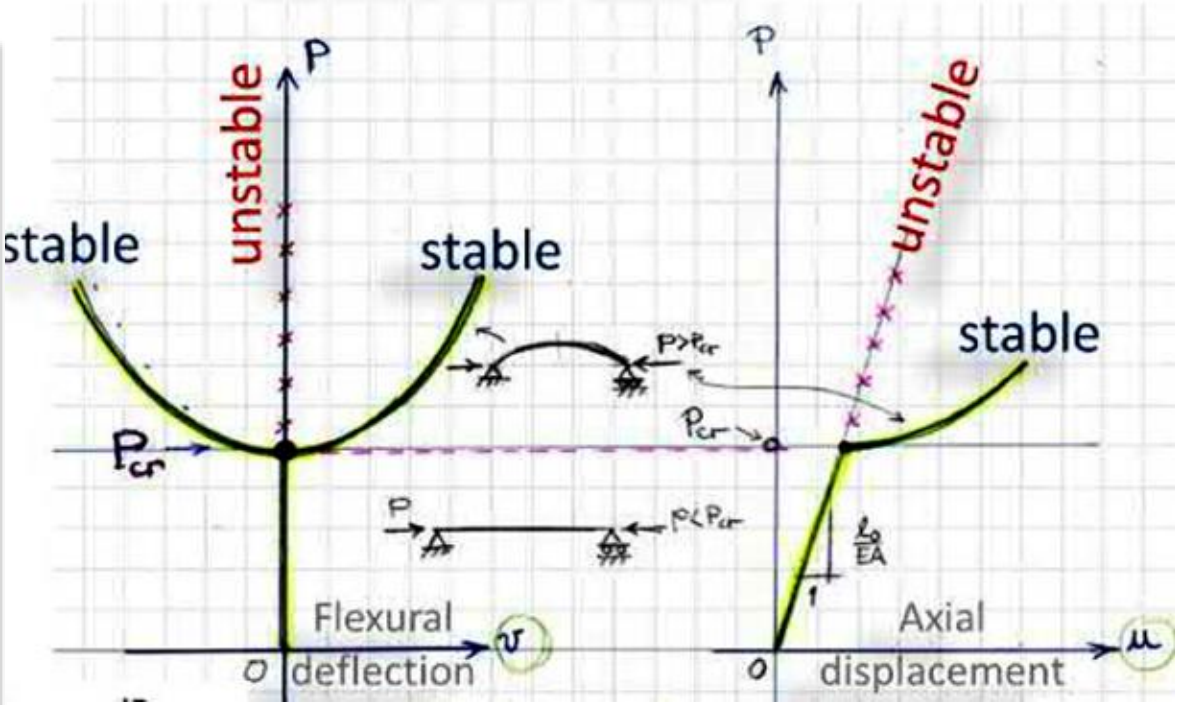
**N. A. Alfutov**

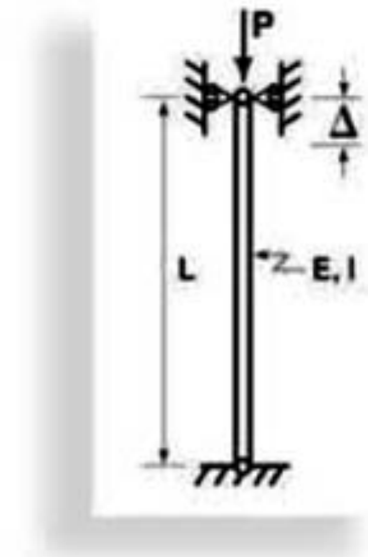
*Stability of Elastic Structures*

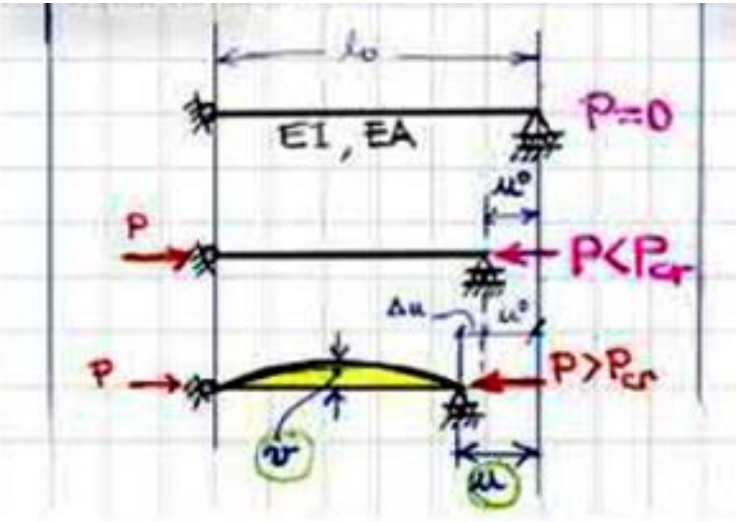
# Equilibrium path and stability loss




## Equilibrium paths

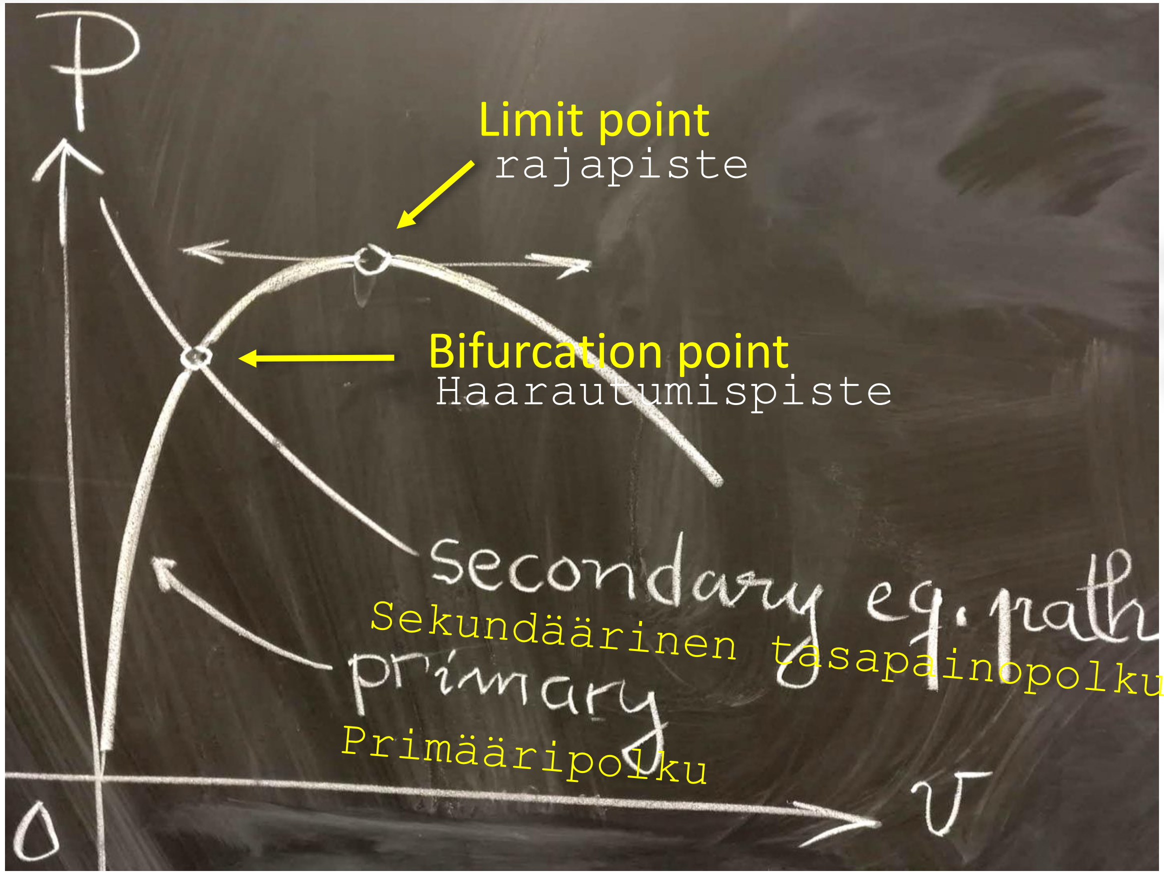


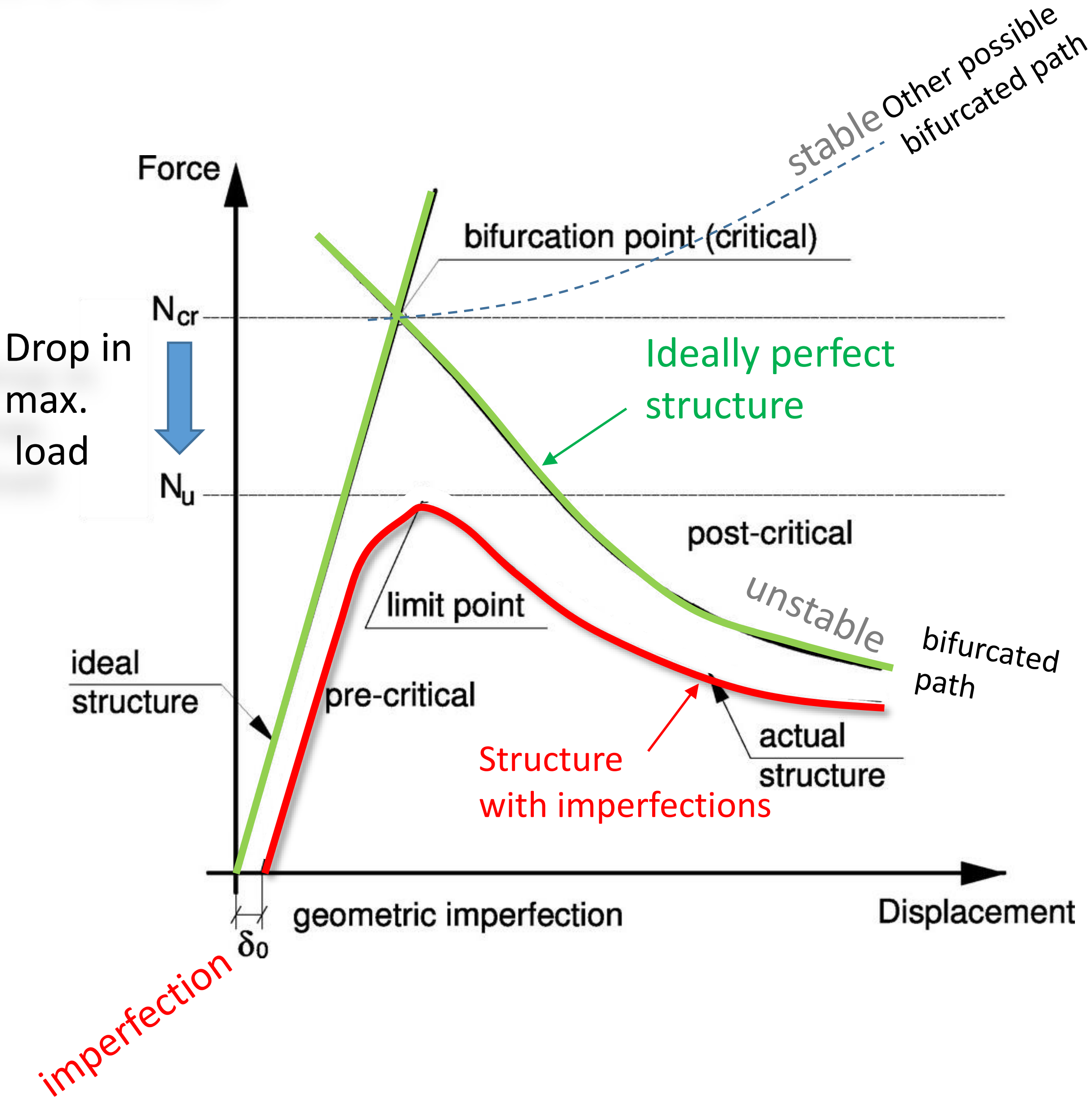



# Critical Equilibrium Points

## Generic equilibrium path




**Critical points**



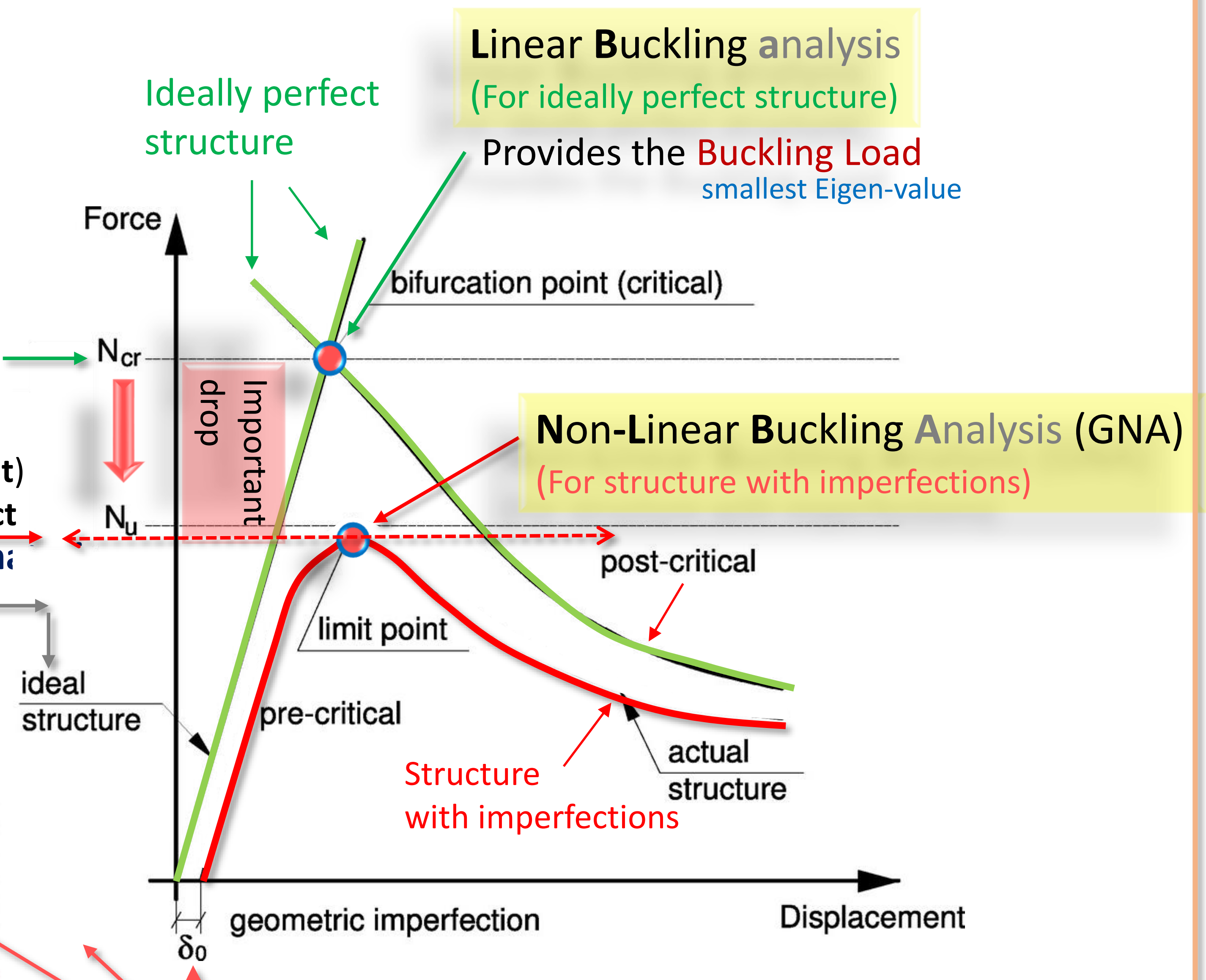
# Linear Buckling Analysis
# Non-Linear Buckling Analysis (GNA)

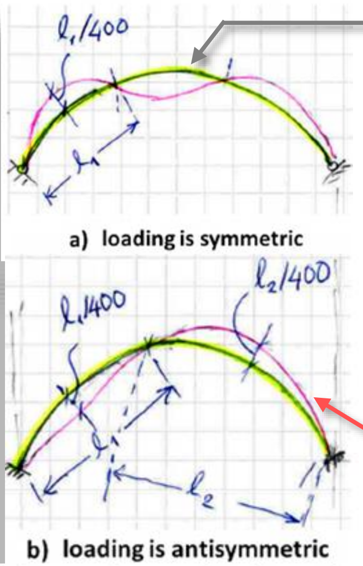
- **Linear buckling** analysis = solving an **eigenvalue problem**. Provides the smallest eigenvalue (**buckling load**) for ideally **perfect** structure

- **A non-linear buckling** analysis = a **geometrically nonlinear analysis** which is <u>not anymore an *eigenvalue-problem*</u> provides **limit-load** (or loosely called buckling load) for the imperfect structure *and the displacement-load curve*

- **How to do GNA?** Non-linear static **+** gradually increasing the loads **+** accounts for initial geometrical imperfections

- In real problems, all other relevant sources of non-linearity can be introduced: **material non-linearity**, contact, etc… **GMNA**

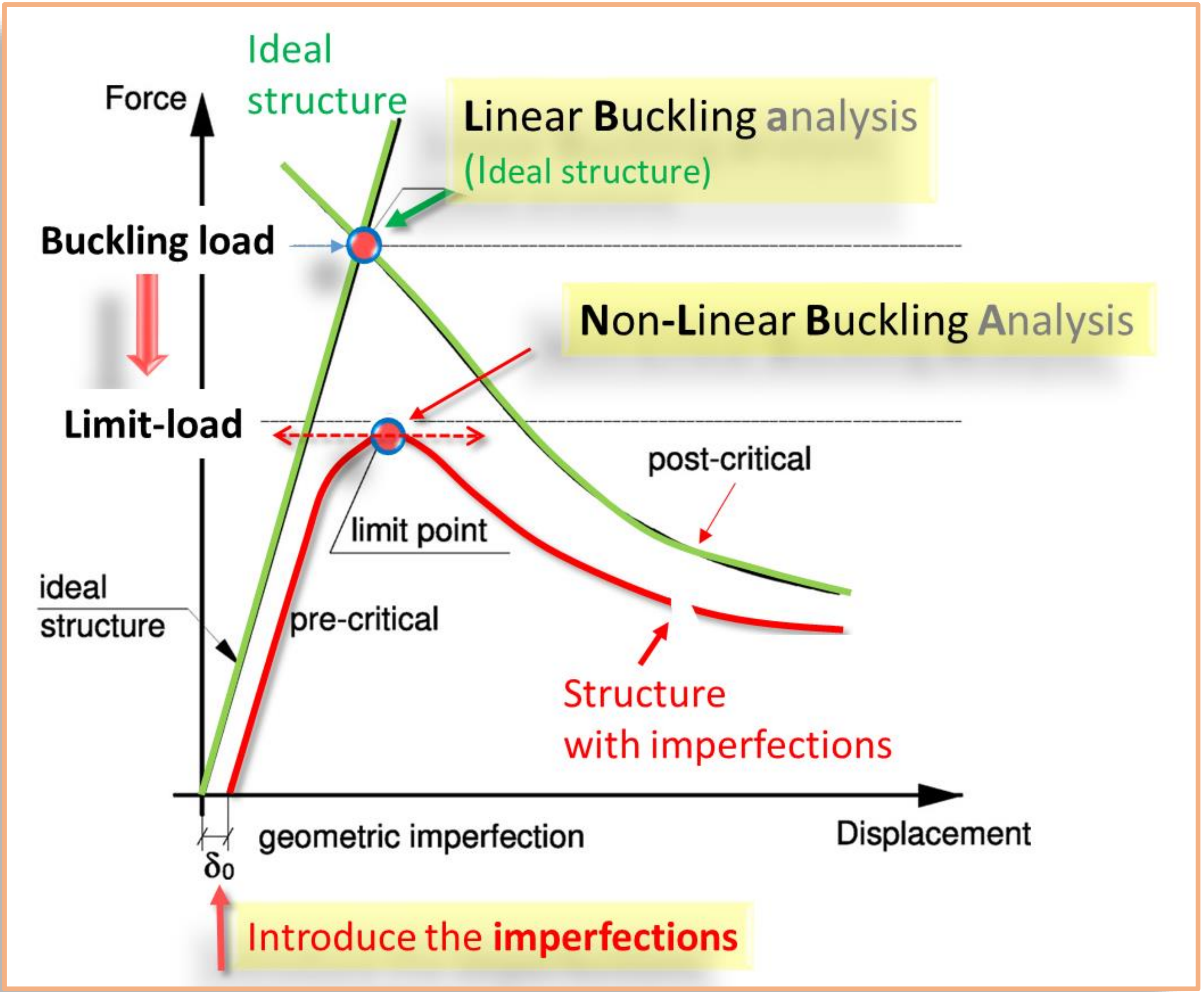



Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Introduce the **imperfections**

GNA = geometrically Non-linear Analysis



Ideal
structure
Force
Linear Buckling analysis
(Ideal structure)
Buckling load
Non-Linear Buckling Analysis
Limit-load
post-critical
limit point
ideal
structure
pre-critical
Structure
with imperfections
geometric imperfection
Displacement
δ0
Introduce the imperfections

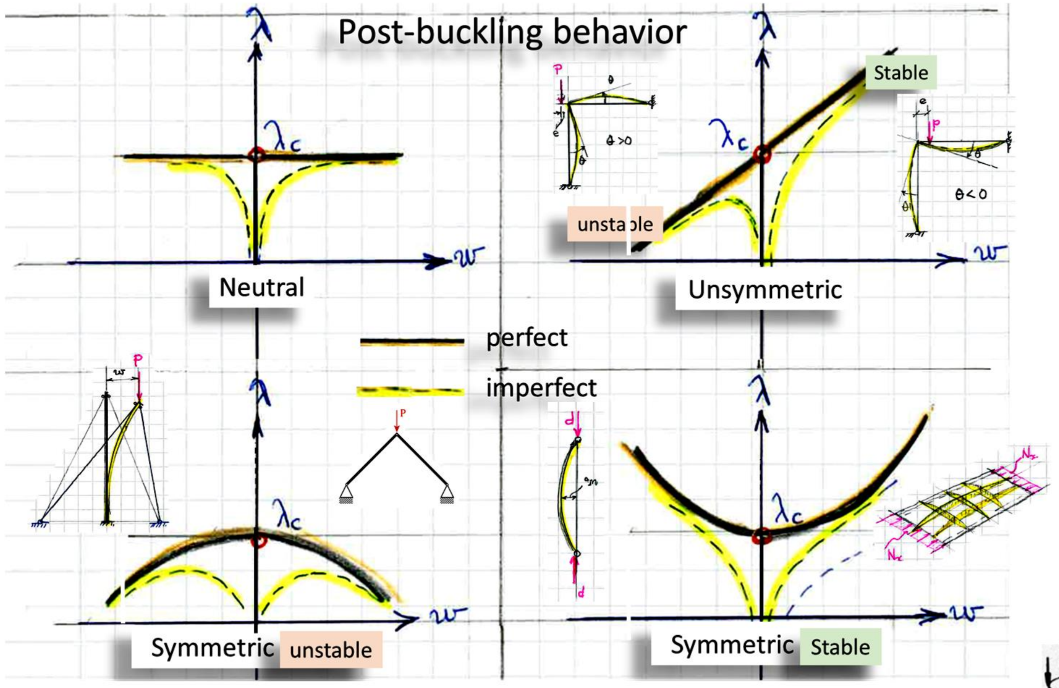
# Types of bifurcational instabilities

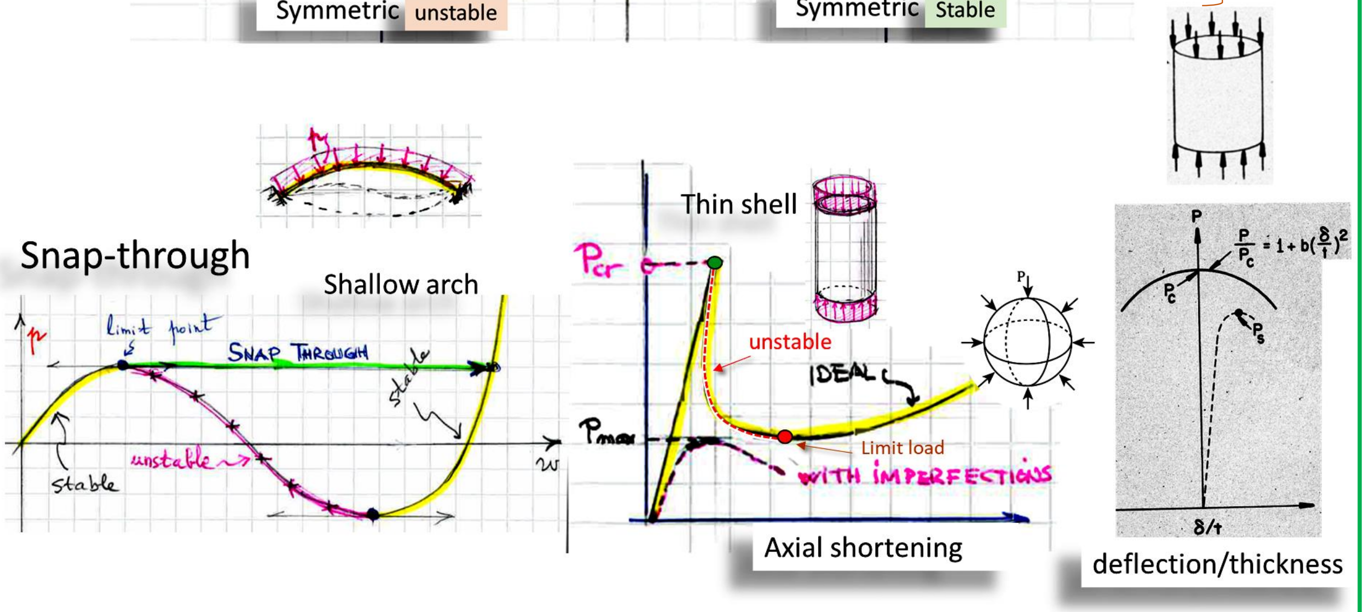
- The nature of post-buckling behavior determines to a large extend safety and the robustness of the structural design

## Basic types of bifurcations Haarautuminen

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*
- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*
- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

- **Snap-through**
  - ✓ Such dynamic behavior is pathological not desired behavior and is locally like an asymmetric branching on equilibrium path




Limit point Rajapiste

Limit point

# Types of bifurcational instabilities

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## Basic types bifurcations

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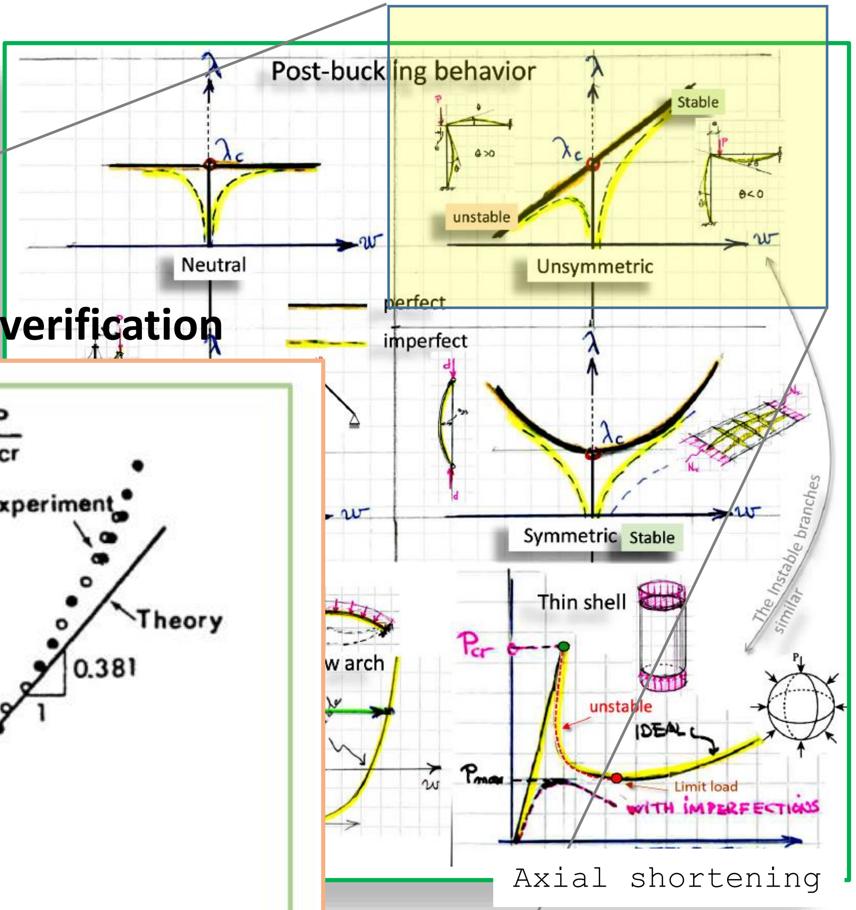
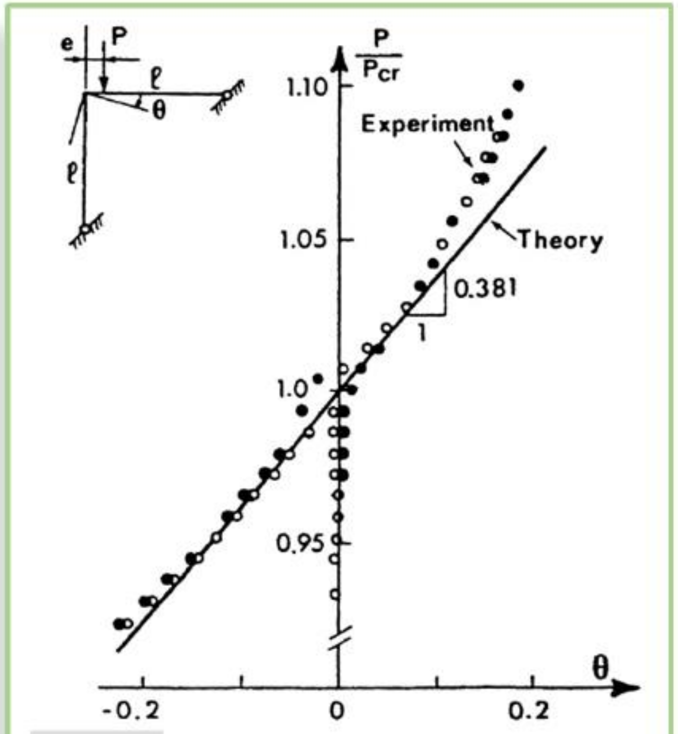



**Roorda's experimental verification**




Roorda's (1971) experimental verification of calculated postcritical response in asymmetric bifurcation of a Γ-frame.

Roorda, 1971, *An experience in equilibrium and stability*, Techn. Note No. 3, Solid Mech. Div., University of Waterloo, Canada.

Bifurcation diagrams are not only theoretical concepts but they really exists

# Example 1: **Buckling of beam-column under self-weight**

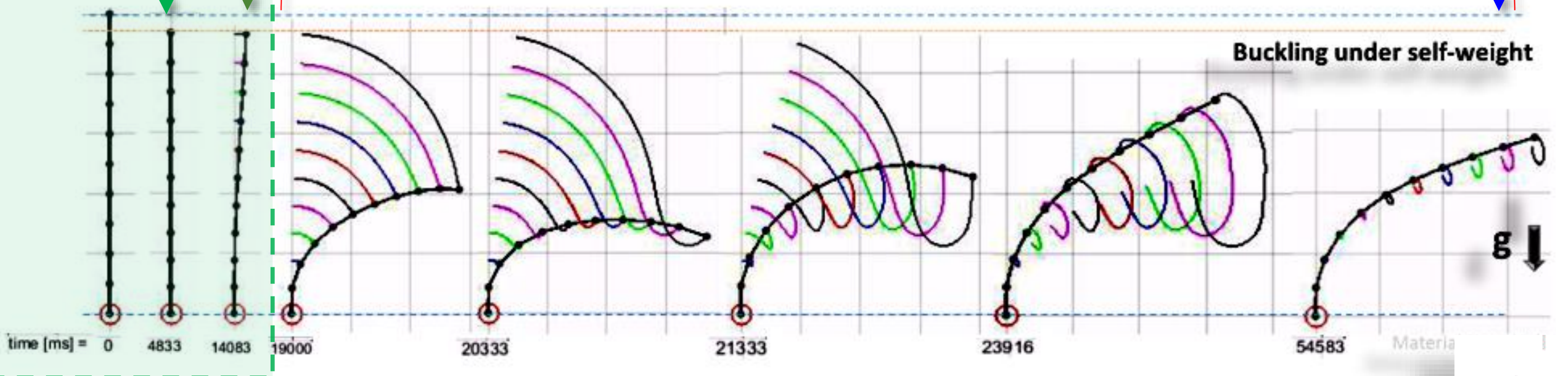
**Dynamic Buckling**

Equilibrium
Straight configuration
*only stretching*

Starts to moves to adjacent dynamic 'equilibrium state'
New flexural configuration
*No more stretching, only flexion, in the beginning*

**Post-buckling behaviour** Both curvature and stretch changes are present

Equilibrium (almost)




**TIME** (ms)

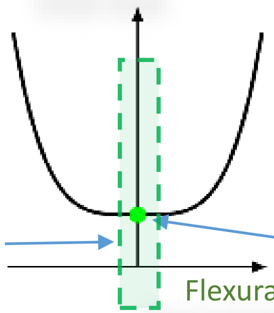
Critical load = $\rho g A L = 7.84 \frac{EI}{L^2}$ (N)

Figure 1.38 – Buckling dynamics of a column under self-weight under a tiny horizontal perturbation. The first configuration is stressless ($t = 0$) then at $t = 0^+$, the column is let to fall freely under its self-weight. The last shown configuration is close to the equilibrium configuration (Fig. 1.36). The colourful lines are trajectories covered during the same laps of time $\Delta t$.

*Ref*: D. Baroudi, 2018 – *full-nonlinear modelling – systematic use of virtual work principle with D'Alembert principle*

- Here one observes '*small*' quasi-static equilibrium configuration change going from straight to adjacent flexural sate (the green box)
- Such small changes are what we study in this course
- The minimum load neads to move from straight to the adjacent (flexural) equilibrium configuration is called buckling (critical) load
- The flexural new equilibrium configuration is called the buckling mode

**Equilibrium path**



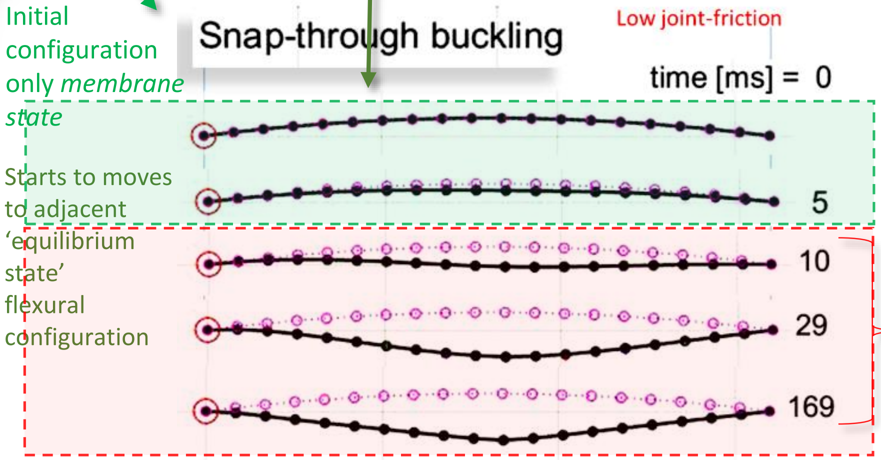
# Example 2: **Snap-through of shallow arch** (symmetric mode)

*Ref*: D. Baroudi, 2018 – *full-nonlinear modelling – dynamics*

**Dynamic Buckling**

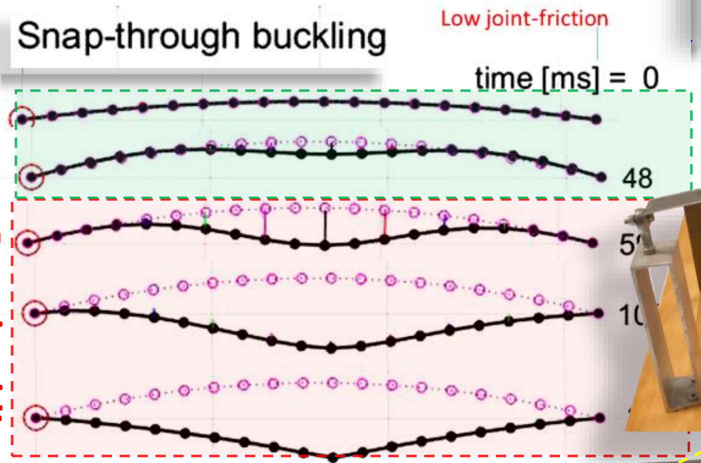
Initial configuration only *membrane state*

Starts to moves to adjacent 'equilibrium state' flexural configuration




TIME (ms)

**Post-buckling** behaviour Both membrane state curvature and



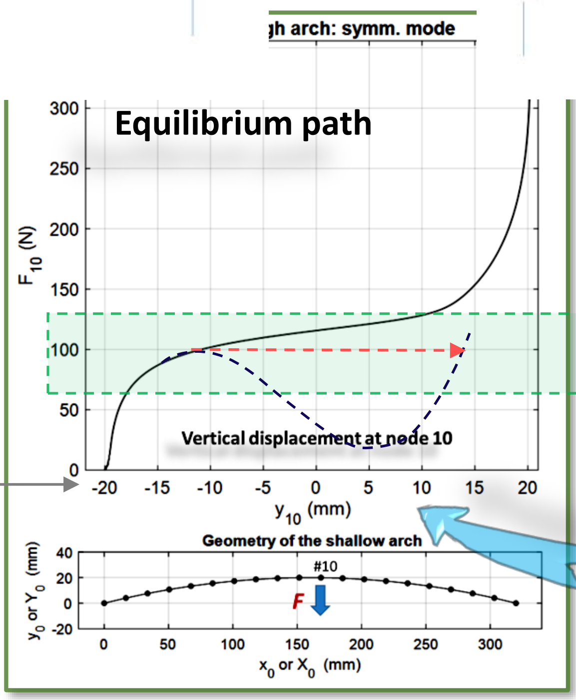


Self-weight and linearly increasing external nodal load at node 10 are acting
In the simulation, the system is to dissipative.

Stable

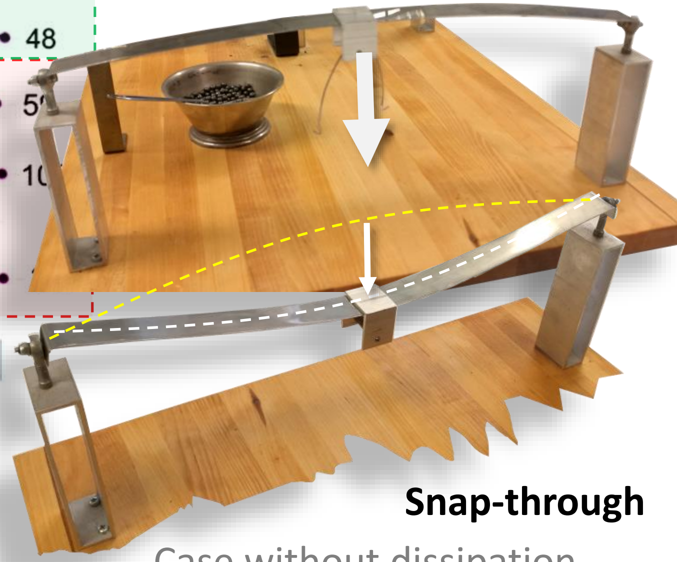
Stable

Painovirhe: These are coordinate of the tip point (not displacements)

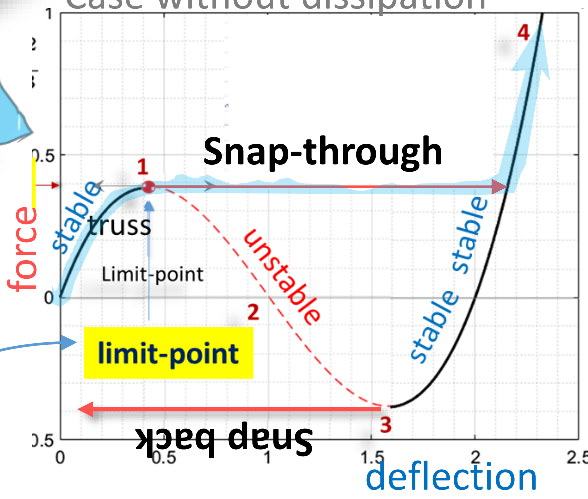



'snap-through' like motion with resisting friction

There is no critical point visible (in this numerical example, I have put too much dissipation; so friction works and resists at joints. So this numerical example is not 100%-elastic as here )




Case without dissipation



# Fundamental questions

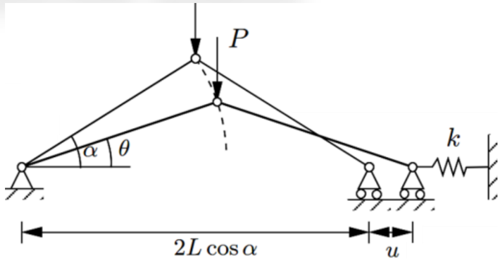
Here the content of this course in four points through questions that will be addressed:

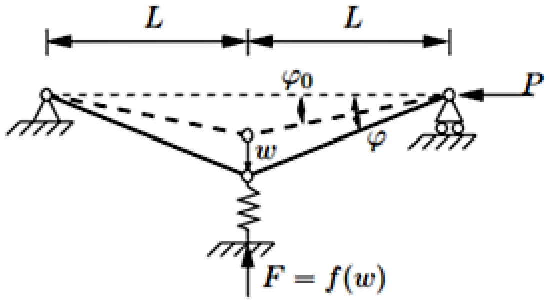
1. can we predict the buckling (critical) load?
2. what happens at the bifurcation (or limit) point?
   (*i.e.*, after the buckling)
3. can we determine the post-critical branches?
   What would be their shape? Nature of stability?
4. what imperfection-sensitive is the structure under study?

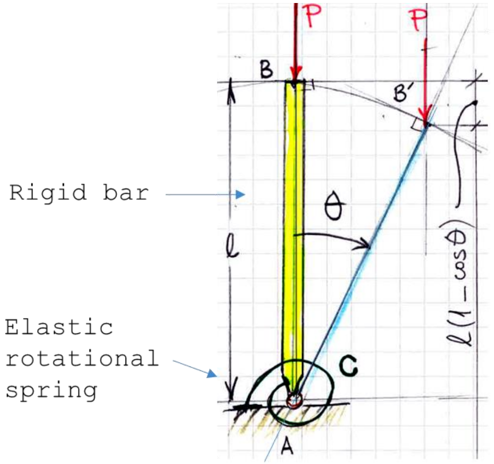
## Equilibrium paths for simple rigid bar systems with springs

- Such simplified models (rigid bars-spring discrete systems) allow to grasp the fundaments or invariants of stability
  - ✓ equilibrium paths
  - ✓ critical points (bifurcation, limit-points)
  - ✓ nature of stability on primary and secondary branches
  - ✓ the principle difference between full-non-linear equations describing the post-critical state and their linearized homogeneous versions in the vicinity of critical points
- With rigid bars, all the strain energy concentrate in the springs (makes the formulation tractable by hand)

Rigid bars-spring discrete systems

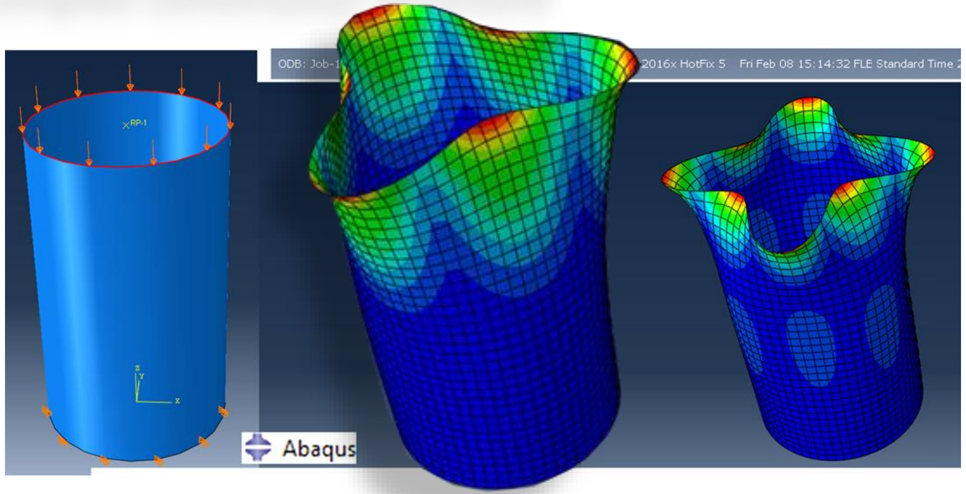


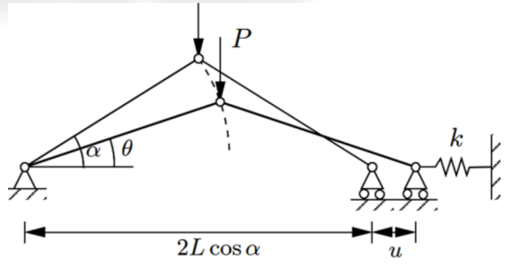

Straight equilibrium configuration (yellow), perturbed (blue).

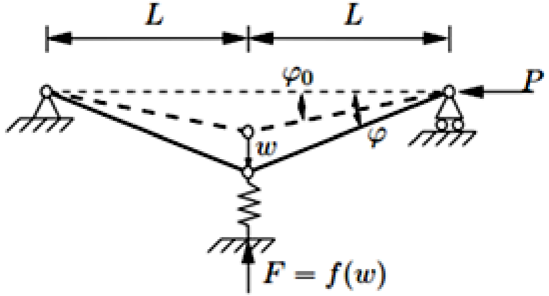
Complex continuous structure

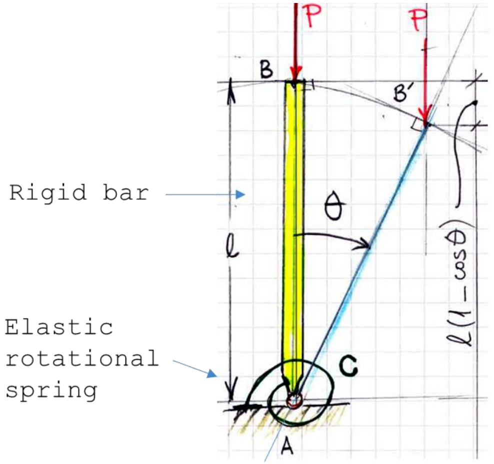


# Equilibrium paths for simple rigid bar systems with springs

Rigid bars-spring discrete systems



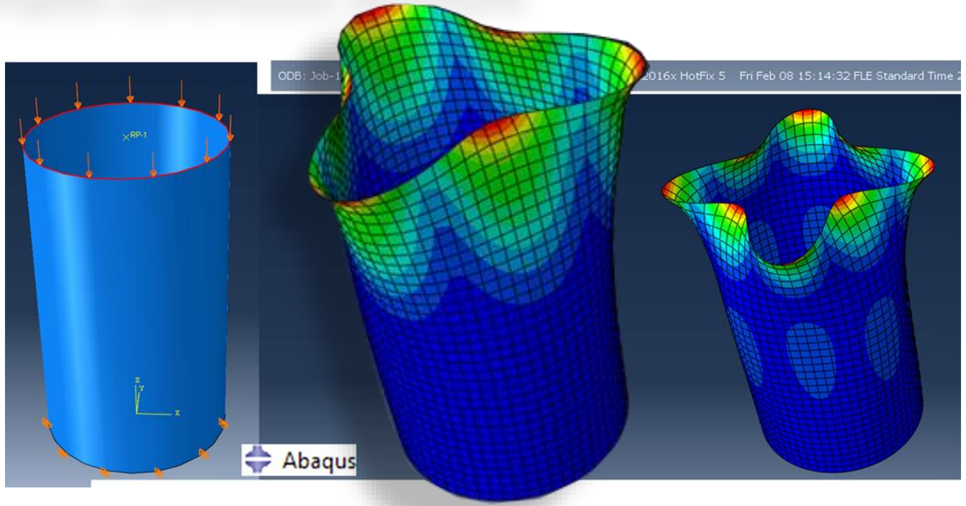




Straight equilibrium configuration (yellow), perturbed (blue).

**Why study rigid-springs systems?** I want to directly study stability of real structures?

- Such simplified models allow to *apprehend* (=grasp) the *key invariant concepts* without being distracted by irrelevant for the task details rising from accounting for more complexity
- These *concepts are general and independent on the level of complexity* of the structure
- Do **not mix** *general concepts* and the *examples* used to illustrate them
- The concept is general and the example is a particular mean for passing the concept

Complex continuous structure



# Equilibrium paths for simple rigid bar systems with springs

## Equilibrium paths

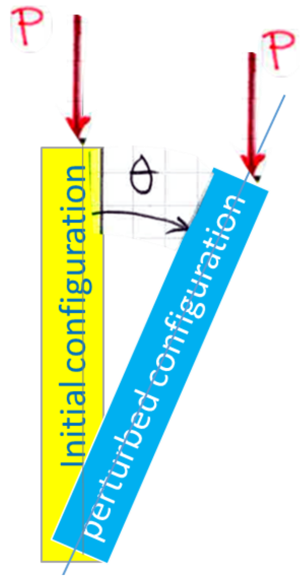
### Stable-symmetric bifurcation

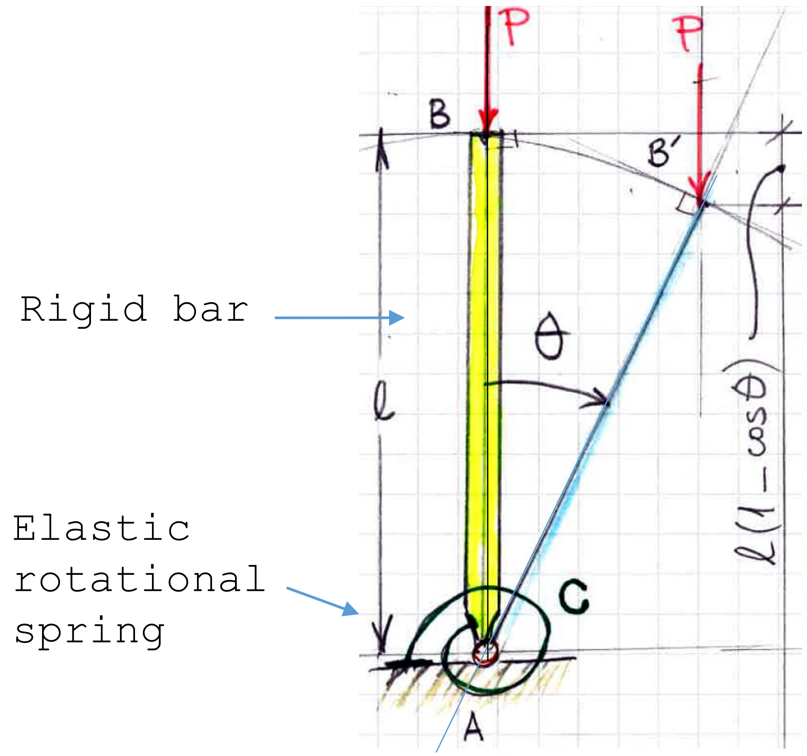
**Tasks**:

- determine all the equilibrium configurations:
  - ✓ Bifurcation (or limit) point
  - ✓ pre-buckling primary or fundamental equilibrium
  - ✓ post-buckling configurations or secondary equilibrium branches

**Increment of total potential energy** between **initial** and **perturbed** equilibrium configurations

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta).$$




Straight equilibrium configuration (yellow), perturbed (blue).

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*
- **Unstable symmetric**
  - ✓ T his gives *imperfection sensitive structures*
- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

# Equilibrium paths

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EIv''^2 dx - P\int_0^\ell \frac{1}{2}v'^2 dx$$

*Compare*: continuous and discrete models

**Increment of total potential energy** $$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta).$$

*Note the non-linearity of this problem*

Equilibrium (stationarity)

$$\delta\Pi = 0 \implies \frac{d\Pi}{d\theta} \equiv \Pi' = c\theta - P\ell\sin\theta = 0.$$

solutions

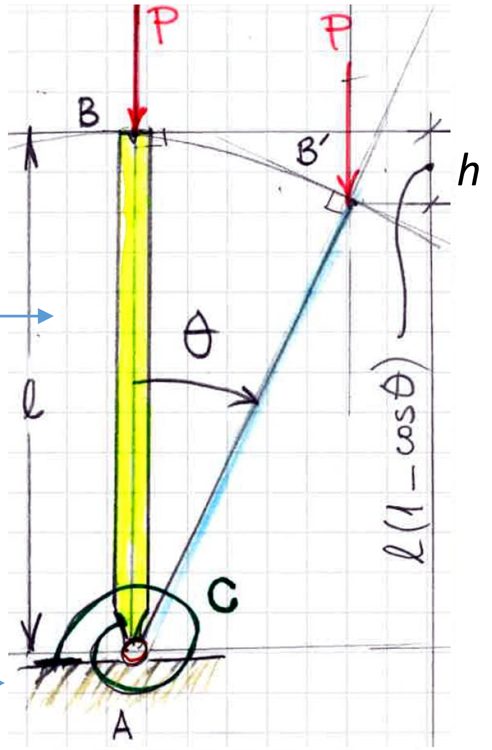
$$\theta = 0 \qquad \text{or} \qquad P = \frac{c}{\ell}\cdot\frac{\theta}{\sin\theta},\ \theta \neq 0.$$

fundamental equilibrium branch
(= pre-buckling branch)

post-buckling equilibrium branch

Rigid bar

Elastic rotational spring




Straight equilibrium configuration (yellow), perturbed (blue).

notation $\frac{c}{\ell} \equiv P_{cr}$

# Equilibrium paths

full non-linear equilibrium equation

**Increment of total potential energy**

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta).$$

*Note the non-linearity of this problem*

Equilibrium (stationarity)

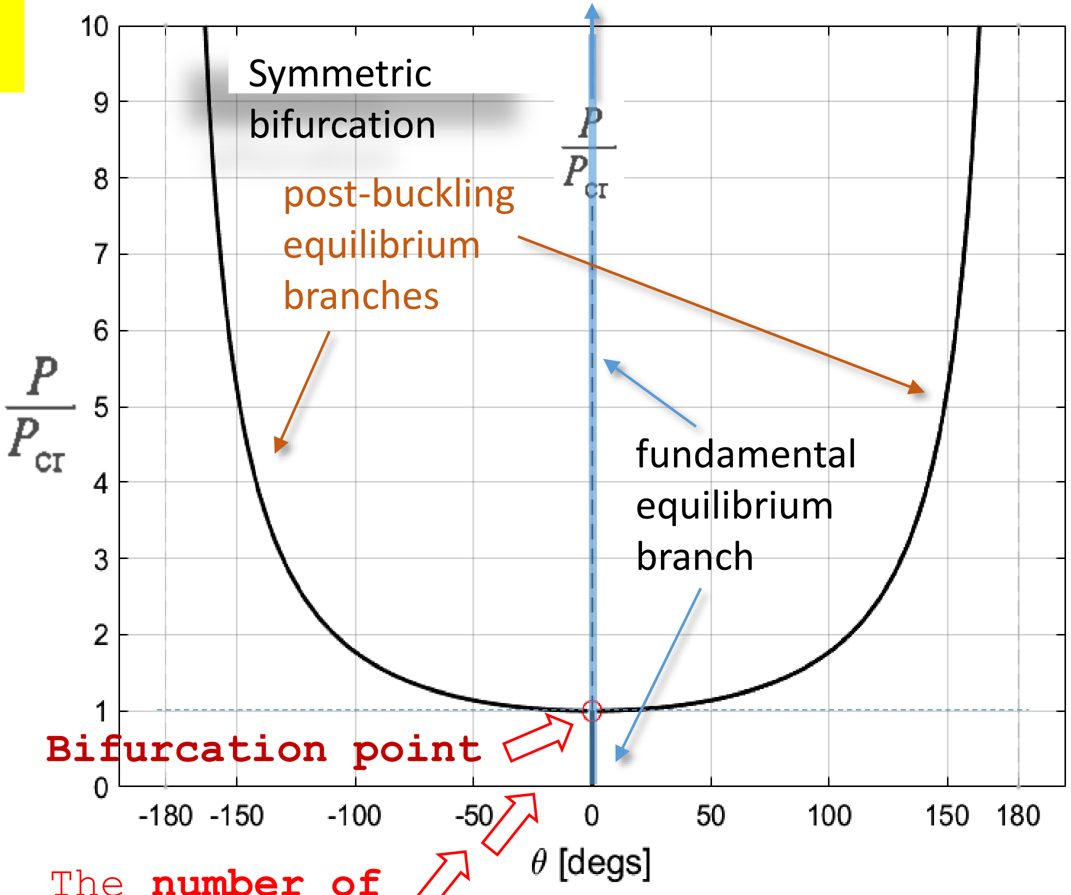
$$\delta\Pi = 0 \implies \frac{d\Pi}{d\theta} \equiv \Pi' = c\theta - P\ell\sin\theta = 0.$$

Equilibrium solutions

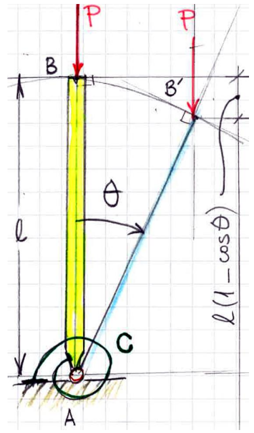
$$\theta = 0 \qquad \text{or} \qquad P = \frac{c}{\ell}\cdot\frac{\theta}{\sin\theta},\ \theta \neq 0.$$

fundamental equilibrium branch or pre-buckled branch

post-buckling equilibrium branch




The **number of solutions changes** from one (primary) to two (secondary branch)




notation $\frac{c}{\ell} \equiv P_{cr}$

The buckling load (bifurcation point) and the nature of stability of all branches will be discussed in next slides …

$$\text{sign } \Pi''|_{\theta=0}? \qquad \text{sign } \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$$

# Stability of Equilibrium

**Lagrange-Dirichlet Theorem**: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$\text{sign}\ \Pi''|_{\theta=0}?$

$\text{sign}\ \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$

$$\Downarrow$$

$$\begin{cases} \Pi'' > 0, & \text{stable}, \\ \Pi'' = 0, & \text{neutral}, \\ \Pi'' < 0, & \text{unstable}. \end{cases} \Longrightarrow$$

**Stability of equilibrium**

Let's study the sign of the second derivative $\Pi''(\theta) = c - P\ell\cos\theta$ (Eq. 1.31)

- **trivial case:** $\theta = 0$ (branches $AB$ and $BC$)
  1.1) $\Pi''(\theta = 0) = c - P\ell = 0 \implies P_{cr} = c/\ell$ (we have a *bifurcation point* at $B$ since the second derivative $\Pi''$ changes sign) ($P_{cr}$ is called the **buckling load**).
- 1.2) $AB$, $\theta = 0$: $\Pi''(0) = c - P\ell > 0 \implies P < P_{cr}$ ($AB$ *stable*)
- 1.3) $BC$, $\theta = 0$: $\Pi''(0) = c - P\ell < 0 \implies P > P_{cr}$ ($BC$ *unstable*)

- **post-bucked case:** $\theta \neq 0$ (branches $BD$ and $BD'$)
  $\rightarrow$ so, the second derivative should be evaluated on the secondary branch $P = \frac{c}{\ell}\cdot\theta/\sin\theta$:

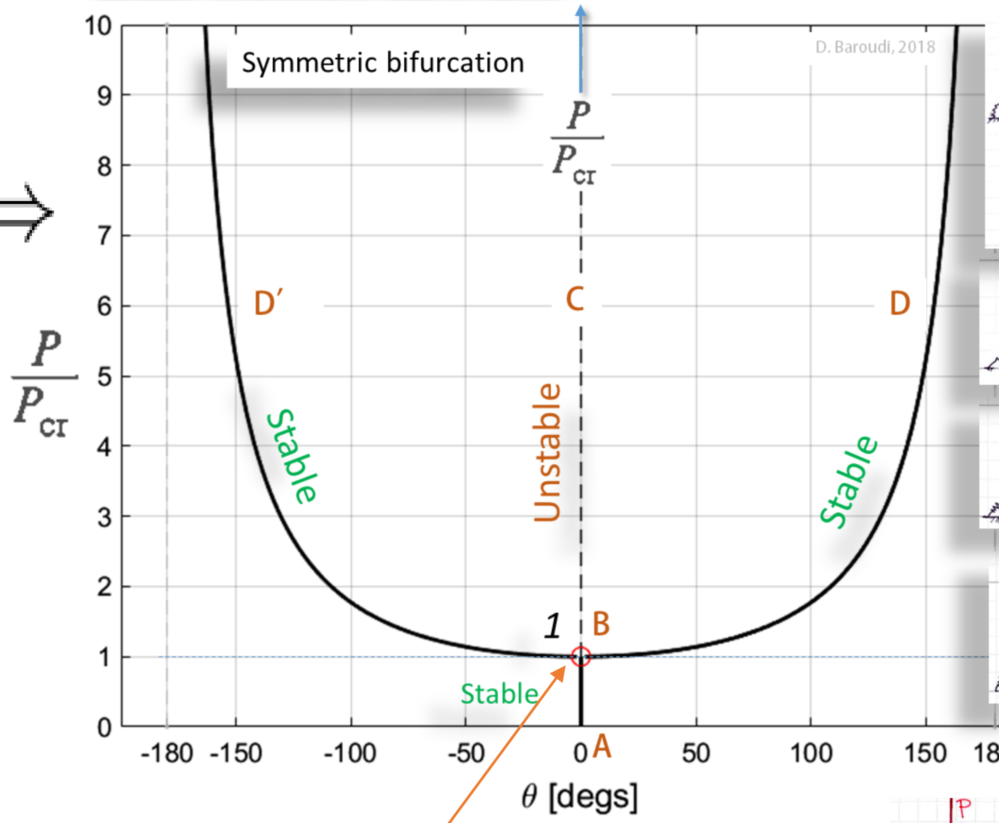
  Eq. (1.31) $\rightarrow$ $\Pi''(\theta) = c - P\ell\cos\theta > 0? \implies$ the sign of $\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$, so we have always $\Pi'' > 0$ for $\theta[\in 0, \pi]$ (see Fig. 1.31)

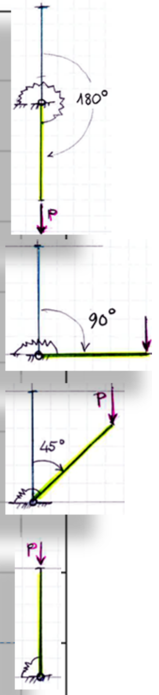
  So $\rightarrow$ $BD$ and $BD'$ *stable*).

$$\Pi'' > 0, \quad \text{stable},$$

Equilibrium paths

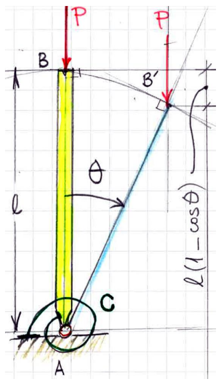
Stable-symmetric bifurcation



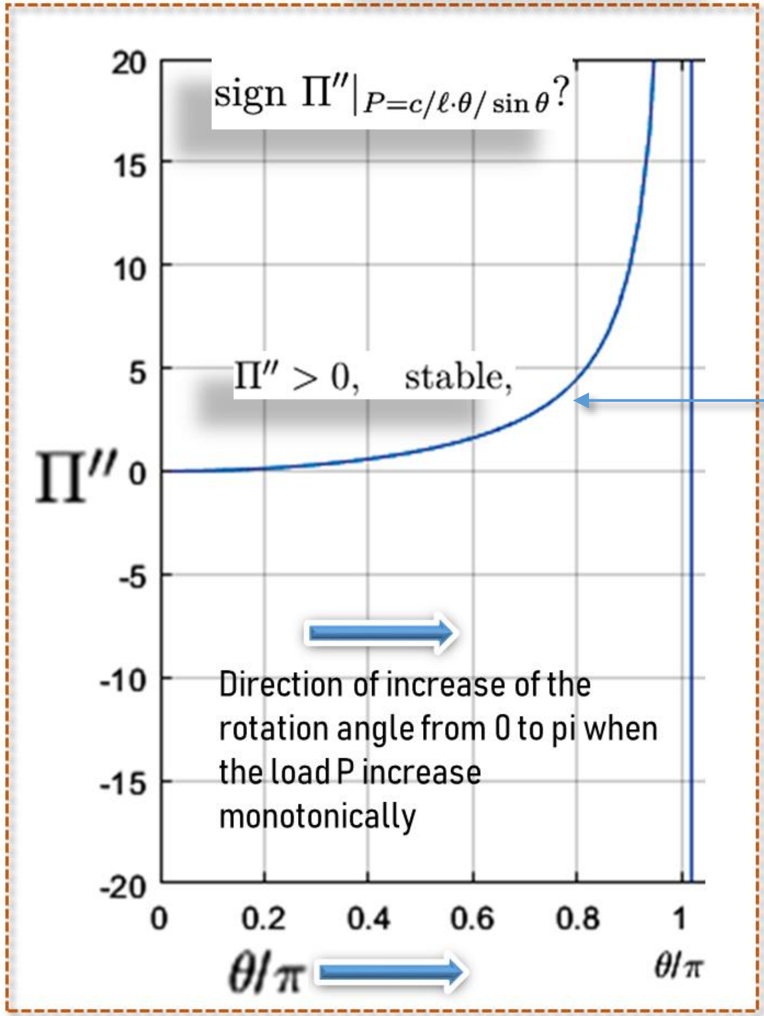

**Bifurcation point**

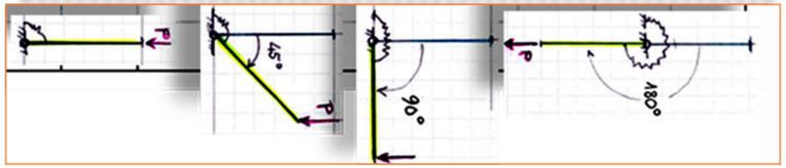
$$\text{notation} \quad \frac{c}{\ell} \equiv P_{cr}$$



$$\Pi''(\theta) = c - P\ell\cos\theta > 0$$

Stability of Equilibrium on the secondary path: sign $\Pi''|_{P=c/\ell\cdot\theta/\sin\theta}$?




$\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$

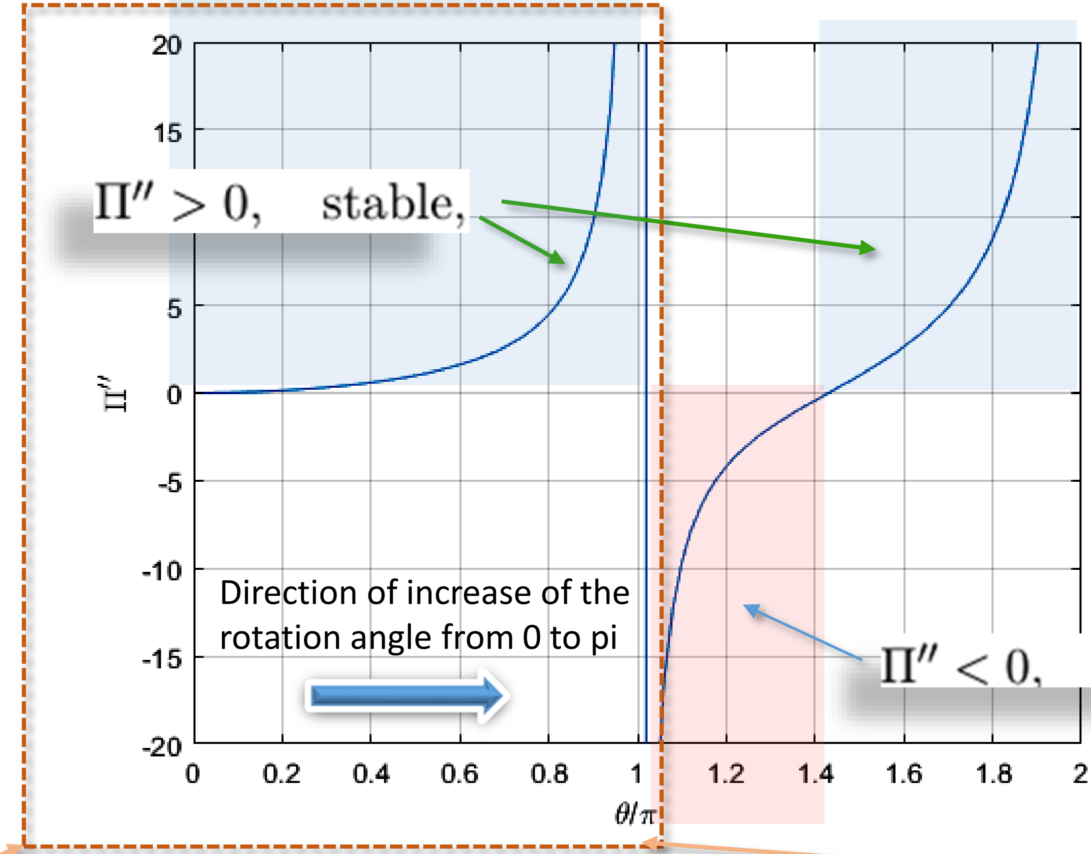
- **post-bucked case:** $\theta \neq 0$ (branches $BD$ and $BD'$)
  → so, the second derivative should be evaluated on the secondary branch $P = \frac{c}{\ell}\cdot\theta/\sin\theta$:

  Eq. (1.31) → $\Pi''(\theta) = c - P\ell\cos\theta > 0$? $\implies$ the sign of $\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$, so we have always $\Pi'' > 0$ for $\theta[\in 0, \pi]$ (see Fig. 1.31)

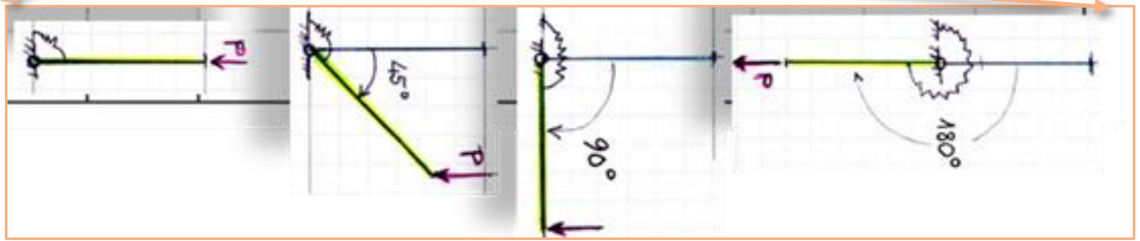
  So → $BD$ and $BD'$ *stable*).

# Stability of Equilibrium

Stability of Equilibrium on the secondary path: $\text{sign}\; \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}$?

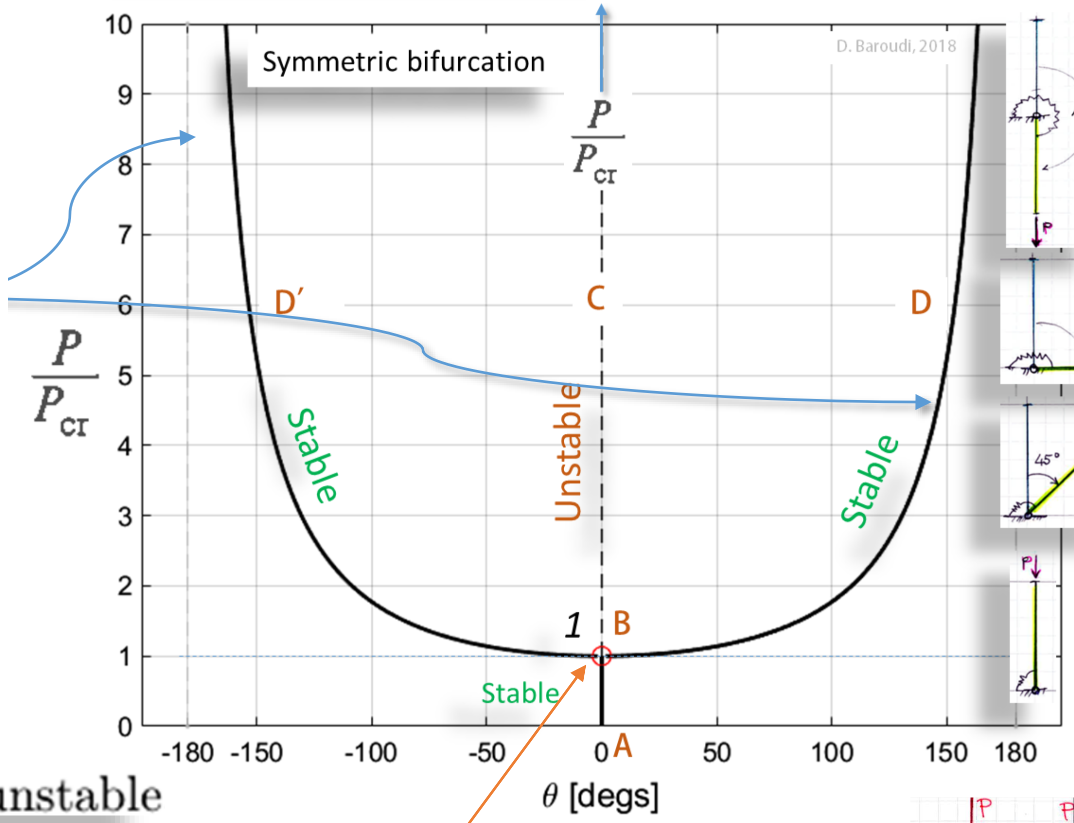


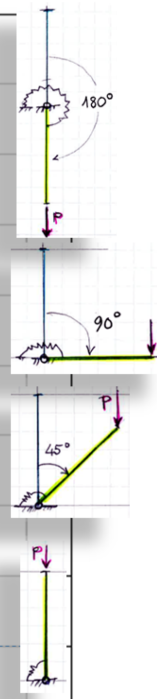

$\Pi'' < 0, \quad \text{unstable}$



Equilibrium paths

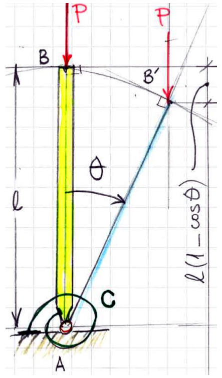
Stable-symmetric bifurcation




**Bifurcation point**

notation $\frac{c}{\ell} \equiv P_{cr}$



# Equilibrium paths for simple rigid bar systems with springs

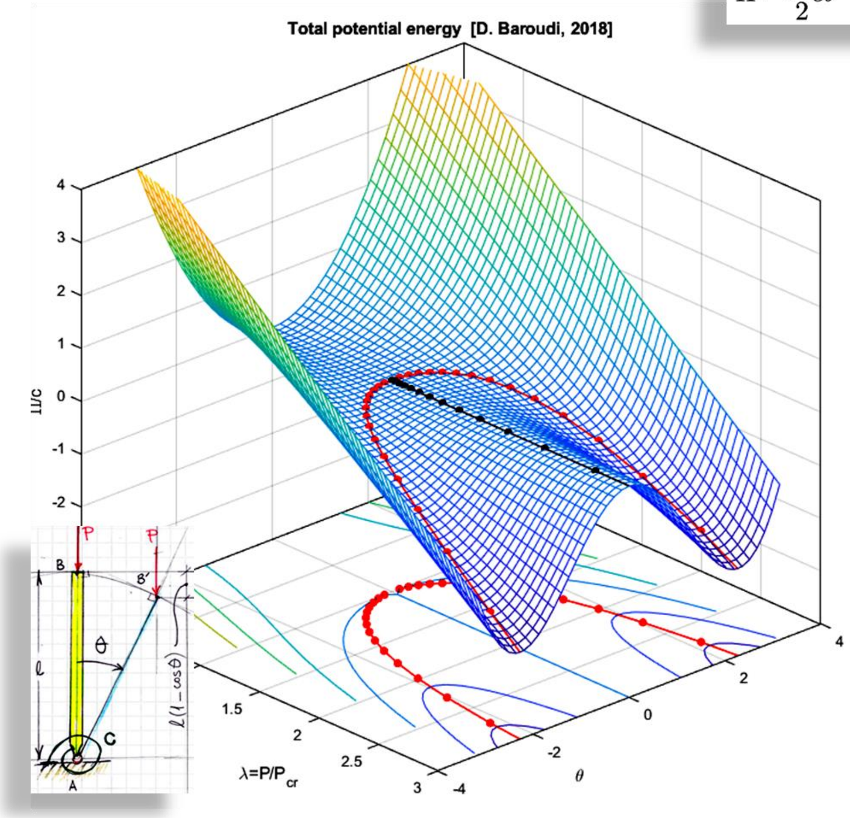
## Equilibrium paths

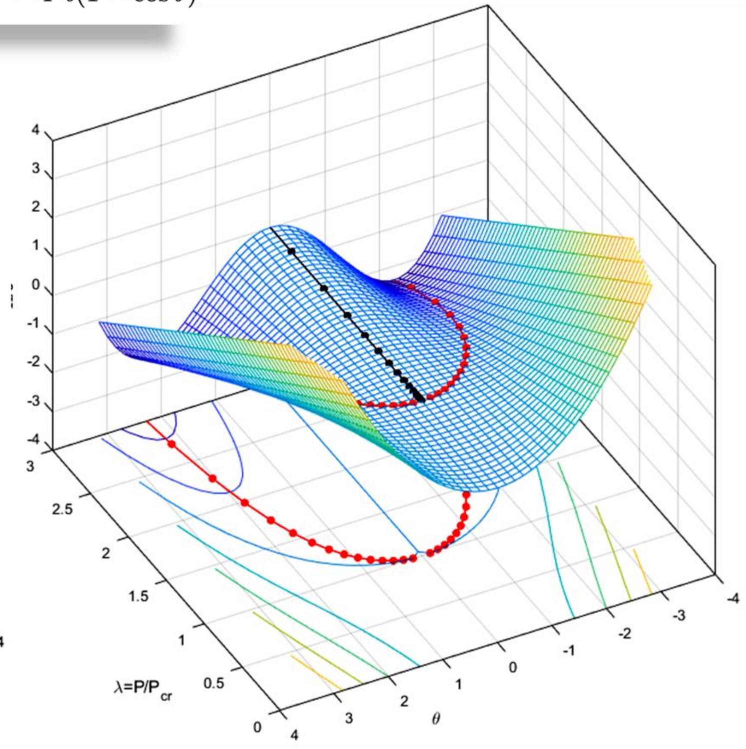
### Stable-symmetric bifurcation

**Increment of total potential energy**

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta).$$

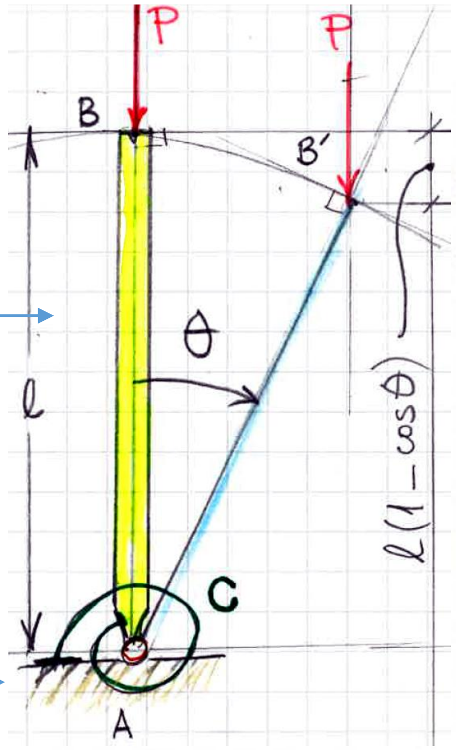
$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta)$$




Rigid bar

Elastic rotational spring

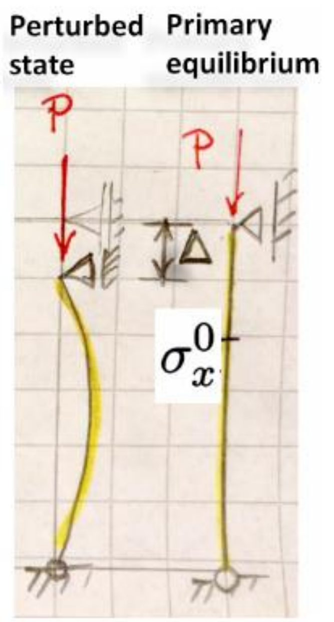



Straight equilibrium configuration (yellow), perturbed (blue).

# Simple examples of a Linear buckling analysis

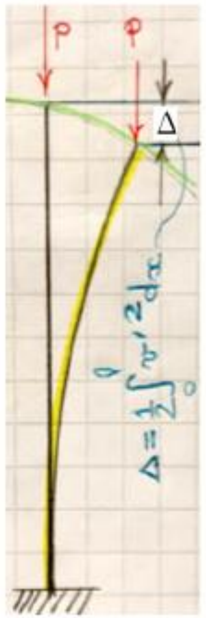
**Determine**

- **the buckling load**
- buckling modes




Additional work $\Delta W_{ext} = P \cdot \Delta$ (*Flexural buckling*)

$\Delta\Pi$




Tip displacement increment after buckling.

Example of use of stability criteria in the form $\delta(\Delta\Pi) = 0$

assuming moderate rotations of bars $(\cos\theta_i \approx 1 - \theta_i^2/2)$

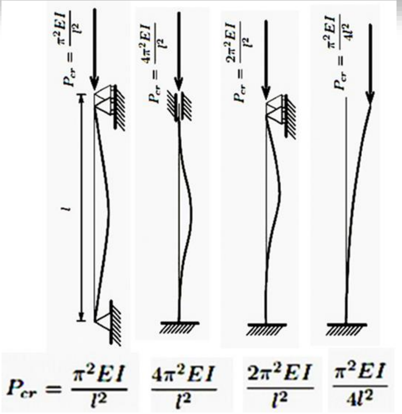
$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x - P\underbrace{\int_0^\ell [\frac{1}{2}(v')^2]\,\mathrm{d}x}_{\Delta},$$

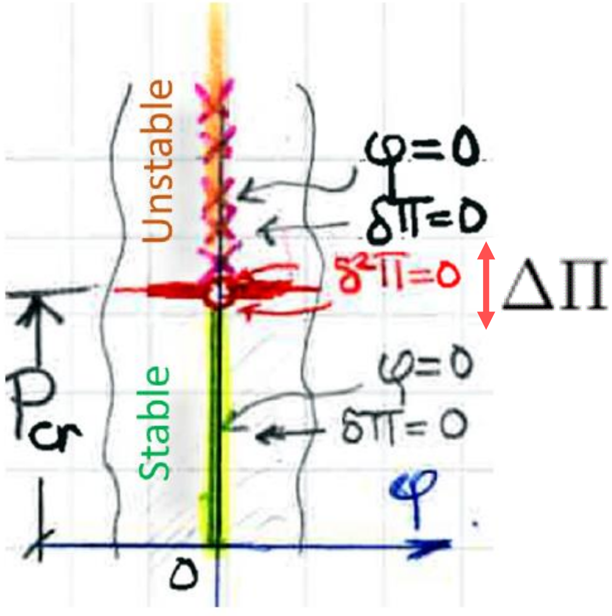
$$\delta(\Delta\Pi[v]) = 0, \forall\delta u \quad \Longrightarrow \quad \delta\left(\tfrac{1}{2}\int_0^\ell EIv''^2\mathrm{d}x - P\int_0^\ell \tfrac{1}{2}v'^2\mathrm{d}x\right) = 0,$$

$$= \int_0^\ell EIv''\delta v''\mathrm{d}x - P\int_0^\ell v'\delta v'\mathrm{d}x = 0$$

**Results in the eigenvalue problem:**

$$(EIv'')'' + Pv'' = 0 \quad \& \quad 4 \quad \mathrm{BC}$$



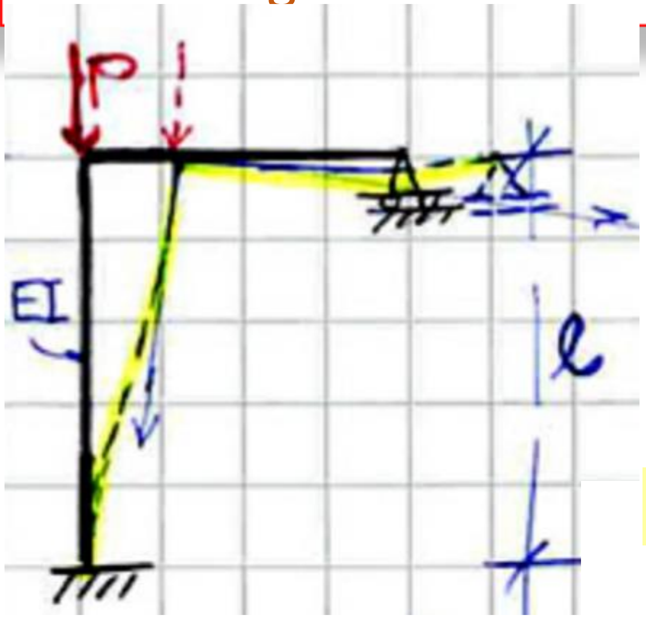



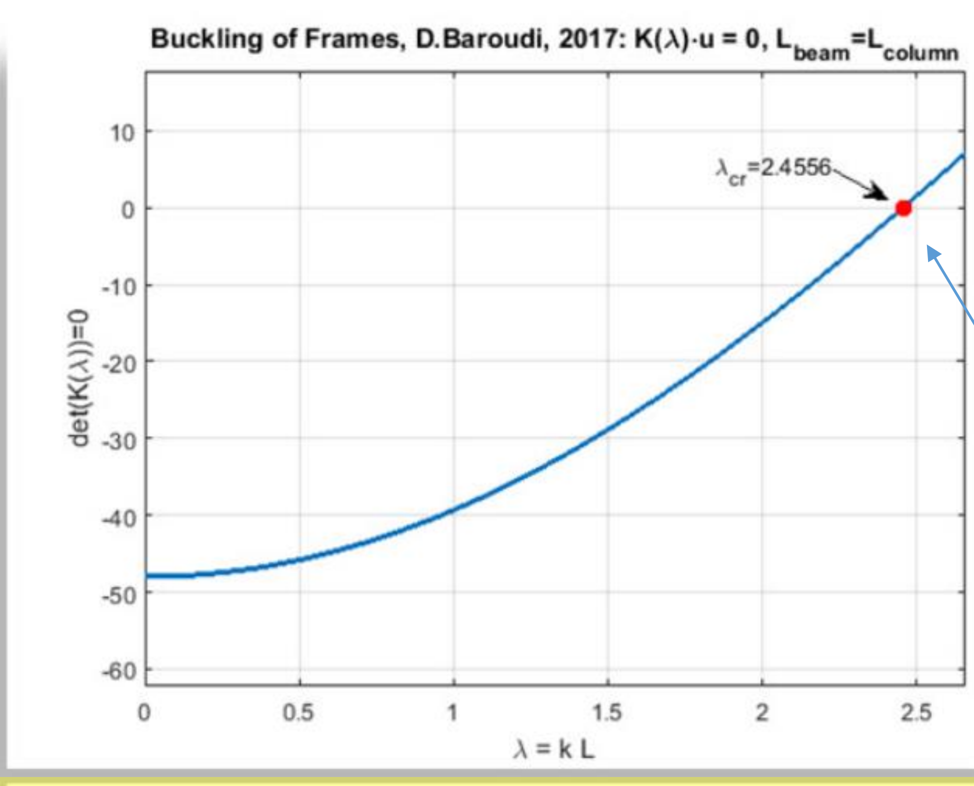

Linearised criterion

# Simple examples of a Linear buckling analysis

## Buckling analysis of side-sway frame

**Determine**
- the buckling load
- buckling modes




Recall from your previous course

**Equilibrium eqations in the tiny buckled configuration**

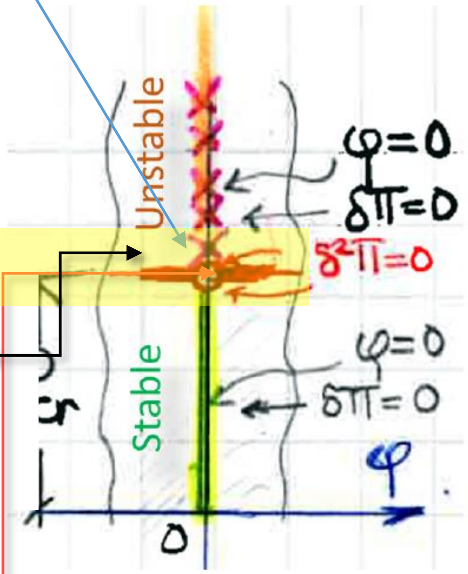
$$\begin{cases} M_{21} + M^0_{23} = 0, \\ Q_{21} = 0, \end{cases}$$

Eigenvalue problem:

$$\begin{bmatrix} A_{21}(\lambda) + a^0_{23} & -C_{21}(\lambda) \\ -C_{21}(\lambda) & +2C_{21}(\lambda)\psi - \lambda^2 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_{cr} = \min .\mathrm{sol.}\{\det(K(\lambda)) = 0\}.$$

$$P_{cr} = \lambda^2_{cr} EI/\ell^2$$




Linearised criterion

# Simple examples of a Linear buckling analysis

**Determine**
- **the buckling load**
- buckling modes

$$\Delta\Pi(v_1, v_2) = \frac{1}{2}kv_1^2 + \frac{1}{2}kv_2^2 - Pu(0).$$

**Linear buckling analysis:** We want to determine the Euler buckling load.

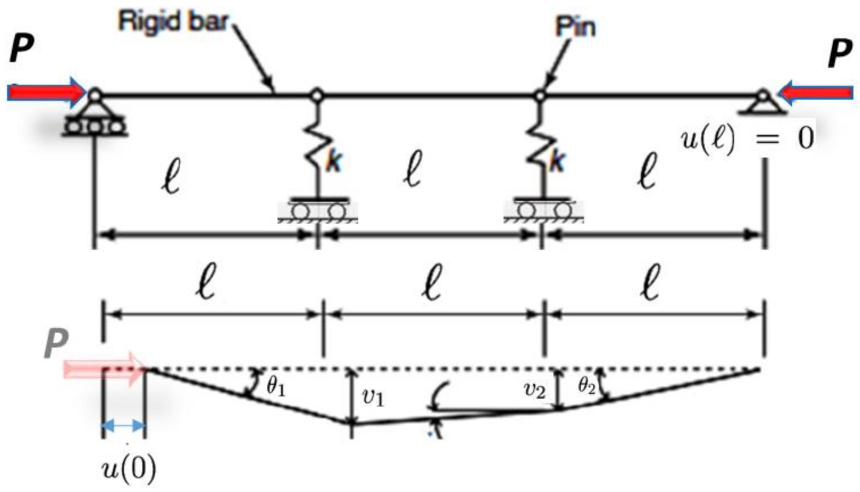
Requiring the neutral equilibrium condition $\delta(\Delta\Pi) = 0$ (for loss of stability) one obtains the eigenvalue-problem

**Results in the eigenvalue problem:**

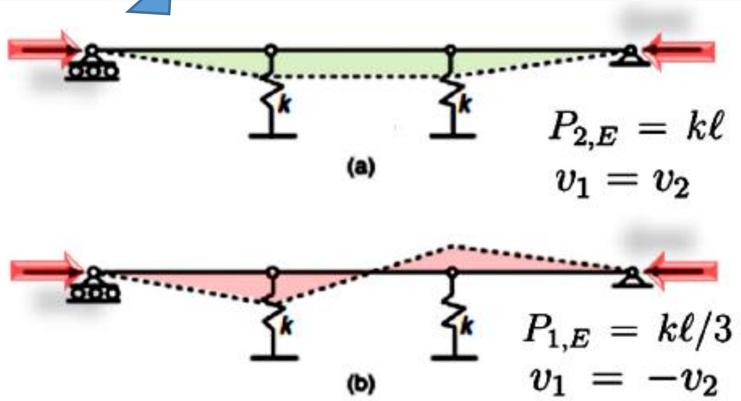
$$\lambda \equiv k\ell.$$

Linearised model

$$\begin{bmatrix} \lambda - 2P & P \\ P & \lambda - 2P \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$




A simple system having two degrees of freedom.

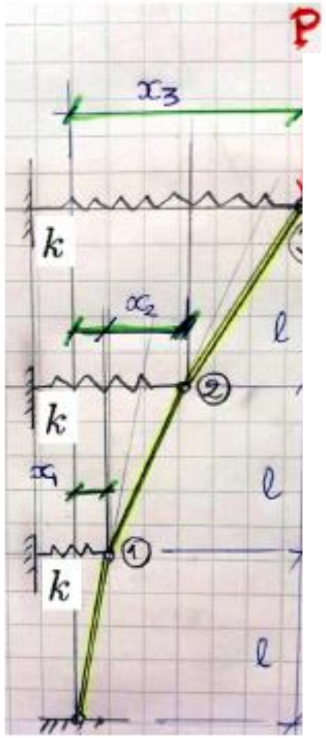


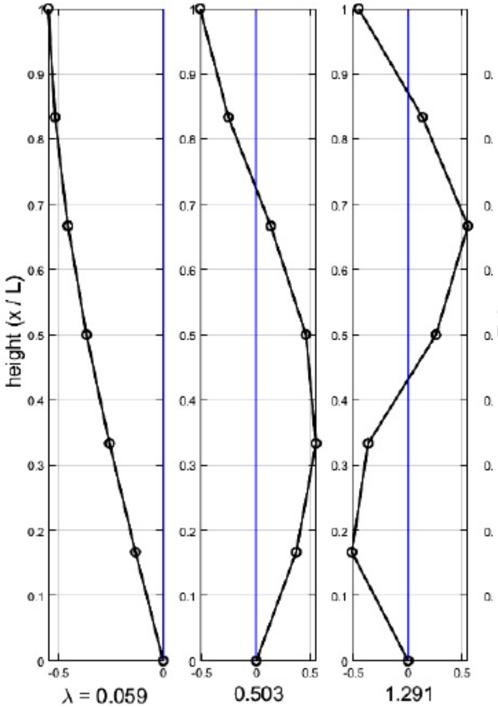

Buckling modes.

# Simple examples of a Linear buckling analysis

**Determine**

- the **buckling load**
- buckling modes




$$\Delta\Pi(x_1, x_2, \dots, x_N) = \frac{1}{2}\sum_{i=1}^{N} k_i x_i^2 + V(P; x_1, x_2, \dots, x_N),$$

assuming moderate rotations of bars $(\cos\theta_i \approx 1 - \theta_i^2/2)$

$$V(P) = -P \cdot \frac{1}{2}\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{\ell_i}, \quad x_0 = 0, \quad i = 1, 2, \dots, N$$

Asking for stationarity at the critical equilibrium point

$$\delta(\Delta\Pi) = 0, \; \forall x_i, \, i = 1, 2, \dots, \implies \frac{\partial(\Delta\Pi)}{\partial x_i} = 0,$$

**Results in the eigenvalue problem:**

So the full eigenvalue problem to solve now is:

$$\left( \begin{bmatrix} k_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & k_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & k_{N-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & k_N \end{bmatrix} - \frac{P}{\ell} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

## Linearised model

**= Linear buckling analysis results in an eigenvalue problem**

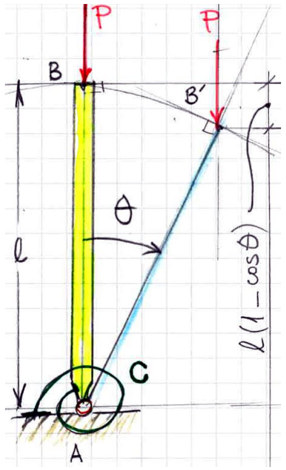
- Expanding total potential energy in Taylor's series around the neutral equilibrium position $\theta = 0$ and keep up-to quadratic terms

?

What is the fundamental difference as regarded to the full **non-linear analysis** performed just above where **all equilibrium branches were** completely **determined**.

$$\Pi(\theta; P) = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta)$$

$$\approx \frac{1}{2}c\theta^2 - P\ell \cdot \frac{\theta^2}{2}$$




**results in an eigenvalue problem**

- **Equilibrium:**

stability loss problem:

$$\delta(\Delta\Pi) = 0 \quad \Longrightarrow \quad \boxed{\Pi' = c\theta - P\ell\theta = (c - P\ell)\cdot\theta = 0}$$

This is a set of *homogeneous linear equations* which forms an **EIGENVALUE problem**

- **Criticality:**

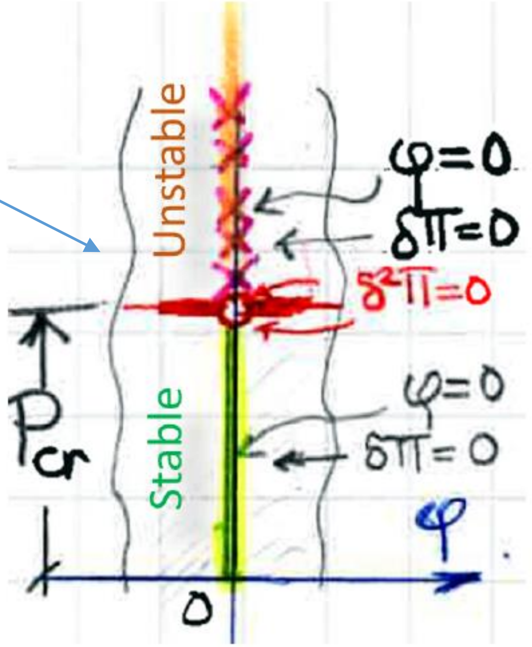
$$\Pi'' = c - P\ell = 0.$$

- **no buckling**: $\theta = 0$, is a solution (trivial initial straight $A - B - C$)
- **buckling**: $\theta \neq 0 \implies c - P\ell = 0 \implies P_{cr} = c/\ell$ (buckling load)

$\Longrightarrow$

**Linear buckling analysis provides ONLY:**

- the **buckling load**
- buckling modes up-to a multiplicative coefficient




Linearised criterion

## Linearised model

= Linear buckling analysis

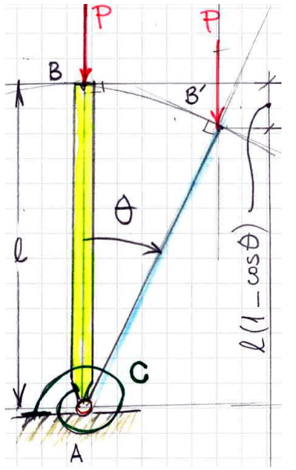
- Equilibrium: $\Pi' = c\theta - P\ell\theta = (c - P\ell)\cdot\theta = 0$
- Criticality: $\Pi'' = c - P\ell = 0.$

$\Longrightarrow$

<u>Linear buckling analysis provides ONLY</u>:
- the **buckling load**
- buckling modes up-to a multiplicative coefficient

$$\Pi(\theta; P) = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta)$$

$$\approx \frac{1}{2}c\theta^2 - P\ell\cdot\frac{\theta^2}{2}$$

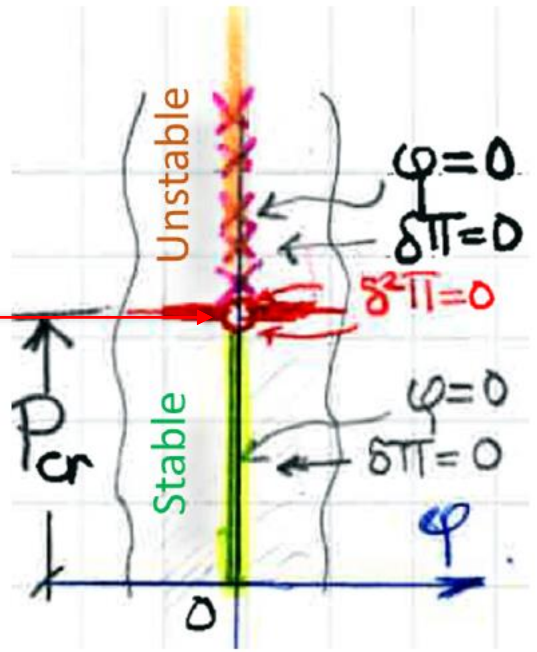



- **What is the nature of stability at the bifurcation point?**
- **Can we answer with the linearized model?** **<u>Answer: NO</u>**

$$\Pi''(\theta = 0; P = P_{cr}) = c - P_{cr}\ell = c - \frac{c}{\ell}\cdot\ell = 0.$$

One may wrongly or too fast, conclude that the equilibrium is *indifferent*. However, this is not true and this result is an artefact of the linearisation. We should take higher order[41] (than quadratic) terms in the expansion of $W_{ext.}(\theta; P) = -P\ell(1-\cos\theta)$ with respect to $\theta$, in order to decide (the sign) of the stability at the bifurcation point. For instance, the expansion $\cos\theta \approx 1 - \theta^2/2 + \theta^4/4!$ can solve the sign problem. This, physically, means that we use a *asymptotic expansion of non-linear equations* and capture the moderate rotations and displacements around $\theta = 0$.[42]

$\Longrightarrow$




Linearised criterion

# Post-buckling analysis - asymptotic non-linear approach

- ***N.B.*** *Linearized buckling analysis* cannot provide information about the post-buckling behavior
- So, needs *post-buckling analysis* **(full non-linear formulation)**

$$\Pi(\theta;P) = \frac{1}{2}c\theta^2 - P\ell(1-\cos\theta)$$

$\Longrightarrow$ Asymptotic post-buckling analysis

*Expand up-to fourth-order terms the total potential energy*

$$\cos\theta \approx 1 - \theta^2/2 + \theta^4/4!$$

$$W_{ext.}(\theta;P) = -P\ell(1-\cos\theta)$$

$$\Longrightarrow \quad \Pi = 1/2c\theta^2 - P\ell\theta^2/2 + P\ell\theta^4/4!$$

$$\mathrm{d}\Pi/\mathrm{d}\theta = (c - P\ell)\theta + P\ell\theta^3/3! = 0.$$

$\theta = 0,\ \forall P$

pre-buckled branch

$P = \dfrac{P_{cr}}{1-\theta^2/3!}$, $\quad P_{cr} = c/\ell,\ \theta \neq 0.$

post-buckling branch

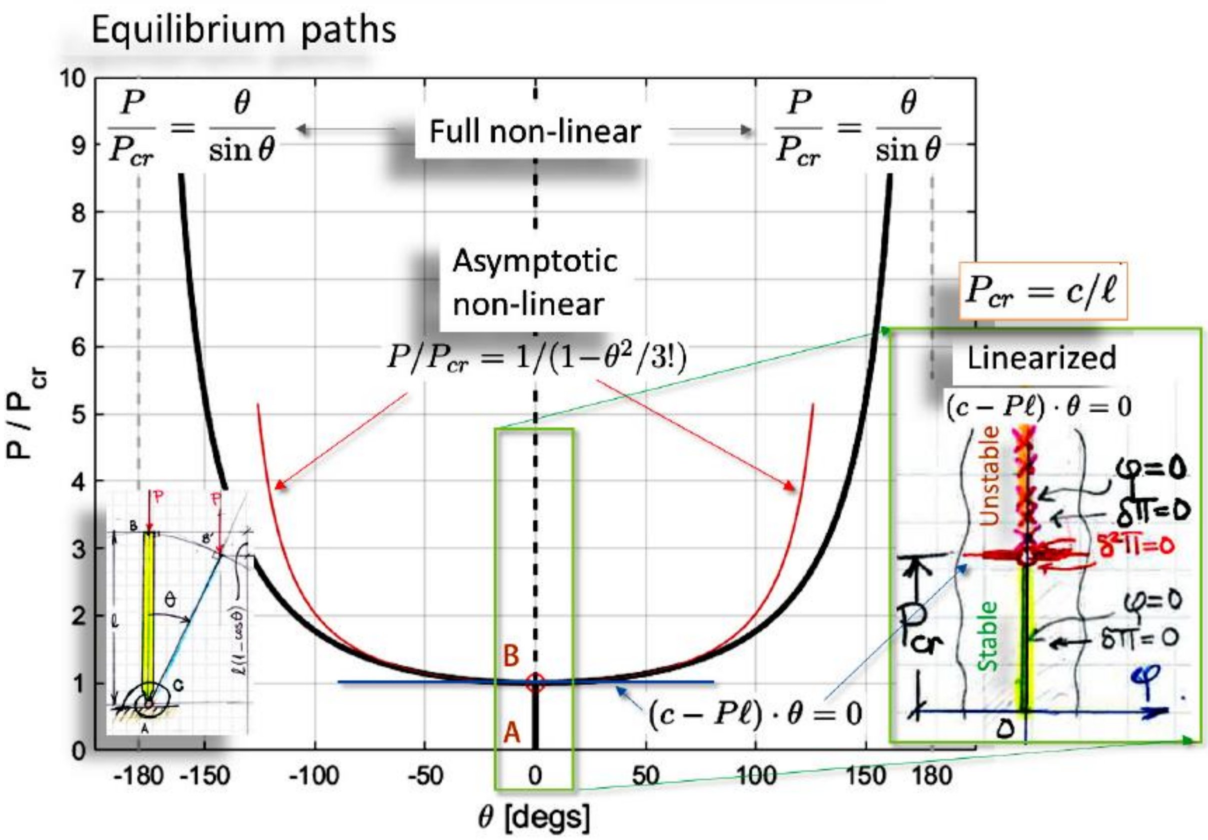



Figure 3.47: Equilibrium paths. Full non-linear model (black), asymptotic non-linear model (orange) and the linearised model (right side).

Note that now, for loading values greater than the buckling load, we obtain the corresponding value for the rotations

# Fundaments of Stability Theory: Example on imperfections

The imperfection parameter is the initial tilt angle

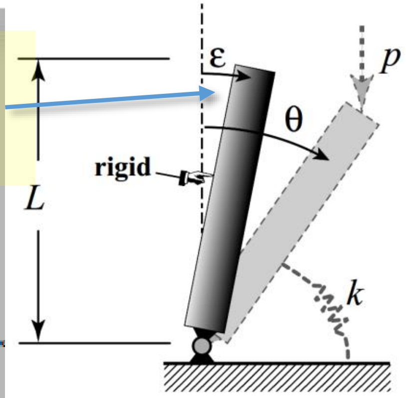



FIGURE 34.1. The impe... l tilt angle $\epsilon$.

One key difficulty is that

We begin the study of the effect of imperfections through several simple yet instructive one-degree-of-freedom (DOF) examples.

## §34.3. The Imperfect Hinged Cantilever

We take up again the critical-point analysis of the hinged cantilever already studied in the previous Chapter. But we assume that this system is *geometrically imperfect* in the sense that the rotational spring is unstrained when the rigid bar "tilts" by a small angle $\epsilon$ with the vertical. By varying $\epsilon$ we effectively generate a *family* of imperfect systems that degenerate to the perfect system when $\epsilon \to 0$.

Denoting again the total rotation from the vertical by $\theta$ as shown in Figure 34.1, the strain energy of the imperfect system can be written

$$U(\theta, \epsilon) = \tfrac{1}{2}k(\theta - \epsilon)^2. \quad (34.1)$$

The potential energy of the imperfect system is

$$\Pi(\theta, \lambda, \epsilon) = U - W = \tfrac{1}{2}k(\theta - \epsilon)^2 - fL(1 - \cos\theta) = k\left[\tfrac{1}{2}(\theta - \epsilon)^2 - \lambda(1 - \cos\theta)\right], \quad (34.2)$$

in which as before we take $\lambda = fL/k$ as dimensionless control parameter.

### §34.3.1. Equilibrium Analysis

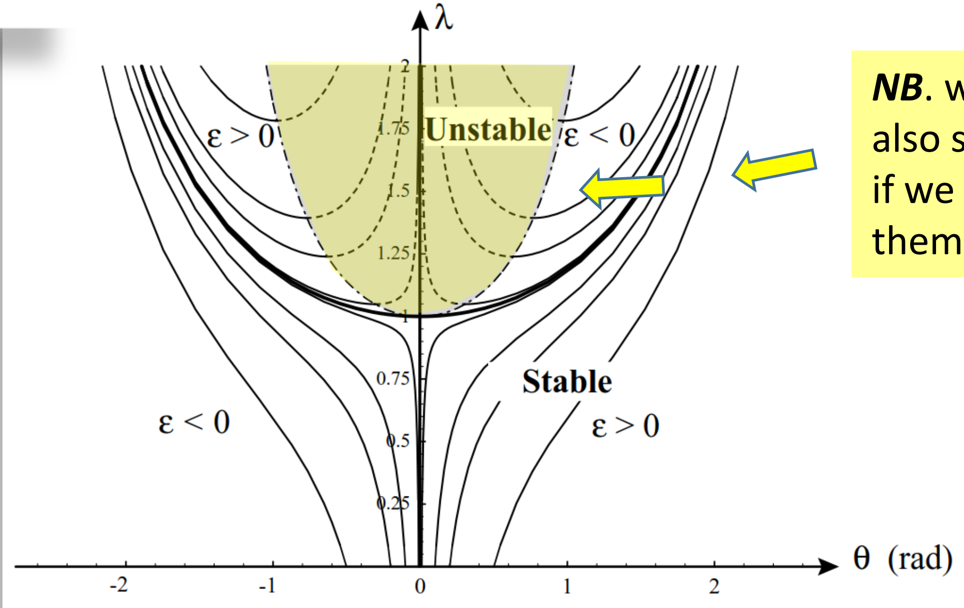
The equilibrium equation in terms of the angle $\theta$ as degree of freedom is

$$r = \frac{\partial \Pi}{\partial \theta} = k(\theta - \epsilon - \lambda \sin\theta) = 0. \quad (34.3)$$

Therefore, the equilibrium path equation of an imperfect system is

$$\lambda = \frac{\theta - \epsilon}{\sin\theta}. \quad (34.4)$$






***NB***. we can have also such '*bassins*' if we can access them

where

$$K = \frac{\partial r}{\partial \theta} = k(1 - \lambda\cos\theta), \quad q = \frac{\partial r}{\partial \theta} = k\sin\theta. \quad (34.6)$$

We have stability if $K > 0$, that is

$$1 - \lambda\cos\theta > 0, \quad (34.7)$$

and instability if $K < 0$, that is

$$1 - \lambda\cos\theta < 0. \quad (34.8)$$

Critical points are characterized by $K(\lambda_{cr}) = 1 - \lambda_{cr}\cos\theta = 0$, or

$$\lambda_{cr} = \frac{1}{\cos\theta}. \quad (34.9)$$

On equating this value of $\lambda$ with that given by the equilibrium solution (32.4) we obtain

$$\theta - \epsilon = \tan\theta. \quad (34.10)$$

This relation characterizes the *locus of critical points* as $\epsilon$ is varied. It is not difficult to show that these critical points are limit points if $\epsilon \neq 0$ (imperfect systems) and a bifurcation point if and only if $\epsilon = 0$ (perfect system).

### §34.3.3. Discussion

The response of this family of imperfect systems is displayed in Figure 34.2.

In this Figure heavy lines represent the response of the peerfect system whereas light lines represent the responses of imperfect systems for fixed values of $\epsilon$. Furthermore continuous lines identify stable equilibrium

$$\Pi(\theta, \lambda, \epsilon) = U - W = \tfrac{1}{2}k(\theta - \epsilon)^2 - fL(1 - \cos\theta) = k\left[\tfrac{1}{2}(\theta - \epsilon)^2 - \lambda(1 - \cos\theta)\right],$$

$$\delta\Pi = 0 \Rightarrow \quad \frac{\partial \Pi}{\partial \theta} = k(\theta - \epsilon - \lambda\sin\theta) = 0. \quad \rightarrow \quad \lambda = \frac{\theta - \epsilon}{\sin\theta}$$

$$\delta^2\Pi(\theta) \Rightarrow \begin{cases} \dfrac{\partial r}{\partial \theta} = k\sin\theta \\ \dfrac{\partial r}{\partial \theta} = k(1 - \lambda\cos\theta), \end{cases}$$

Pos.def?

REF: https://www.colorado.edu/engineering/cas/courses.d/NFEM.d/Home.html

# Example of FE-based post-buckling analysis

For real problems, one should rely on **computational** technology and **experimental** approach

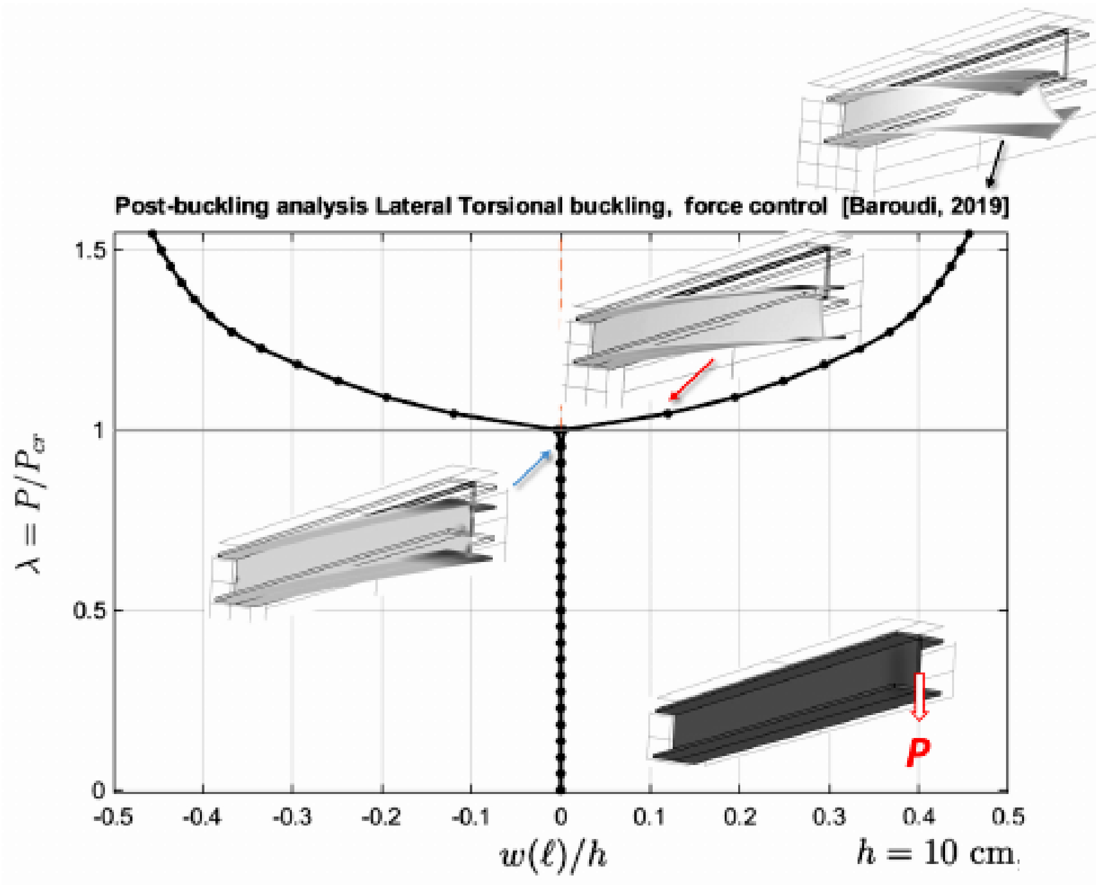
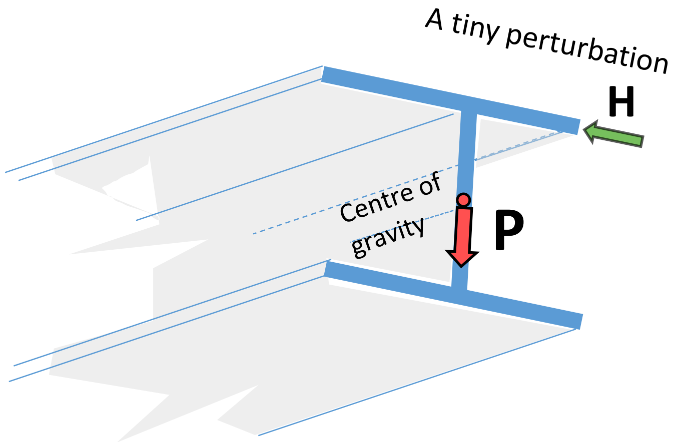
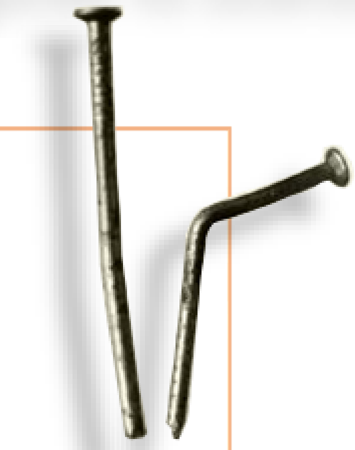



Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

- **Validate computational models** you are using before making predictions
- There can be physics that computational models do not see (not or cannot be accounted for), which effects can be accessed only **experimentally**



# Effect of imperfections Robustness of design?



All **real structural** systems are **imperfect**

- ✓ in form,
- ✓ in material properties,
- ✓ in the sense of residual stresses
- ✓ in the way the loads are applied

[12] *It may be safely said that all real structural systems are imperfect in form, imperfect in material properties, imperfect in the sense of residual stresses and imperfect in the way the loads are applied.* **Roorda** (1980)

# Effects of imperfections

## Symmetric Stable bifurcation

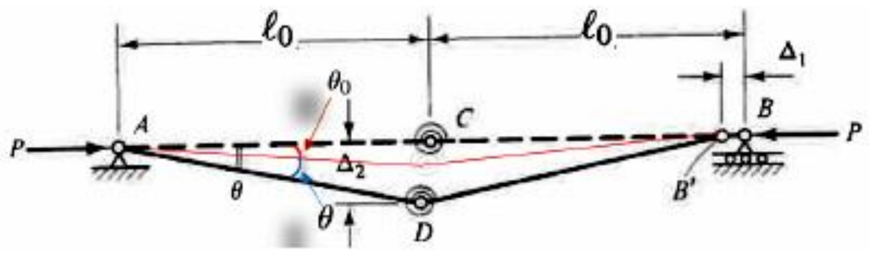
- Axially loaded structure with imperfections
  - ✓ imperfection in horizontality (or verticality for a column)

$$\Pi = \frac{1}{2}c[2(\theta - \theta_0)]^2 - 2P\ell_0(\cos\theta - \cos\theta_0)$$

$$\frac{d\Pi}{d\theta} = 0 \implies \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin\theta},$$

$$P_{cr} = \frac{2c}{a}$$

**Initial imperfection in horizontality**




- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*
- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*
- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

# Effects of imperfections

## Symmetric Stable bifurcation

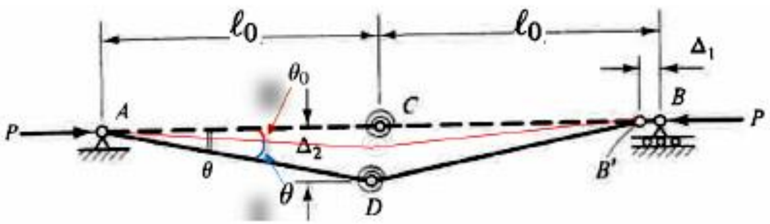
- Axially loaded structure with imperfections
  - ✓ imperfection in horizontality (or verticality for a column)

$$\Pi = \frac{1}{2}c[2(\theta-\theta_0)]^2 - 2P\ell_0(\cos\theta - \cos\theta_0)$$

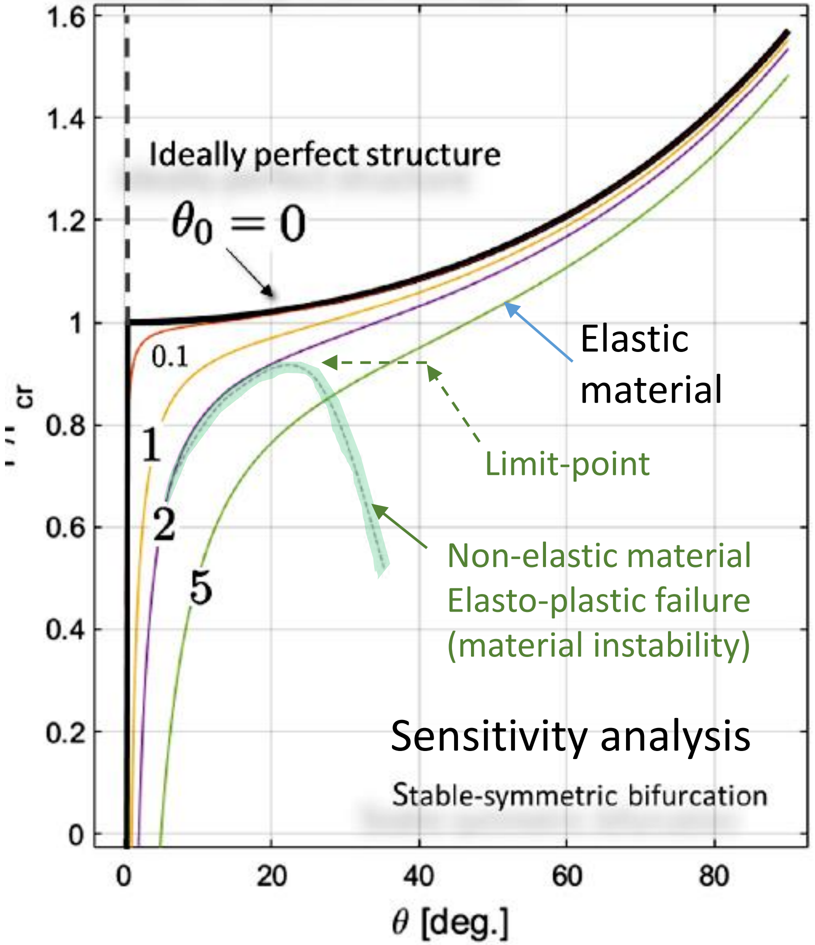
$$\frac{\mathrm{d}\Pi}{\mathrm{d}\theta} = 0 \implies \frac{P}{P_{cr}} = \frac{\theta-\theta_0}{\sin\theta},$$

$$P_{cr} = \frac{2c}{a}$$

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*



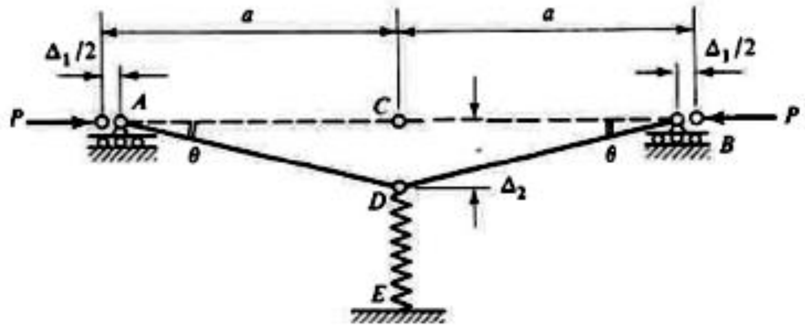

Effect of initial shape imperfection on the maximum compressive limit load



# Unstable-symmetric bifurcation model

Axially loaded perfect structure:

$$\Pi = \frac{1}{2}kv^2 - P\ell_0(1-\cos\theta)$$
$$= \frac{1}{2}k\ell_0^2\sin^2\theta - 2P\ell_0(1-\cos\theta)$$



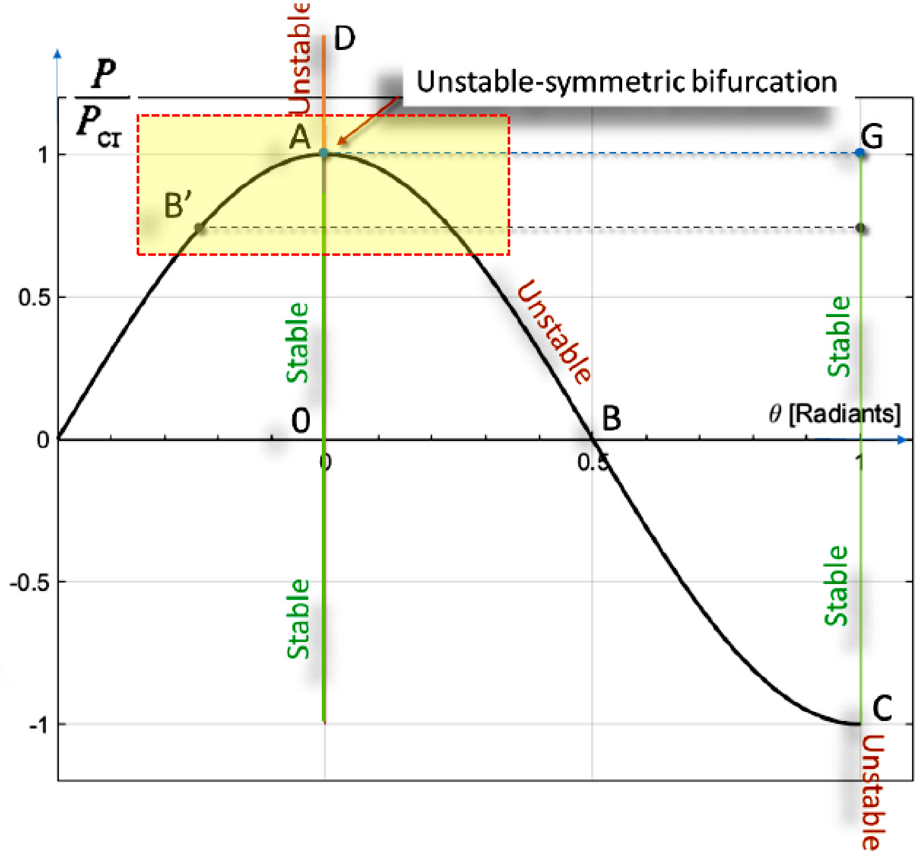
The equilibrium paths

$$\frac{d\Pi}{d\theta} = k\ell_0^2 \sin\theta\cos\theta - 2P\ell_0.\sin\theta = 0$$

$$\Downarrow$$

$$P = \frac{k\ell_0}{2}\cos\theta = P_{cr}\cos\theta,$$

where $P_{cr} \equiv \dfrac{k\ell_0}{2}$.



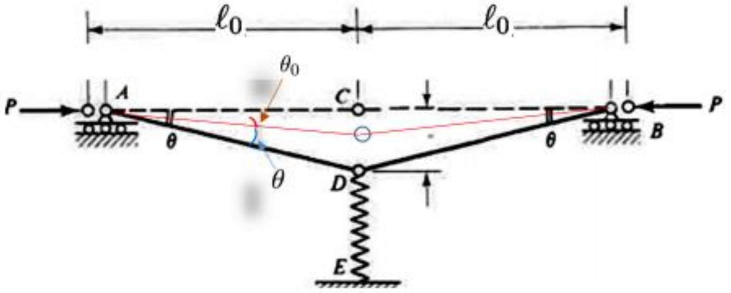

Equilibrium path. Unstable-symmetric.

**Stability is determined by studying the sign** $d(d\Pi/d\theta) = d^2\Pi/d\theta^2$

If the second derivative vanishes, then one should take higher derivatives till non-zero[45] value is achieved, for this case, for the sign of $\delta(\Delta\Pi)$.

## Unstable-symmetric bifurcation model

$$\Pi = \frac{1}{2}k\ell_0^2(\sin\theta - \sin\theta_0)^2 - 2P\ell_0(\cos\theta_0 - \cos\theta).$$

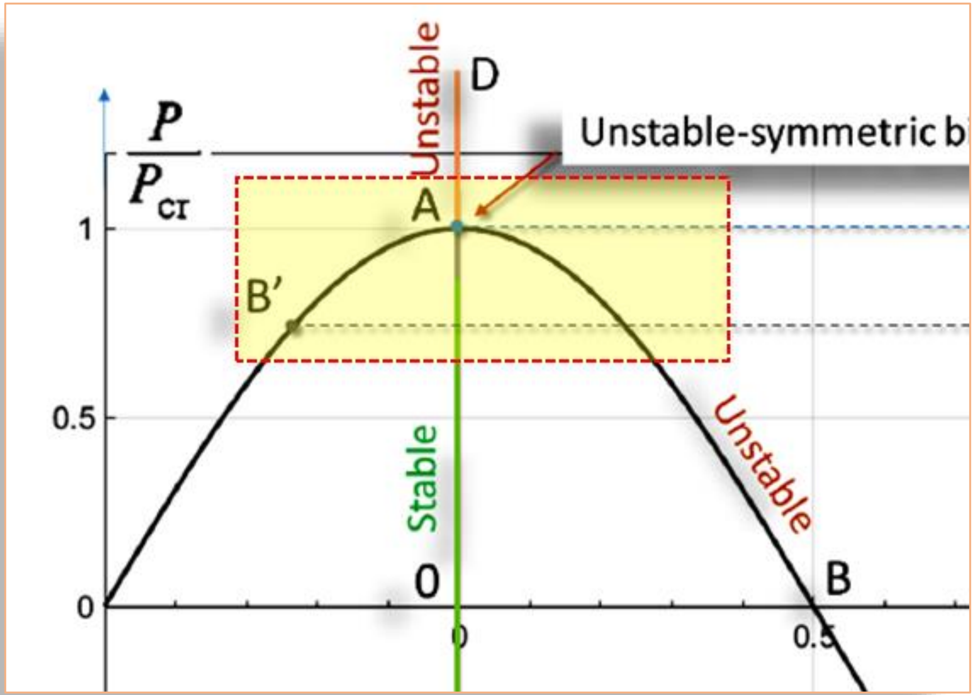



: Axially loaded structure with initial imperfection (
The spring stiffness is $k$.

The equilibrium paths

$$\frac{P}{P_{cr}} = \left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\cos\theta, \quad P_{cr} = k\ell_0/2.$$

$$\Downarrow$$

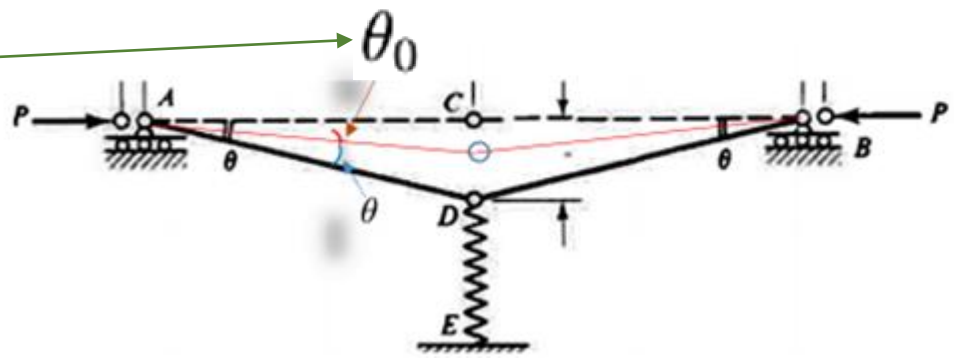



**Stability** **is determined by studying the sign** $\mathrm{d}(\mathrm{d}\Pi/\mathrm{d}\theta) = \mathrm{d}^2\Pi/\mathrm{d}\theta^2$

If the second derivative vanishes, then one should take higher derivatives till non-zero[45] value is achieved, for this case, for the sign of $\delta(\Delta\Pi)$.

# Unstable-symmetric bifurcation model

Axially loaded structure with imperfections: $\theta_0$

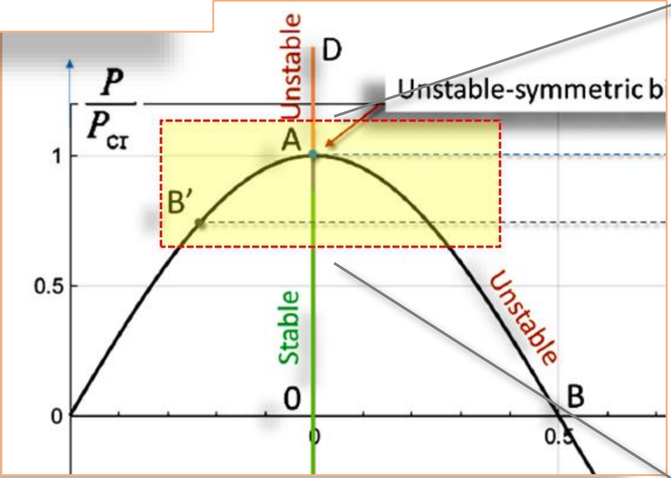



$$\Pi = \frac{1}{2}k\ell_0^2(\sin\theta - \sin\theta_0)^2 - 2P\ell_0(\cos\theta_0 - \cos\theta).$$

$\Downarrow$

The equilibrium paths

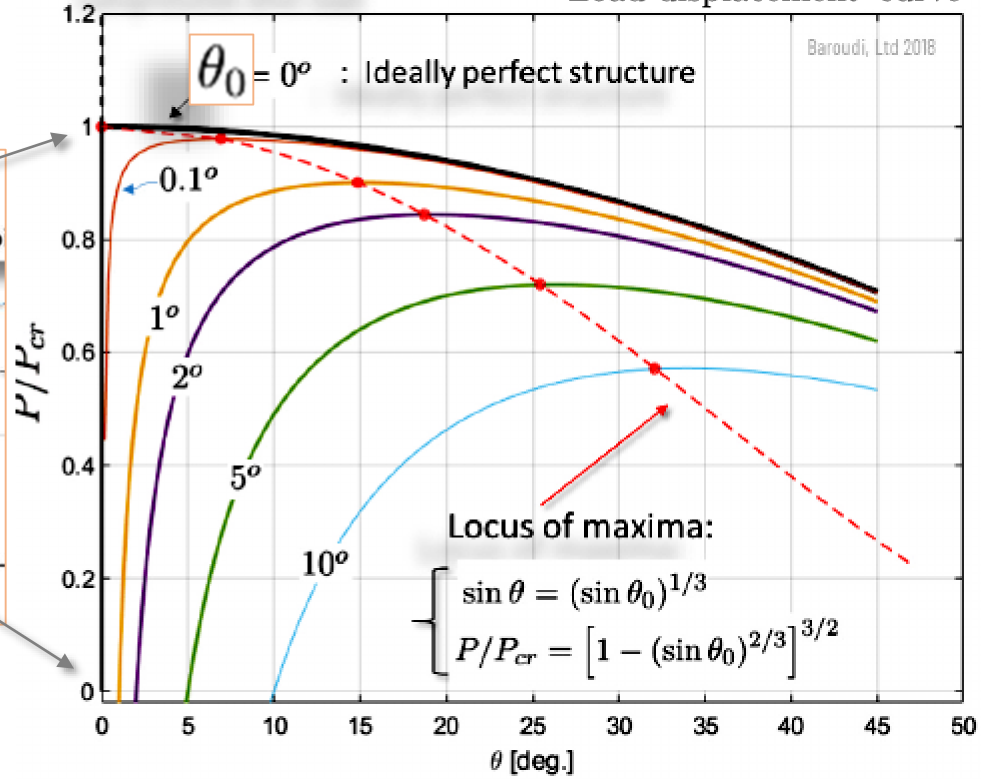
$$\frac{P}{P_{cr}} = \left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\cos\theta, \quad P_{cr} = k\ell_0/2.$$




Perfect structure

Effect of initial shape imperfection on the maximum compressive limit load

Load-displacement curve




Imperfect structure

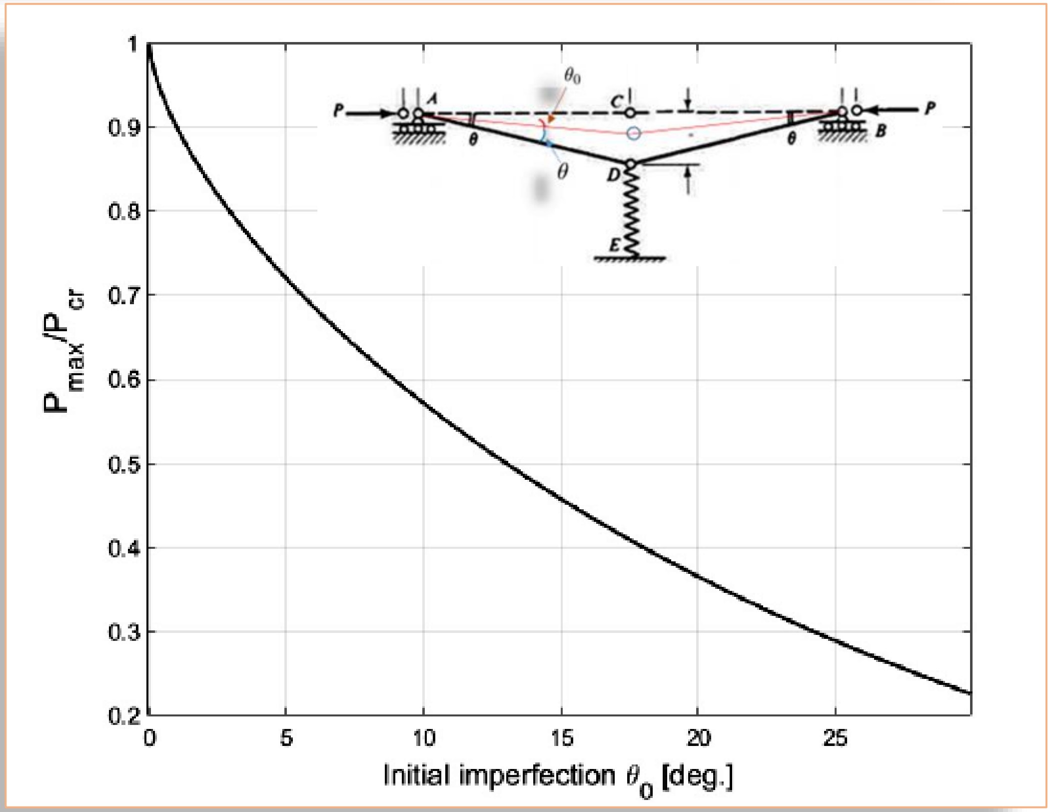
- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

$\Longrightarrow$ $\Longrightarrow$

# Unstable-symmetric bifurcation model

Axially loaded structure with imperfections: $\theta_0$

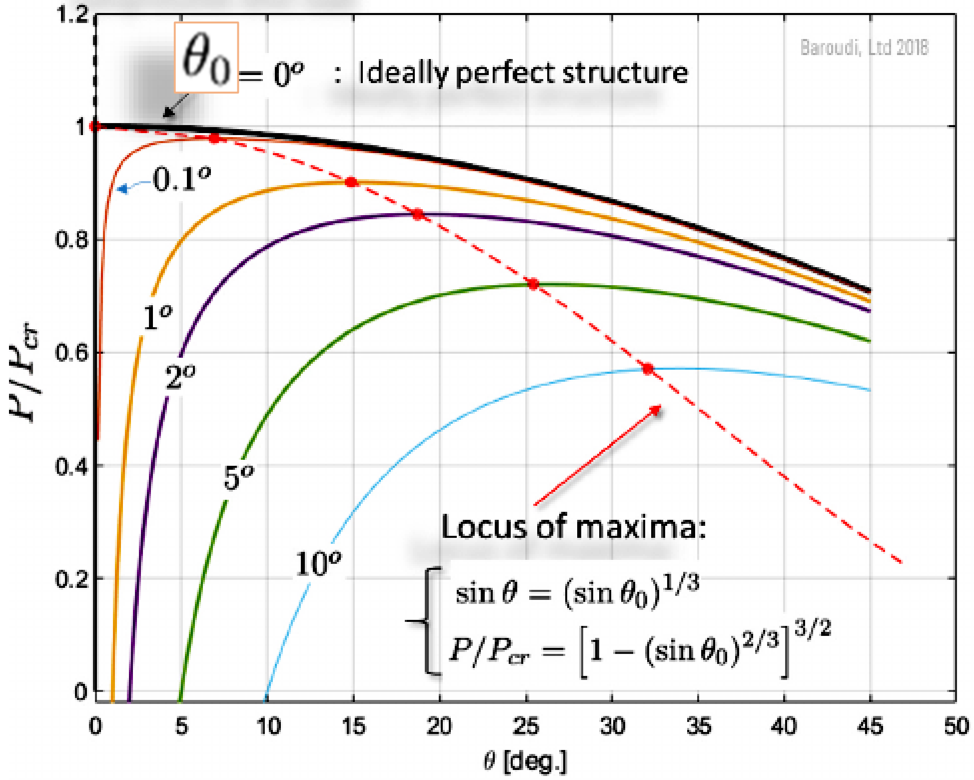
- **Unstable symmetric**
  - ✓ T his gives *imperfection sensitive structures*




$\Longrightarrow$

$\Longleftarrow$

Effect of initial shape imperfection on the maximum compressive limit load




Maximum axial force reduction with respect to the amplitude of initial imperfection. $P_{cr}$ is the collapse or buckling load of the perfect structure.

Imperfect structure

# Asymmetric bifurcation model

**Limit-load**, raja-kuorma

## Snap-through model

Simplified example to illustrate the concept of ***limit-load***

- Rigid truss: two straight rigid bars of equal length connected to each other by a hinge. One support allows free lateral movement restrained by a elastic spring *k*
- Load *P* is kept increasing quasi-statically and we want to solve the force-displacement curve (equilibrium paths)

$$\Pi = \frac{1}{2}k(2L)^2(\cos\theta - \cos\alpha)^2 - PL(\sin\alpha - \sin\theta).$$

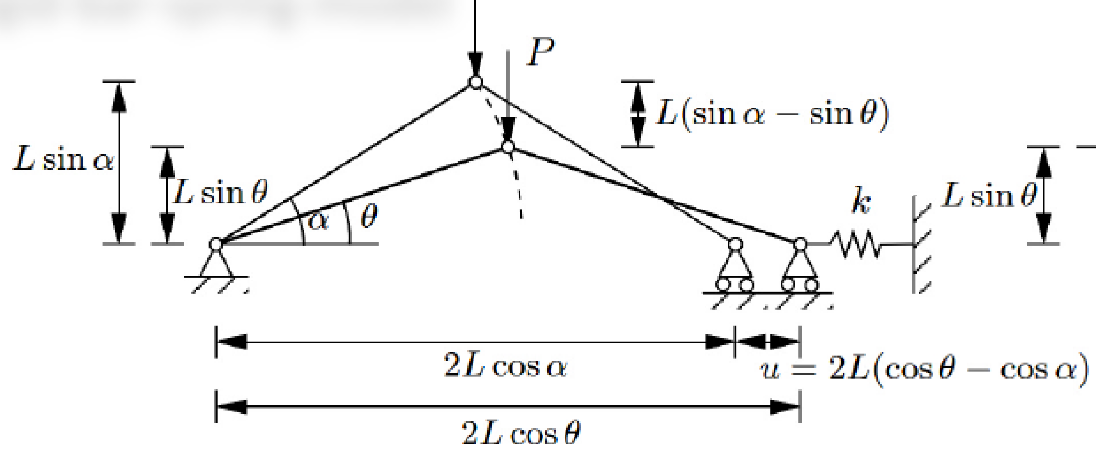
**2) Equilibrium paths**

The equilibrium $\mathrm{d}\Pi/\mathrm{d}\theta = 0$

$\Downarrow$ load-displacement 'curve' = equilibrium paths

$$\frac{P}{4kL} = \sin\theta - \tan\theta\cos\alpha = \sin\theta(1 - \cos\alpha/\cos\theta)$$

Rigid-bar-spring-model




**2) Stability of equilibrium**

$\Downarrow$

$$\mathrm{d}^2\Pi/\mathrm{d}\theta^2 = 4kL^2(\cos\alpha/\cos\theta - \cos^2\theta)$$

STABLE $\Longleftarrow$ $\Pi'' > 0, \quad \text{when } \theta < 0 \text{ and } \theta > \theta_B,$

UNSTABLE $\Longleftarrow$ $\Pi'' < 0, \quad \text{when } -\theta_B < \theta < \theta_B.$

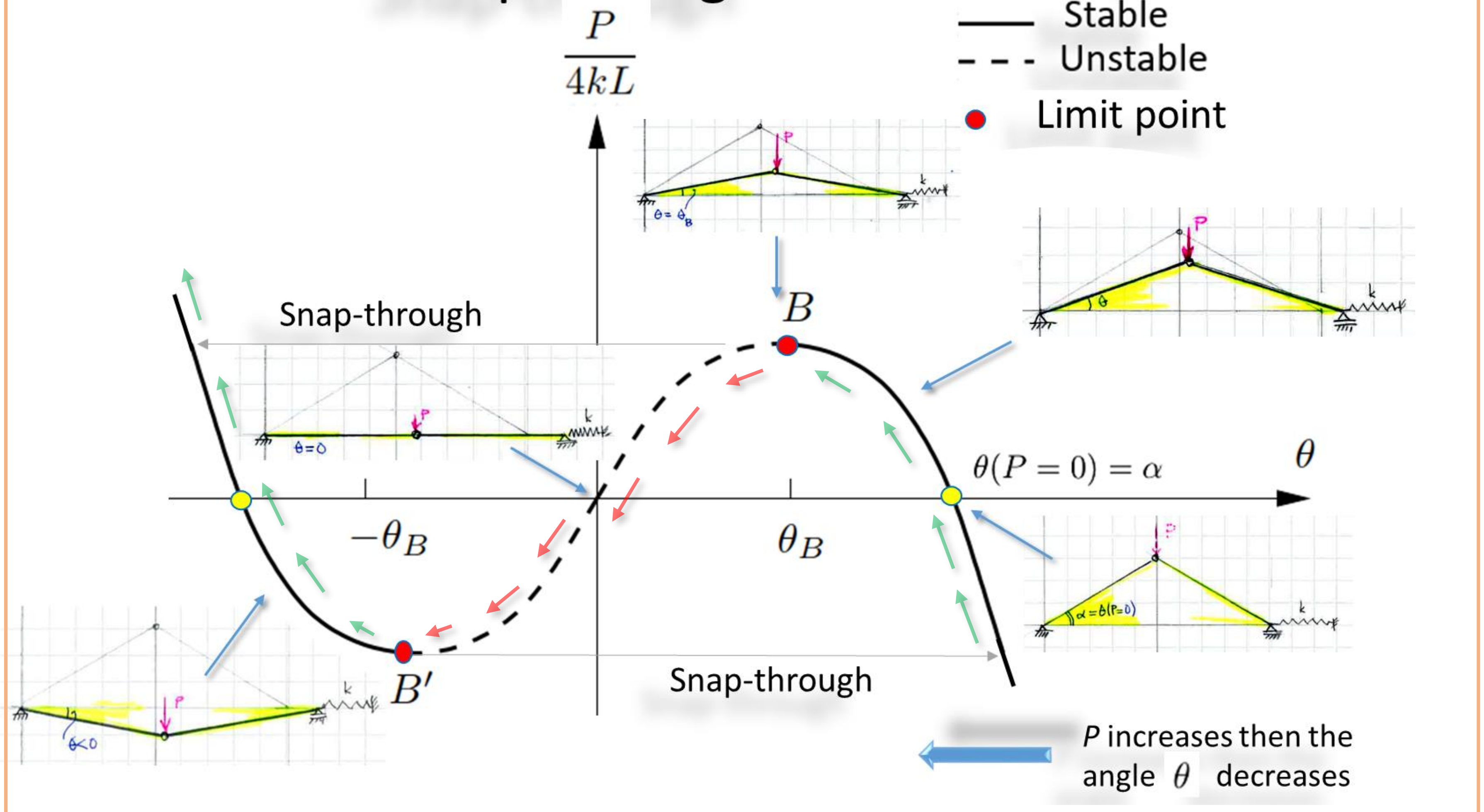
The zeros of the second derivative are

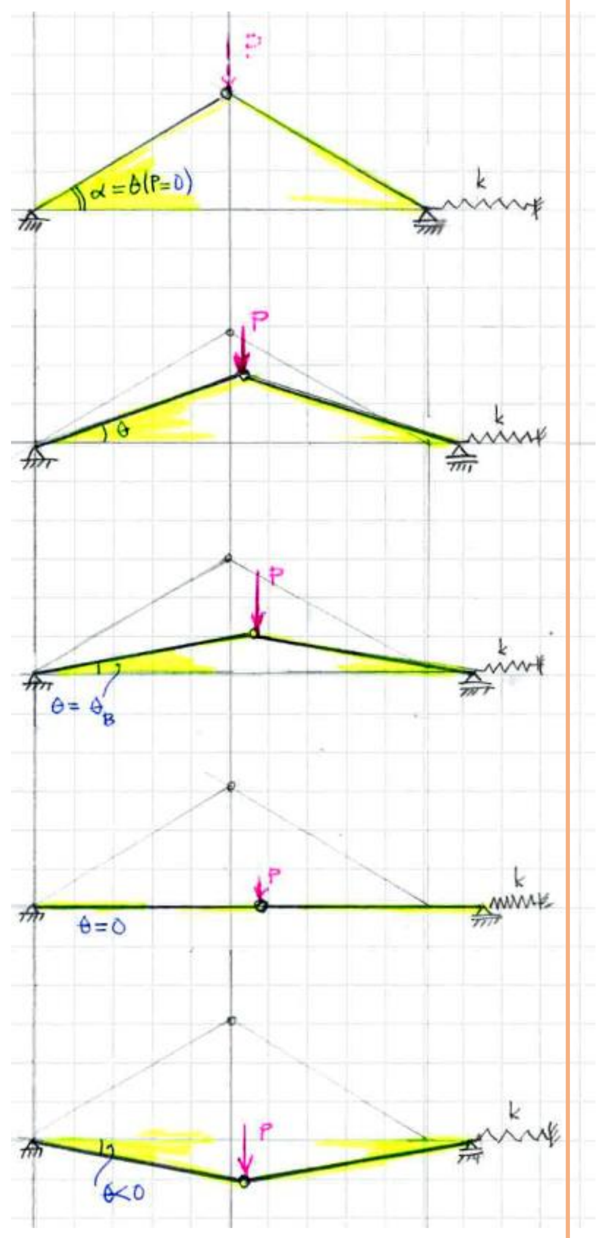
$$\theta = \pm\arccos[(\cos\alpha)^{1/3}] \equiv \pm\theta_B$$

To fix the ideas,
for an initial angle $\alpha = 30^o \Longrightarrow \theta_B = \pm 17.6^o$
$P = \pm 0.028 \cdot 4kL.$

load-displacement 'curve'

# Snap-through



To fix the ideas, for an initial angle $\alpha = 30^o$ $\Longrightarrow$ $\theta_B = \pm 17.6^o$, $P = \pm 0.028 \cdot 4kL$.

# Asymmetric bifurcation model

Limit-load, raja-kuorma

## Snap-through model

The following example illustrate the concept of ***limit-load***

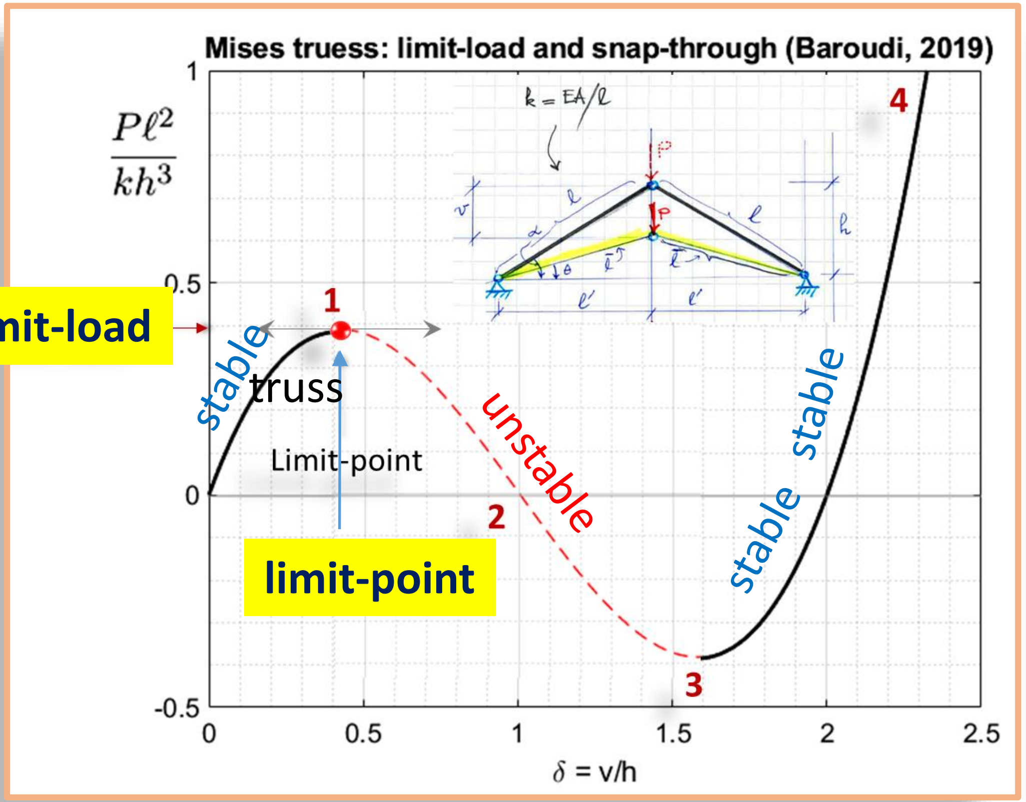
- **Mises** truss: two straight elastic bars of equal length connected to each other by a hinge to fixed supports allowing free rotations only

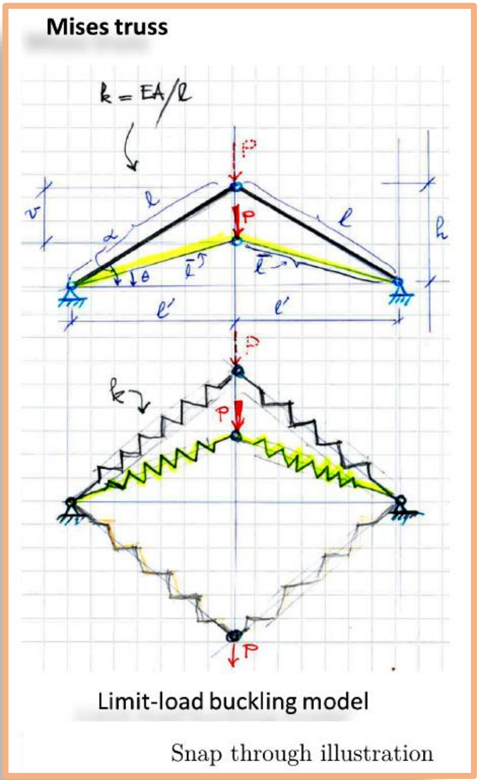
- Load $P$ is kept increasing quasi-statically and we want to solve the force-displacement curve (equilibrium paths)

- The truss so shallow that no buckling of separate bars occurs: only snap-through (consequently, only vertical component of the tip-displacement occurs. If truss enough high then one should consider the horizontal component as well.)

load-displacement 'curve'

$$P\frac{\sqrt{\ell^2+v^2-2vh}}{2k(h-v)} = \ell - \sqrt{\ell^2+v^2-2vh}$$




A shallow truss

load-displacement 'curve' for *shallow trusses* $h \ll \ell$

$$\frac{P\ell^2}{kh^3} = 2\delta - 3\delta^2 + \delta^3, \qquad \delta = v/h.$$