A? MyCourses

CIV-E4100 - Stability of Structures D, 01.03.2021-18.04.2021

Content of the 2nd week lectures:

Content

0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability

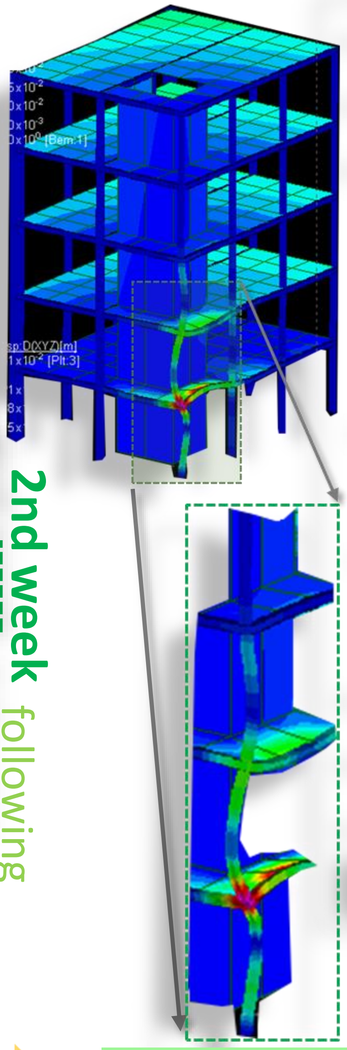
First week

1. **Flexural buckling (nurjahdus)**

2nd week

2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

following weeks



- General Energy criteria of loss of stability
- Trefftz stability loss criteria
- Flexural buckling
  - Buckling of beam-column
  - Timoshenko column
  - Buckling of beam-column on elastic foundation
- Energy approach- examples
- Effects of imperfections
  - ❑ Ayreton-Perry formula & Eurocode buckling curves
- Linear buckling analysis
- Post-buckling analysis
- Finite element method – a hand-version for buckling analysis (= linearised slope deflection method)

The method:
two consecutive lectures & two cessions of guided exercises for doing the weekly compulsory homework

One topic per week

Mo	Tu	We	Th	Fr	Sa	Su	
1	2	3	4	5	6	7	March
8	9	10	11	12	13	14	
15	16	17	18	19	20	21	
22	23	24	25	26	27	28	
29	30	31	1	2	3	4	
5	6	7	8	9	10	11	April
12	13	14	15	16	17	18	

Lecturer
Djebar Baroudi, Dr.
Civil Engineering Department
Aalto University

version 10.3.2021 -

The energy principles – all in one slide

1) The stationary total potntial energy:

(valid only for conservative systems in statics)

$$\forall \delta \mathbf{v} \quad \delta(\Delta\Pi) = 0 \implies \delta(\delta^2\Pi) = 0, \Longleftarrow$$

= stationaity condition

= neutral equilibrium condition

Trefftz stability loss criterion (less general)

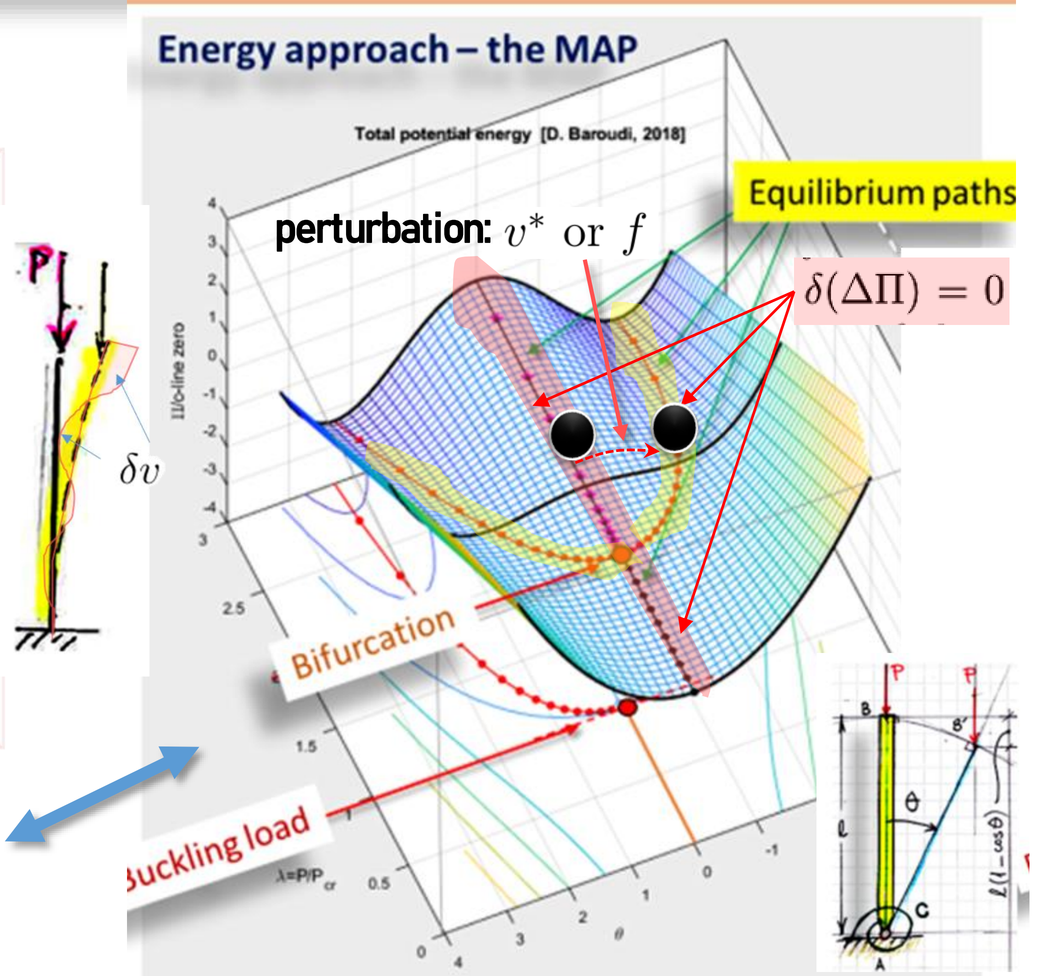
2) The virtual work principle: $\delta(\Delta\Pi) = 0$ $\Uparrow$ $\forall \delta \mathbf{v}$

(valid for all systems systems consevatives, non-consevatives, statics, dynamics)

$$\delta(\Delta W_{int}) + \delta(\Delta W_{ext}) = \delta(\Delta W_{acc.})$$

- It is this energy form $\delta(\Delta\Pi) = 0$ criticality condition that will be used systematically thorough this course to derive the stability loss equations (=eigenvalue problem=buckling equations in the differential form) for all our structures
- It is again this energy form that will be used to obtain good approximations for the buckling loads (for hand- calculations & Finite Element formulation)

Physically speaking, these two conditions mean that the perturbed state is also an equilibrium state; thus another neighbouring (or far) equilibrium exists and a tiny 'kick' can moves the system there (=loss of stability)



Elastic Stability of Structures

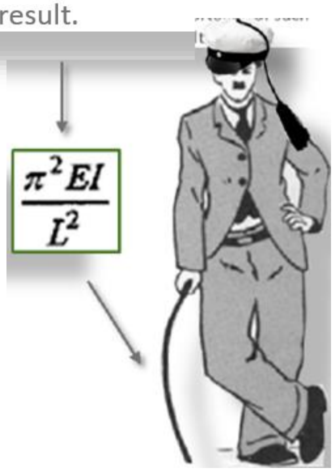
Content

Topics of the lectures and homework

Pieter van Musschenbroek (1692 – 1761)

Leonard Euler

He derived the theoretical critical load for buckling of a column already in **1774**! At that time no one understood the importance of such result.



Observed that the maximum compressive load a column can sustain prior to failure is proportional to $1/\ell^2$

- Performed experiments on column buckling (1729)

The key stability question in structural design

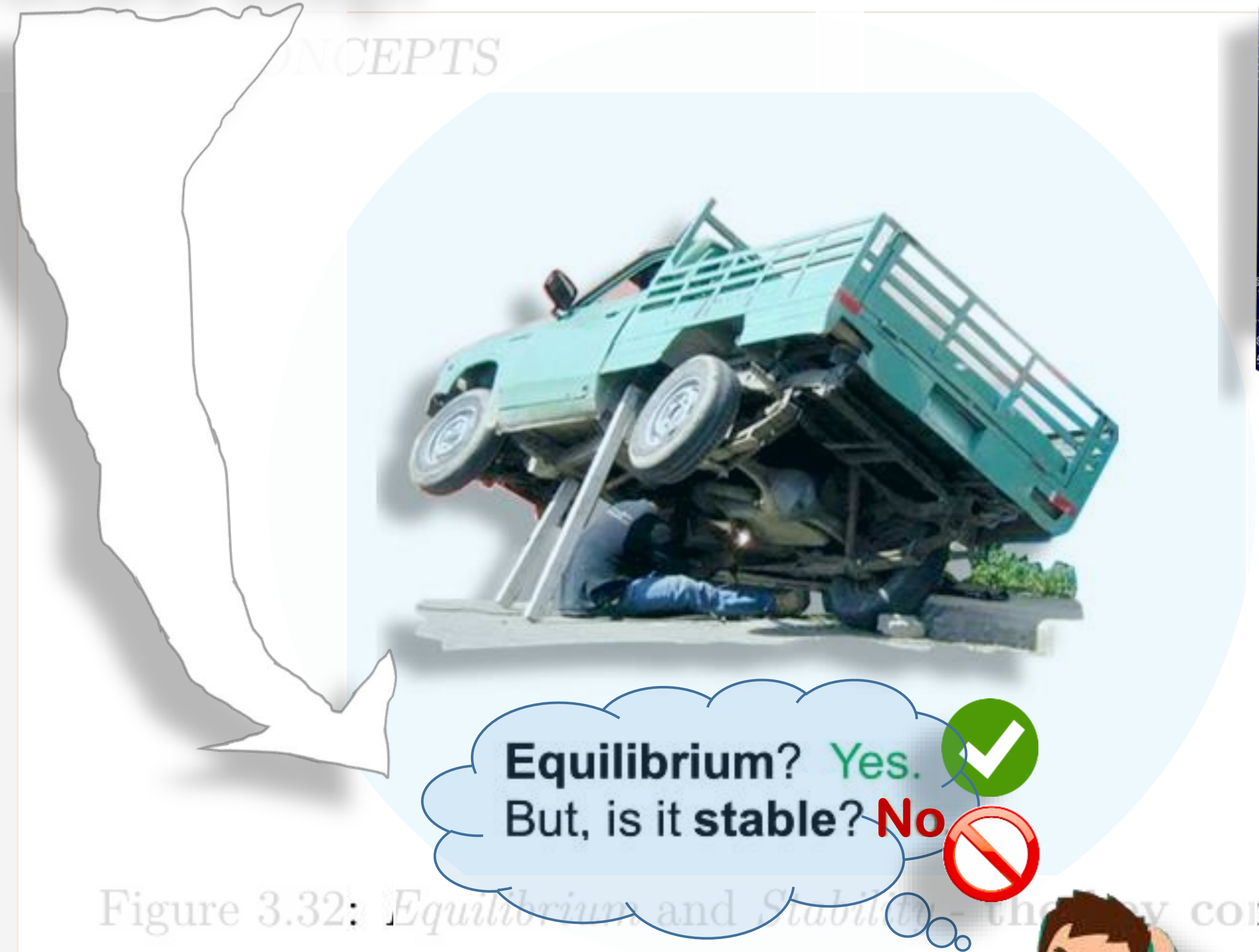
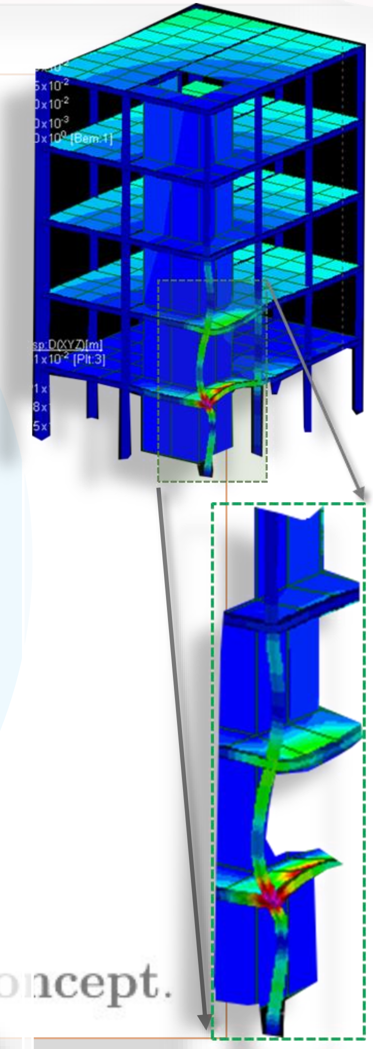
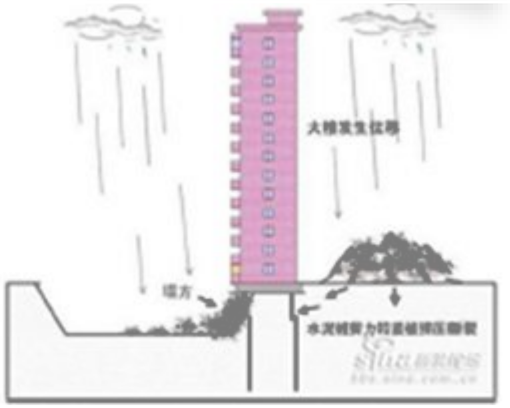


Figure 3.32: *Equilibrium* and *Stability* - the key concept.





Soil material unstability

Lecture slides for internal use only

D. Baroudi, Dr.

The Fundamental Questions

Here the content of this course in four points through questions that will be addressed:

1. can we predict the buckling (critical) load?
2. what happens at the bifurcation (or limit) point? (*i.e.*, after the buckling)
3. can we determine the post-critical branches? What would be their shape? Nature of stability?
4. what imperfection-sensitive is the structure under study?

- **analytical approach** for very simple problems **possible** like buckling of simple members or substructrues, columns, frames, plates, symmetrical shells...
- analytical approach for very simple structures - assymptotic analysis

- for many practical problems, need **numerical** and **experimental** approaches for usuable results in structural design

why need for theory then?

Understanding the theory is necessary 1) to correctly design experiments and 2) to interpret their results.

This is even more true, for designing and doing correctly (=reliably) the numerical simulations and interpreting correctly their resuts.

All **real structural** systems are **imperfect**

- ✓ in form,
- ✓ in material properties,
- ✓ in the sense of residual stresses
- ✓ in the way the loads are applied

Structural design and stability

Standards: design of steel structures

- Local buckling .... EN 1993-1-5
- Flexural buckling .... EN 1993-1-1 hot rolled columns
- Lateral torsional buckling .... EN 1993-1-1 beams
- Lateral

- Flexural torsional buckling ....
- Local-global ....
- Distortional ....
- Shear buckling

EN 1993-1-3
EN 1993-1-5

- Shell buckling .... EN 1993-1-6
  - Linear elastic Bifurcation Analysis (LBA) (= linear buckling analysis)
  - Geometrically Non-linear Analysis (GNA)
  - Geometrically Non-linear Analysis with Imperfections
  - ... LA , LBA , GNA , GNIA, ...

(= post-buckling analysis for perfect structure and structure with imperfections)

* * *

Standards: design of wood structures

- Stability issues & imperfections .... EN 1995-1-1

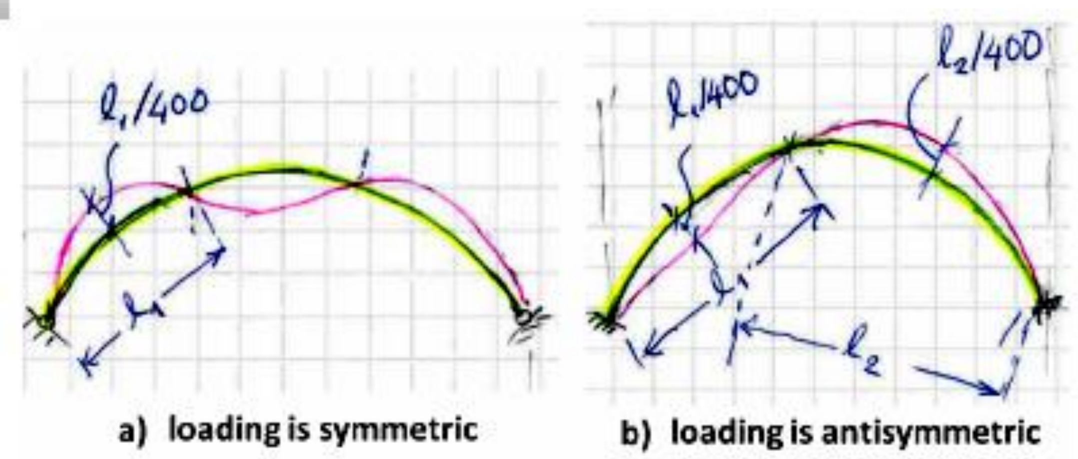
Standards: design of concrete structures

- **Sect. 5.8 Second order effects with axial load..... EN 1992-1-1**

Some standards related to stability issues in structural design.

+ Eurocode 7, geotechnical design

- Slope stability
- Pile stability (foundations)
- ...

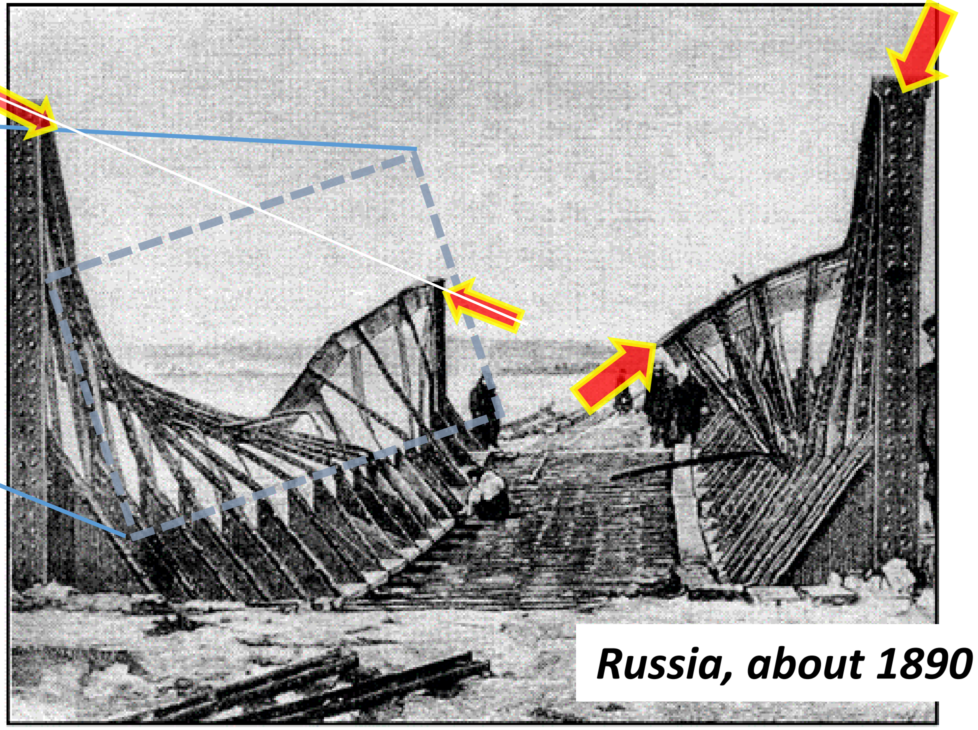


Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Example of initial shape imperfections in wooden arches to be accounted in the structural analysis.

Foot bridge (ramp) collapse in Jiujiang City (China's Jiangxi)

Railway bridge collapse, Russia ~1890

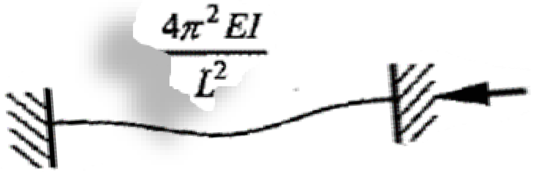


Russia, about 1890

fig. 8.3. - Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).

buckling

Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).

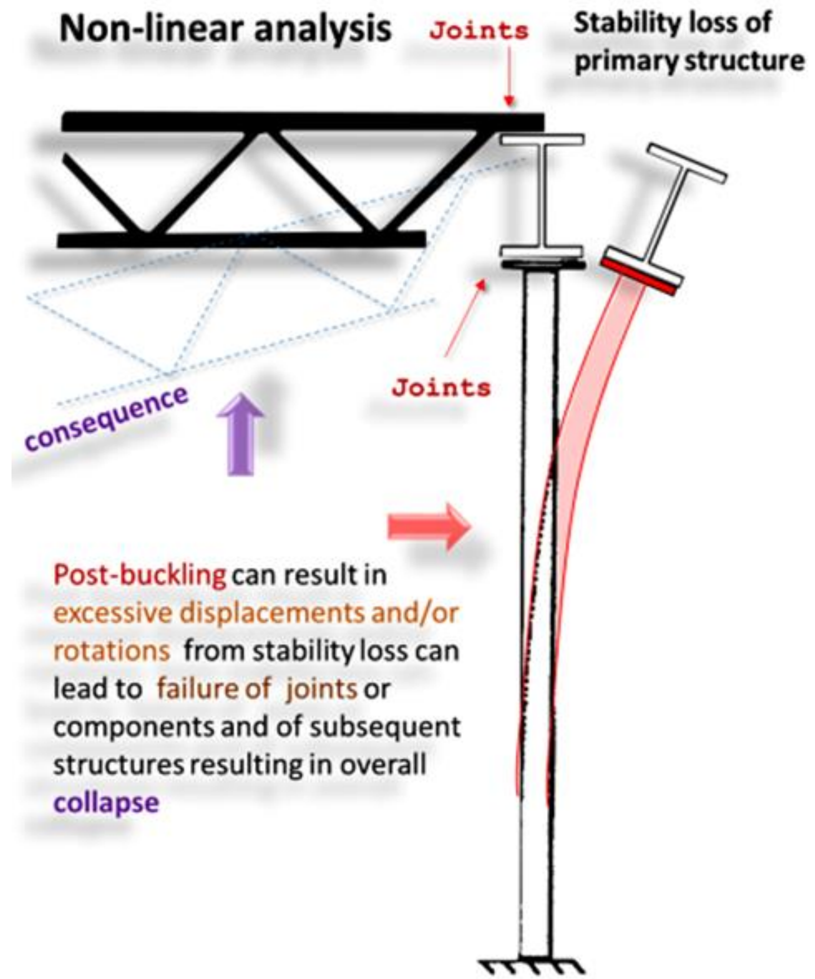


$$P_{cr} = \mu\pi^2 \frac{EI}{L^2}$$

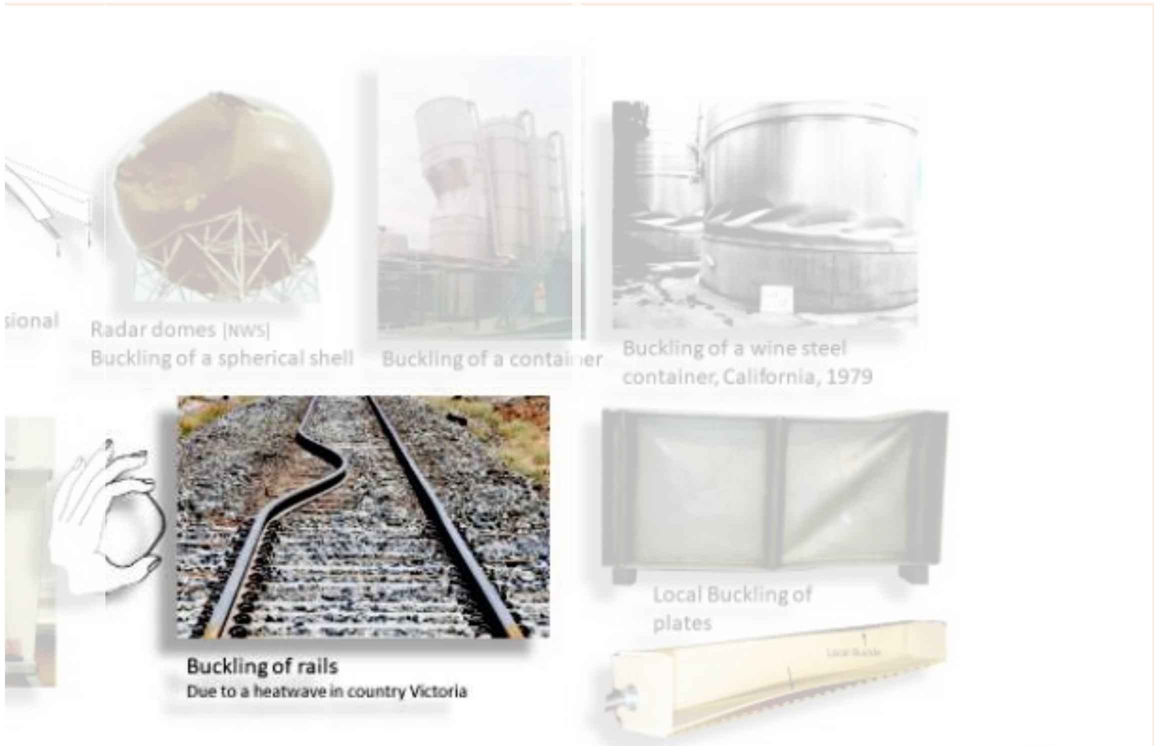
The mechanical cause of the collapse is the same: flexural buckling of compressed upper chord of the truss (yläpaarteen nurjahdus)

Structural design and stability

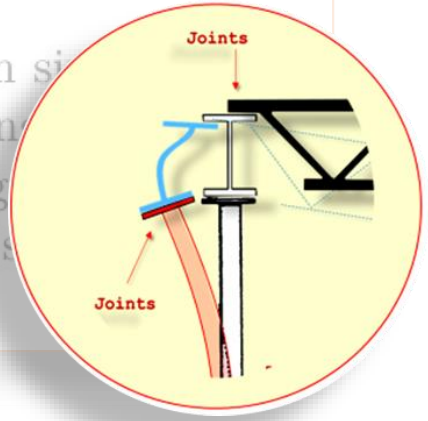
Flexural buckling

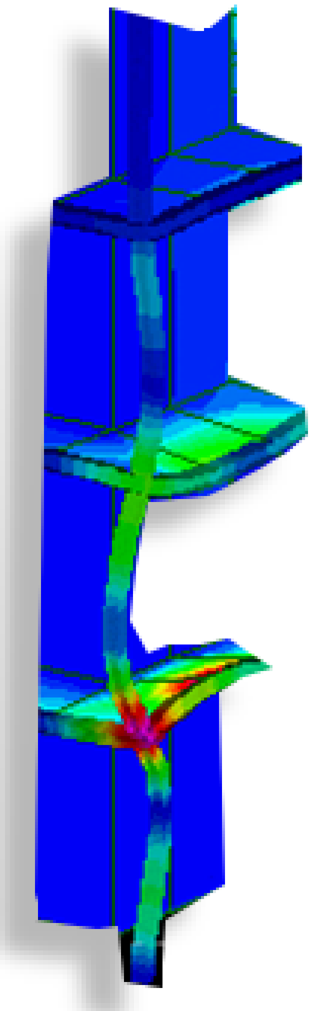


Post-buckling can result in excessive displacements and/or rotations from stability loss can lead to failure of joints or components and of subsequent structures resulting in overall collapse



es of various types of loss of stability in si
ight: lateral-torsional buckling, bucklin





DESPITE OUR INTEREST FOR POST-BUCKLING BEHAVIOR, IN *STRUCTURAL DESIGN*, **STABILITY LOSS** IS AN UNWANTED EVENT.

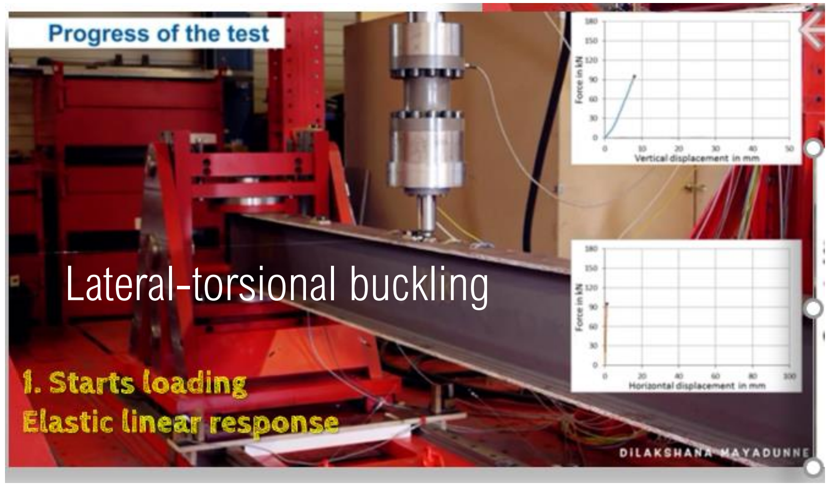
However, bifurcational buckling exists only for a non-existing perfect structure and thus GNA should be performed to find the limit-load, if any.

Mechanical meaning of stability loss

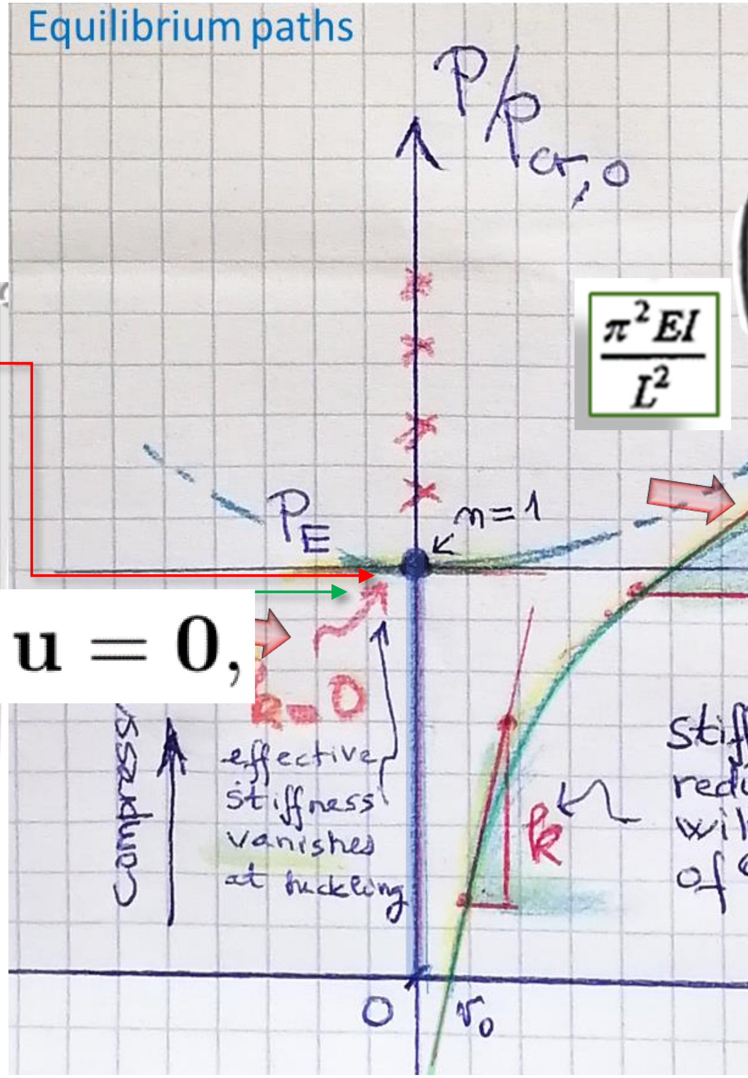
Loss of stability means loss of effective (apparent) rigidity K of the stucture = (nearly or) horizontal tangent on the load-displacement curve (=stiffness matrix K becomes singular)

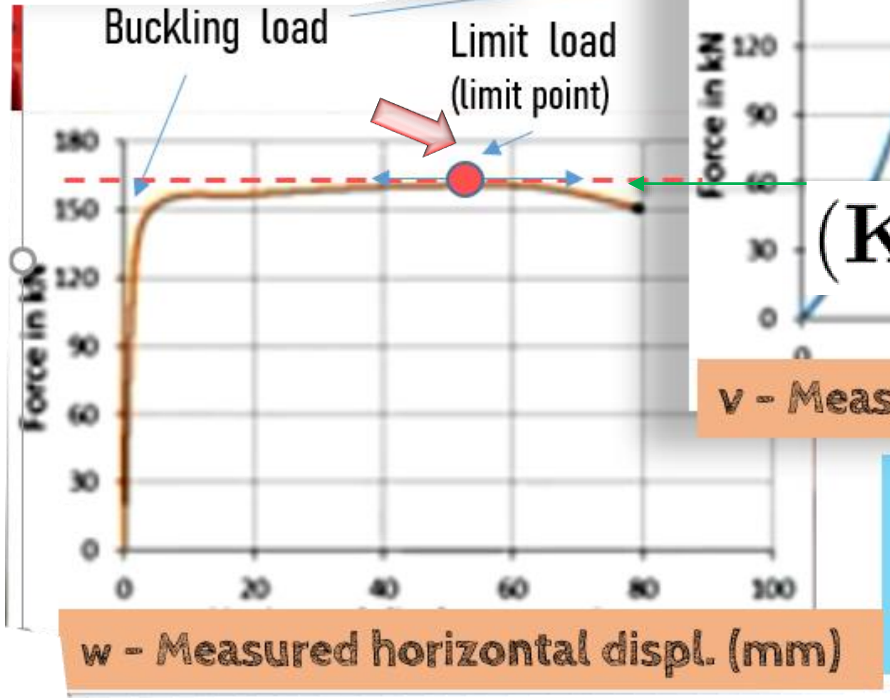
$$(\mathbf{K} - \lambda \mathbf{K}_G)\,\mathbf{u} = \mathbf{0}$$

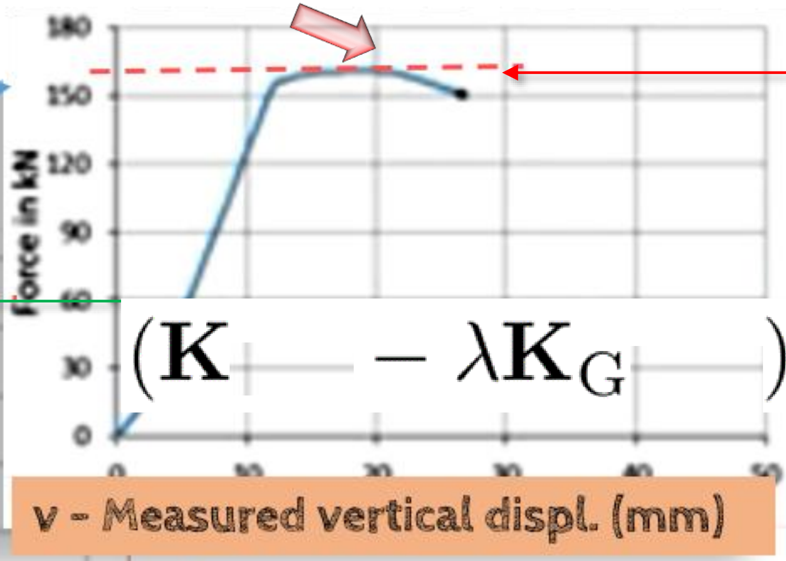
effective rigidity



Load-displacement curve







$$(\mathbf{K} - \lambda \mathbf{K}_G)\,\mathbf{u} = \mathbf{0},$$

Experimental load-displacement curves = equilibrium paths

The following slides:

Stability theorem of **Lagrange-Dirichlet** &
Trefftz stability loss criteria

are mainly a recall from last week meant for **Self-reading**

the reader can jump directly to
the new topic (slide 18):

Post-buckling analysis:

GOTO slide 18

Stability theorem of **Lagrange-Dirichlet** & **Trefftz** stability loss criteria

Self-reading

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).* $\Longrightarrow$

Trefftz *stability loss* criterion

$$\delta(\delta^2\Pi) = 0.$$

(This theorem is more general than **Trefftz** stability loss criteria)

$$\Delta\Pi = \Pi(u^0+\delta u) - \Pi(u^0) = \underbrace{\delta\Pi|_{u_0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u_0} + \frac{1}{3!}\delta^3\Pi|_{u_0} + \ldots$$

stability loss criteria

$$\Pi'' = 0 \longleftrightarrow \delta(\Delta\Pi) = 0 \Longrightarrow \delta\left(\frac{1}{2}\delta^2\Pi|_{u_0} + \frac{1}{3!}\delta^3\Pi|_{u_0} + \ldots\right) = 0$$

$$\delta\Pi^0 = \delta\Pi|_{u_0} = 0 \ (u^0 \text{ -equilibrium initial state})$$

keeping only the quadratic terms, one obtains the energy criterion

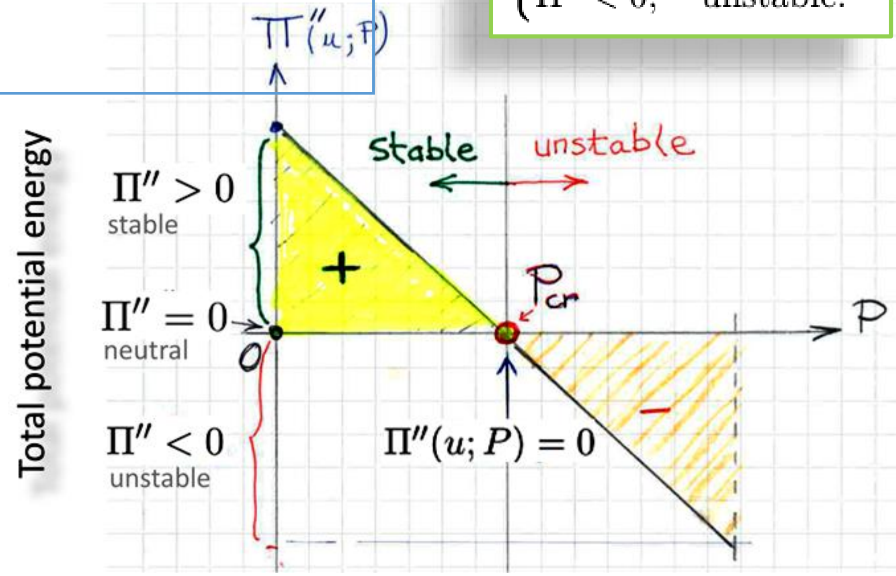
$$\Longrightarrow \delta(\Delta\Pi) = 0 \Longrightarrow \delta(\delta^2\Pi) = 0,$$

More general criterion than Trefftz

Trefftz *stability loss* criterion

stability loss criteria

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$



Trefftz is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

It is this form of criticality condition that will be used systematically thorough this course to derive the stability loss equations for all our structures

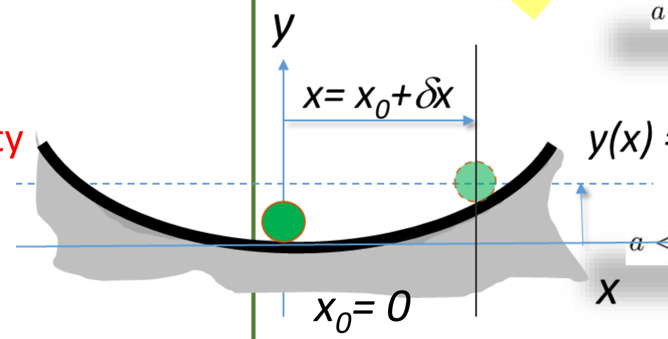
Energy criteria for determination of instability of elastic structures

Self-reading

RECALL

Let's illustrate mathematically the basic **stability types**

- stable
- Indifferent $\Delta\Pi = 0.$ ← this will be one condition for loss of stability
- unstable
- keeping a simplified example of the rigid ball (null strain energy)

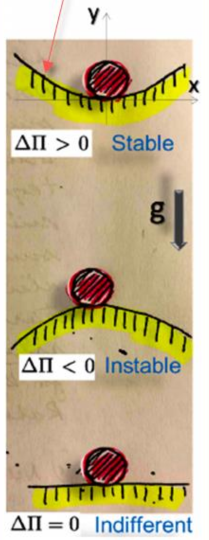


Geometry locally approximated $y(x) = ax^2$

$a > 0$ for convex,

$a < 0$ for concave

$a = 0$ for the neutral



Three various types of equilibrium configurations.

The total potential energy of the system $\Pi(x) = \Pi_0 + mgax^2$

Initial total potential energy (Π_0); potential energy of gravitation ($mgax^2$)

perturbed equilibrium position

$$\Pi(x_0+\delta x) = \Pi(x_0) + \underbrace{\frac{\mathrm{d}\Pi(x)}{\mathrm{d}x}|_{x_0}\delta x}_{\delta\Pi|_{x_0}} + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}|_{x_0}(\delta x)^2}_{\delta^2\Pi|_{x_0}} + \frac{1}{3!}\underbrace{\frac{\mathrm{d}^3\Pi(x)}{\mathrm{d}x^3}|_{x_0}(\delta x)^3}_{\delta^3\Pi|_{x_0}} + \ldots$$

Initial equilibrium position (x_0)

$$\equiv \Pi(x_0) + \delta\Pi|_{x_0} + \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \ldots$$

Since x_0 is an equilibrium then $\delta\Pi|_{x_0} = 0.$

The sign of $\Delta\Pi$ **gives the full information about the stability behavior**

$$\Longrightarrow \Delta\Pi = \Pi(x_0+\delta x) - \Pi(x_0) = \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \ldots$$

The sign will provides us the nature of **stability**

The idea is the make the study of stability in terms of *variational calculus*

Equilibrium? Yes.
But, is it **stable**? No.

$\Pi'' = 2mga.$

… or equivalently

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

Energy criteria for determination of instability of elastic structures

Self-reading

First, keep only up-to the second order[21] term:

$$\Delta\Pi = \frac{1}{2}\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}\Big|_{x_0}(\delta x)^2 = mga(\delta x)^2 + O(\delta x)^3.$$

Consequently, the initial equilibrium x_0 is stable when $a > 0$ (locally convex surface), unstable for $a < 0$ (locally concave surface) and indifferent when $a = 0$.

Bellow follows a résumé: At the critical points (equilibrium points), studying the sign of the increment of total potential energy $\Delta\Pi$, makes it possible to make statements on the nature of the actual equilibrium:

1. **stable**: (stabiili) $\Delta\Pi > 0$
2. **indifferent** : (indiferentti) $\Delta\Pi = 0$. Often, the total potential energy increment $\Delta\Pi$ is expanded to second order only (squares of small displacements). In this case, $\delta^2\Pi = 0$ and therefore, higher order terms should be included in the Taylor expansion to decide of the sign of $\Delta\Pi$ to disclose the character of indifferent equilibrium.
3. **unstable**: (labiili, epästabiili) $\Delta\Pi < 0$

So, the criticality condition: $\Delta\Pi = 0.$

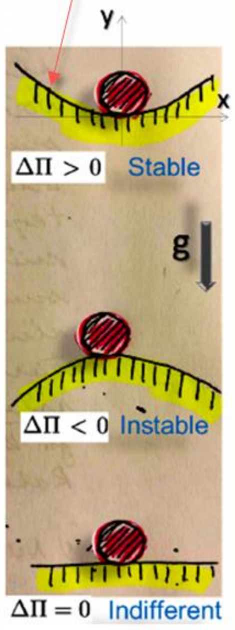
RECALL

Geometry locally approximated $y(x) = ax^2$

$a > 0$ for convex,

$a < 0$ for concave

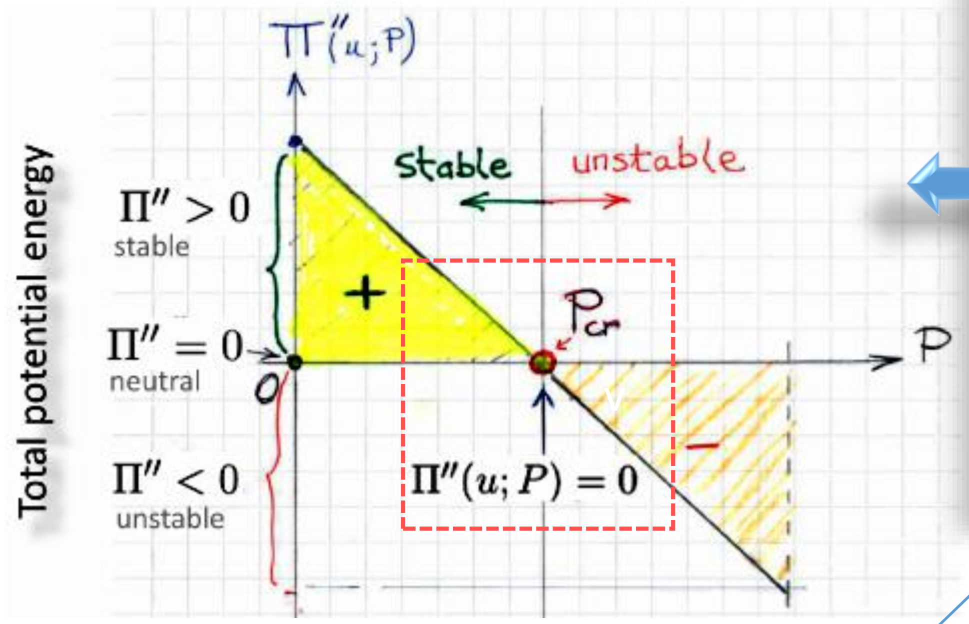
$a = 0$ for the neutral



Three various types of equilibrium configurations.

Equilibrium? Yes.
But, is it **stable**? No.

The criteria of loss of stability



This is a **Taylor expansion** of a function

$$\Pi(\underbrace{\mathbf{q}^0+\delta\mathbf{q}}_{\mathbf{q}}) = \Pi(\mathbf{q}^0) + \sum_{i=1}^{N}\frac{\partial\Pi}{\partial q_i}\Big|_{\mathbf{q}^0}\cdot\delta q_i + \frac{1}{2!}\sum_{i,j=1}^{N}\underbrace{\frac{\partial^2\Pi}{\partial q_i\partial q_j}\Big|_{\mathbf{q}^0}}_{\equiv\mathbf{H}(\mathbf{q}^0)}\cdot\delta q_i\delta q_j + \dots$$

Self-reading

$$\approx \Pi(\mathbf{q}^0) + \underbrace{\underbrace{[\nabla\Pi(\mathbf{q}^0)]^{\mathrm{T}}}_{=0,\ \text{equilibrium}}\delta\mathbf{q}}_{\equiv\delta\Pi} + \underbrace{\frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}}_{\equiv\delta^2\Pi} + \mathcal{O}(\|\delta\mathbf{q}\|^3),$$

at equilibrium $(\delta\Pi = 0)$.

$$\Delta\Pi = \delta^2\Pi + \mathcal{O}(\|\delta\mathbf{q}\|^3) \sim \frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}$$

More suitable form for finite number of dofs and continuous case

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system contain-ing ... if the total potential energy of the system has a strict minimum ...*

A **Taylor expansion** of a function

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

Leading term for sign change in the increment of total potential energy

$$\delta(\Delta\Pi) = 0 \implies \delta(\delta^2\Pi) = 0,$$

Trefftz condition

$$\Pi''(u;P) = 0 \text{ or more generally, } \delta(\Delta\Pi) = 0,$$

It is tis form of criticality condition that will be used systematically thorough this course to derive the stability loss equations for all our structures

Physically speaking, this condition means simply that the perturbed state is also an equilibrium state; thus an neighboring equilibrium exists

Self-reading

Why saying that $f'(x_0) = 0$ is equivalent to say that $\Delta f|_{x_0} = 0$.

- stable
- Indifferent
- unstable

$\Delta\Pi = 0.$

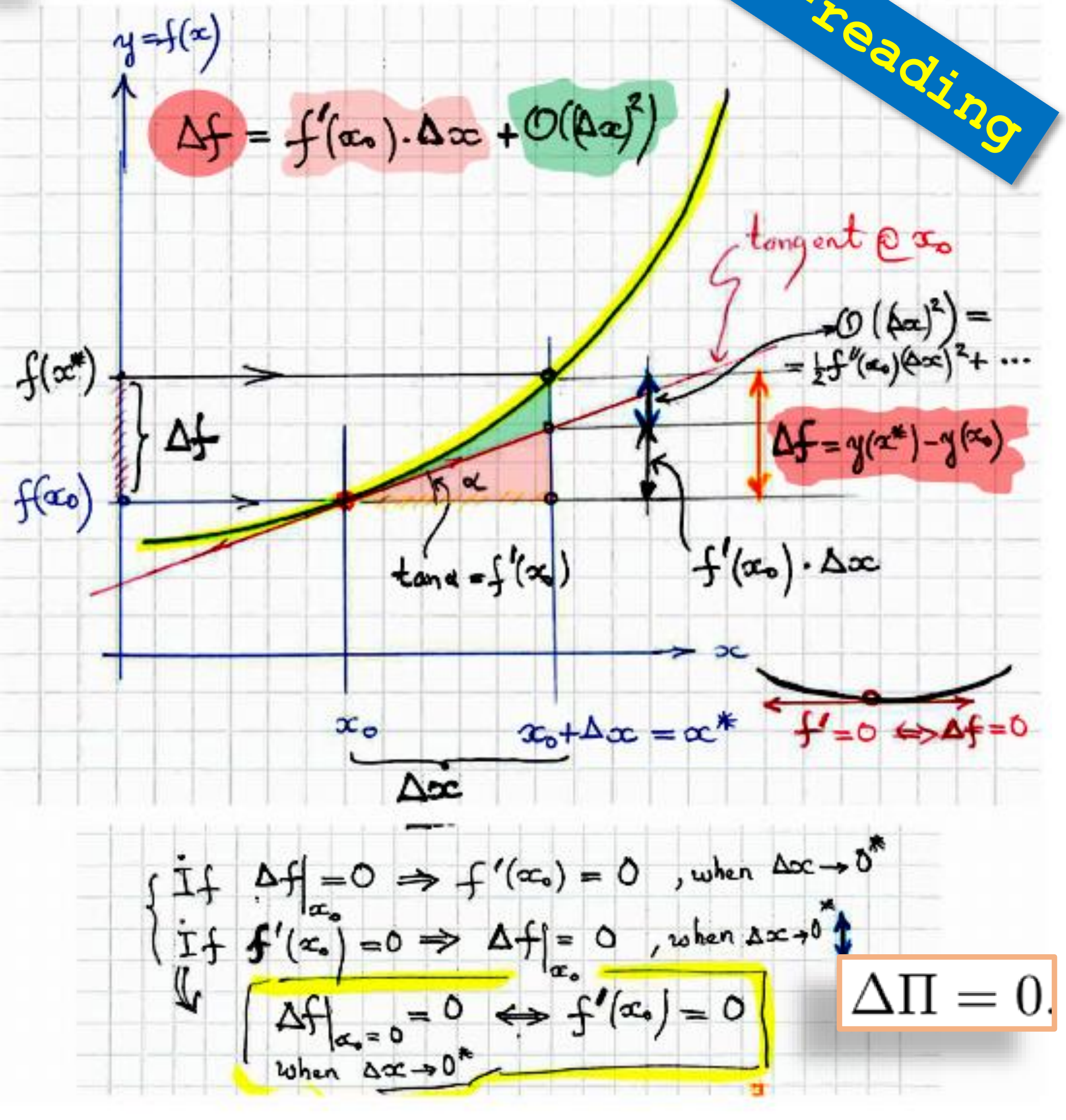
position

$$\Pi(x_0 + \delta x) = \Pi(x_0) + \underbrace{\frac{\mathrm{d}\Pi(x)}{\mathrm{d}x}\Big|_{x_0}\delta x}_{\delta\Pi|_{x_0}} + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}\Big|_{x_0}(\delta x)^2}_{\delta^2\Pi|_{x_0}} + \frac{1}{3!}\,\mathrm{d}$$

Initial equilibrium position

$$\equiv \Pi(x_0) + \delta\Pi|_{x_0} + \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \dots$$

Since x_0 is an equilibrium then $\delta\Pi|_{x_0} = 0$.



$\Delta\Pi = 0.$

Linear buckling analysis

About the criteria of loss of stability – Example with two dofs

$$\Delta\Pi(\epsilon_1,\epsilon_2)=\frac{1}{2}k\ell^2(\epsilon_1^2+\epsilon_2^2)-P\ell\cdot\left(\left[1-\sqrt{1-\epsilon_1^2}\right]+\left[1-\sqrt{1-(\epsilon_2-\epsilon_1)^2}\right]+\left[1-\sqrt{1-\epsilon_2^2}\right]\right)$$

the relative shortenings are defined as $\epsilon_1 = v_1/\ell$ and $\epsilon_2 = v_2/\ell$.

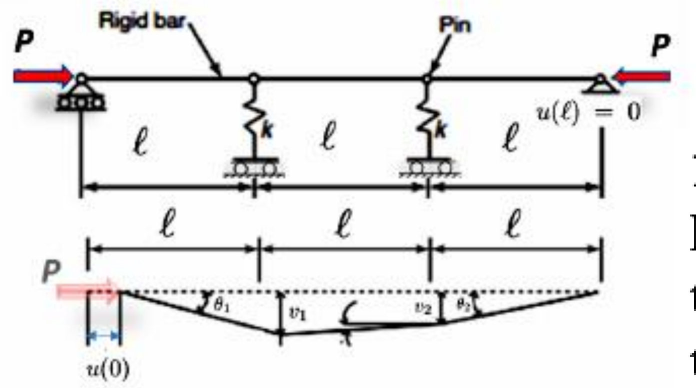


Figure 1.42: A simple system having two degrees of

1) Linear buckling analysis: We want to determine the Euler buckling load. In such analysis we have, by definition, both relative shortening of the column $\epsilon_1 \ll 1$ and $\epsilon_2 \ll 1$, so as the reader may recall, one expands the total potential energy increment into *Taylor expansion up-to quadratic* terms in v_1/ℓ and v_2/ℓ (or ϵ_1 and ϵ_2). So,

$$\Downarrow$$

$$\Delta\Pi(v_1,v_2)=\frac{1}{2}k(v_1^2+v_2^2)-P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2-v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2}{\ell}\right)^2\right]$$

Requiring the neutral equilibrium condition $\delta(\Delta\Pi)=0$ (for loss of stability)

the loss of stability condition in its *variational form* $\delta(\Delta\Pi)=0$

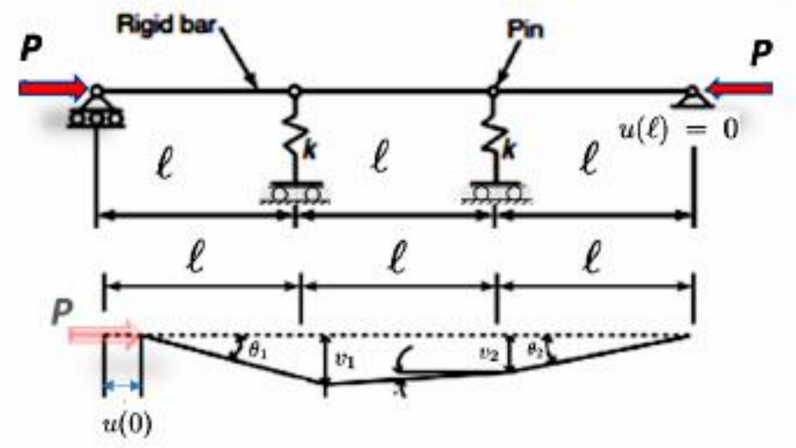
Self-reading

Linear buckling analysis

About the criteria of loss of stability –
Example with two dofs

RECALL

$$\Delta\Pi(v_1,v_2) = \frac{1}{2}k(v_1^2+v_2^2) - P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2 + \frac{1}{2}\left(\frac{v_2-v_1}{\ell}\right)^2 + \frac{1}{2}\left(\frac{v_2}{\ell}\right)^2\right]$$



...re 1.42: A simple system having two degrees of freedom.

$$\Delta\Pi(v_1,v_2) = \frac{1}{2}\begin{bmatrix} v_1 & v_2 \end{bmatrix} \underbrace{\left(\underbrace{\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}}_{\mathbf{K}} - \underbrace{\frac{P}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\mathbf{S}(P)} \right)}_{\mathbf{H}(0,0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (1.68)$$

o, one obtains the *quadratic form*

$$\Delta\Pi(\mathbf{q}) = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}, \quad (1.69)$$

ʼhere $\mathbf{q}$ being a tiny deviation from trivial equilibrium configuration $\mathbf{q}^0 = \mathbf{0}$ nd

$$\mathbf{H} = \begin{bmatrix} \lambda - 2P & P \\ P & \lambda - 2P \end{bmatrix}. \quad (1.70)$$

We can also write directly the loss of stability condition in its *variational form* $\delta(\Delta\Pi) = 0$ and obtain

$$\delta(\Delta\Pi) = \frac{1}{2}\delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q} + \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\delta\mathbf{q} = \delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q} = 0, \forall\delta\mathbf{q} \implies \quad (1.71)$$

$$\implies \mathbf{H}\mathbf{q} = \mathbf{0}\text{, which is linear Eigen-value problem.} \quad (1.72)$$

Requiring the neutral equilibrium condition $\delta(\Delta\Pi) = 0$ (for loss of stability)

Note that the coefficient matrix of the associated Eigen-value problem (Equation 1.66) is the same[60] than our Hessian matrix So loss of stability occurs when

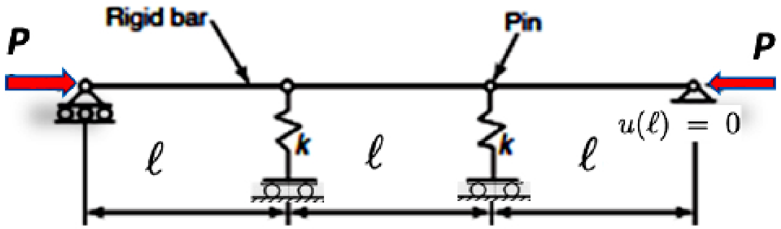
$$\Pi'' = 0 \sim \det\{\mathbf{H}\} = 0 \quad (1.73)$$

Self-reading

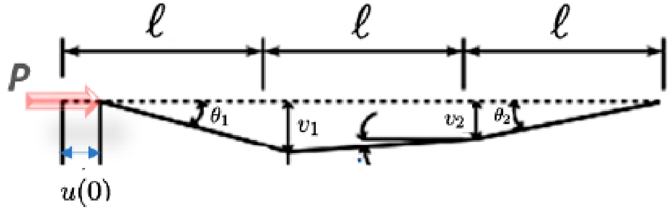
Post-buckling analysis:

$$\Delta\Pi(\epsilon_1,\epsilon_2)=\frac{1}{2}k\ell^2(\epsilon_1^2+\epsilon_2^2)-P\ell\cdot\left(\left[1-\sqrt{1-\epsilon_1^2}\right]+\left[1-\sqrt{1-(\epsilon_2-\epsilon_1)^2}\right]+\left[1-\sqrt{1-\epsilon_2^2}\right]\right)$$

the relative shortenings are defined as $\epsilon_1 = v_1/\ell$ and $\epsilon_2 = v_2/\ell$.



smallest buckling load: $P_{1,E} = k\ell/3$



Post-buckling analysis: *What is the nature of the bifurcated branch just in the near neighbourhood of the bifurcation point $P_{1,E} = k\ell/3$?* For that, we do an asymptotic analysis and take up-to the fourth-order in the Taylor expansion of $\Delta\Pi$. In addition, since we are in the neighbourhood of the buckling load, the ratio $v_1 = -v_2$ as given by the corresponding buckling mode, remains unchanged if we limit ourselves to very small additional deflections v_1 and v_2 from the neutral configuration. (so ratios $v_1/\ell \ll 1$ and $v_2/\ell \ll 1$). Consequently,

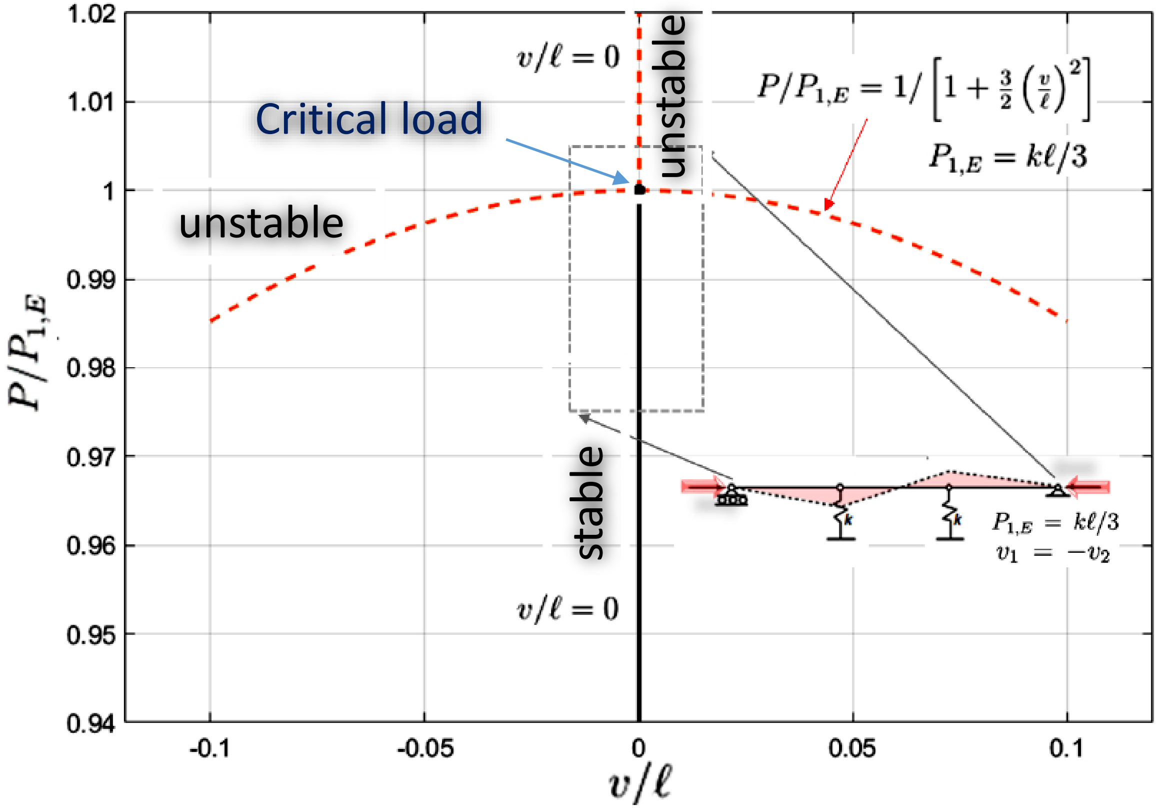
$$\Delta\Pi(v_1,v_2)=\frac{1}{2}k(v_1^2+v_2^2)-P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2+\frac{1}{8}\left(\frac{v_1}{\ell}\right)^4+\frac{1}{2}\left(\frac{v_2-v_1}{\ell}\right)^2+\frac{1}{8}\left(\frac{v_2-v_1}{\ell}\right)^4+\frac{1}{2}\left(\frac{v_2}{\ell}\right)^2+\frac{1}{8}\left(\frac{v_2}{\ell}\right)^4\right].$$

Inserting the relation $v \equiv v_1 = -v_2$, one finally obtains

$$\Delta\Pi(v)=k\ell^2\left(\frac{v}{\ell}\right)^2-3P\ell\left(\frac{v}{\ell}\right)^2-\frac{9}{4}P\ell\left(\frac{v}{\ell}\right)^4$$

$$\delta[\Delta\Pi(v)]=0 \implies [\Delta\Pi]'=0$$

$$\implies k\ell\left(\frac{v}{\ell}\right)\left[1-\frac{P}{P_{1,E}}\left(1+\frac{3}{2}\left(\frac{v}{\ell}\right)^2\right)\right]=0$$



Equilibrium path (asymptotic post-buckling analysis)

NEW Material starts from here …

$$\Delta\Pi[v] = \frac{1}{2}\int_0^\ell EIv''^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2 \mathrm{d}x =$$

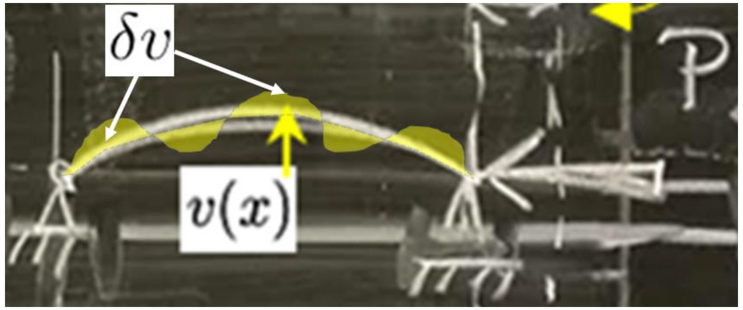
Stability (loss) energy criterion

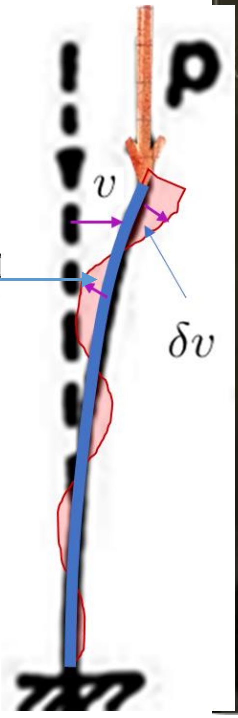
$$\delta(\Delta\Pi[v]) = 0, \forall \delta v$$

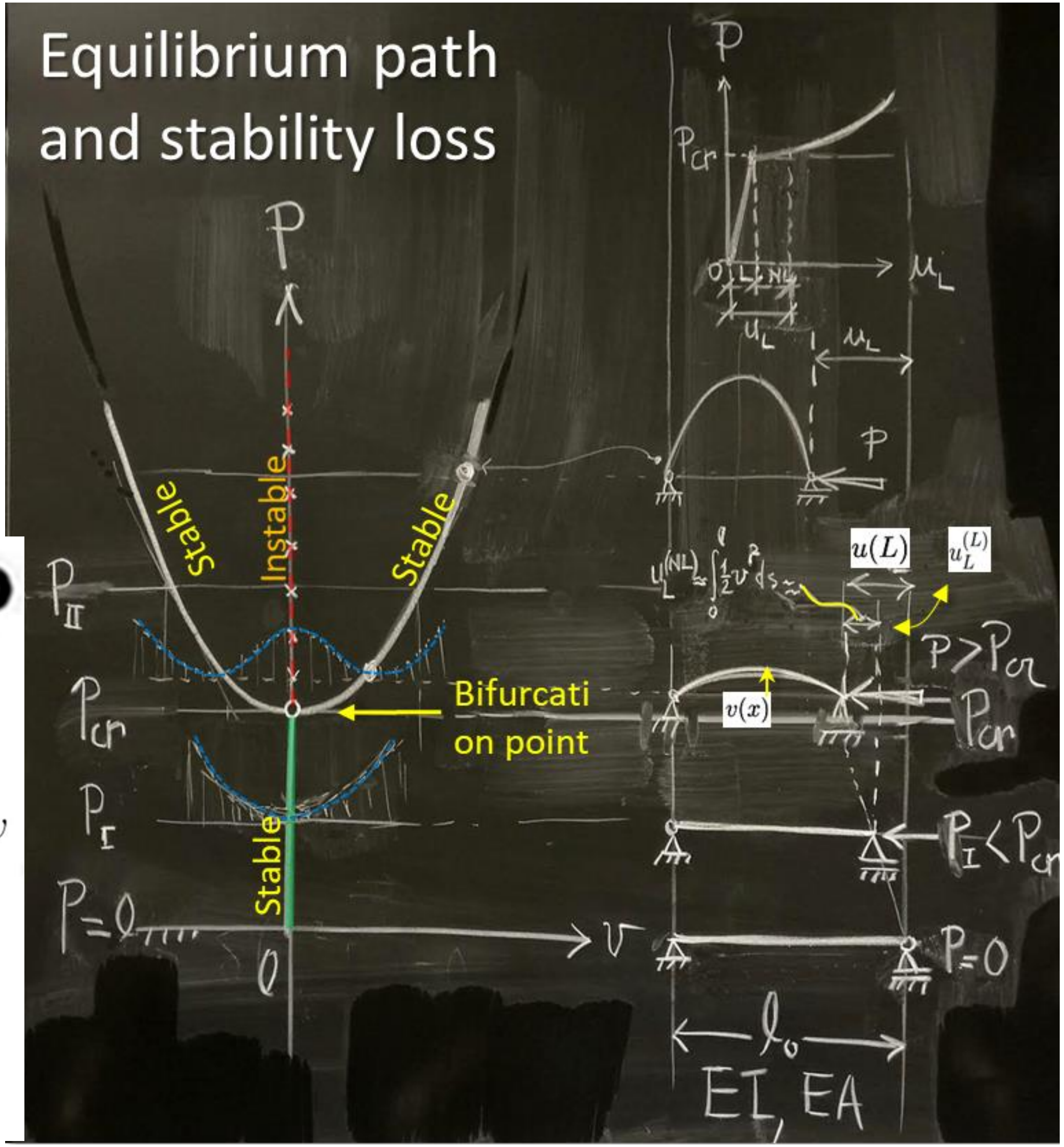
$$\Downarrow$$

Euler-Lagrange equations stability of a colu

$$(EIv'')'' + Pv'' = 0 \quad \& \quad 4 \quad \text{BCs.}$$





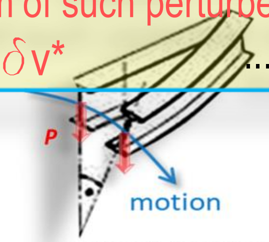


Energy criteria for determination of instability of elastic structures

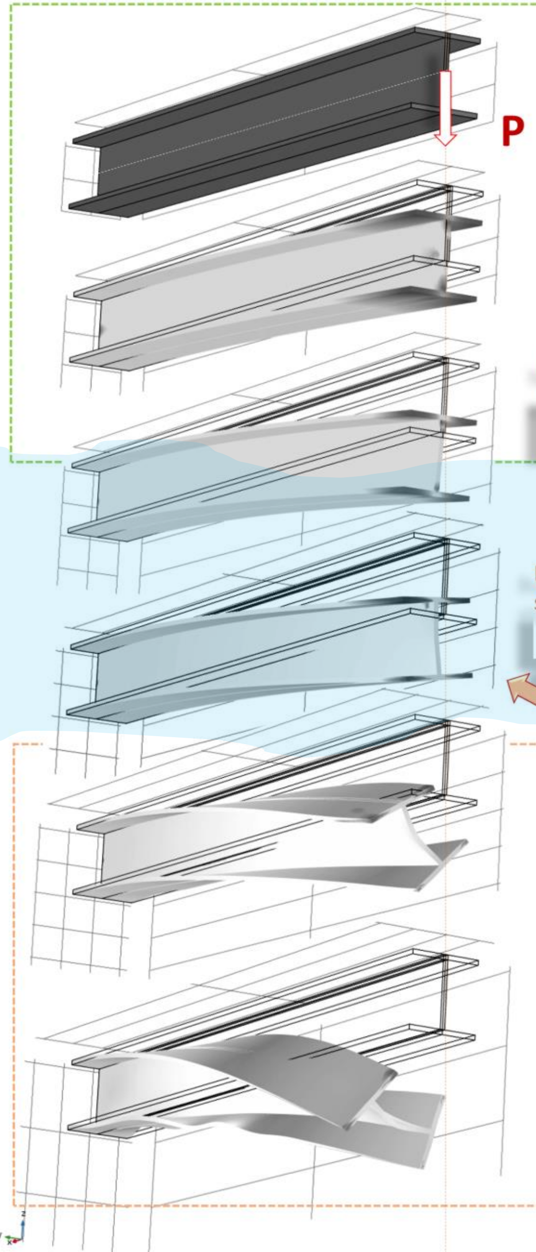
Change of total potential energy **between which** two **states**?

?

N.B. The perturbed configuration [.]* can be thought achieved keeping the load constant and for instance, giving a tiny kinematical perturbation to a an adjacent configuration v* . After that, we test for the equilibrium of such perturbed state by giving it a tiny virtual perturbation δv^* ... this is why we vary the change $\Delta\Pi$



Lateral torsional buckling



Primary *pre-buckling* state

No buckling yet Π^0

Buckling just started Π^*

Lateral-torsional buckling occurs

Post-buckling behavior

$$\Delta\Pi = \Pi^* - \Pi^0$$

$$\Delta\Pi = \Pi(u^0+\delta u) - \Pi(u^0) = \underbrace{\delta\Pi|_{u_0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u_0} + \dots$$

$$\delta(\Delta\Pi) = 0 \implies \delta\left(\frac{1}{2}\delta^2\Pi|_{u_0} + \dots\right) = 0$$

General *stability loss* criterion

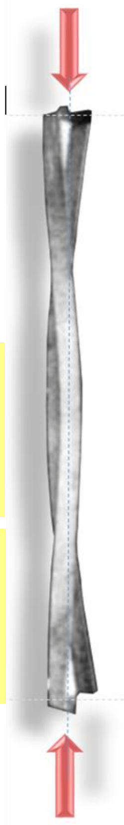
Keeping up to quadratic terms

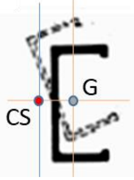
$$\implies \delta(\delta^2\Pi) = 0$$

Trefftz *stability loss* criterion

This criticality condition for bifurcation provides the Buckling Equations

Torsional buckling





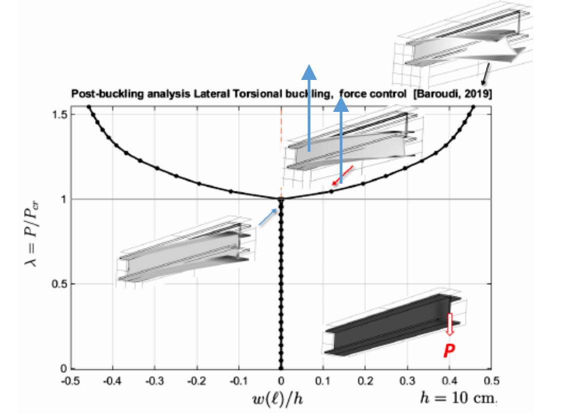


Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

Energy criteria for determination of instability of elastic structures

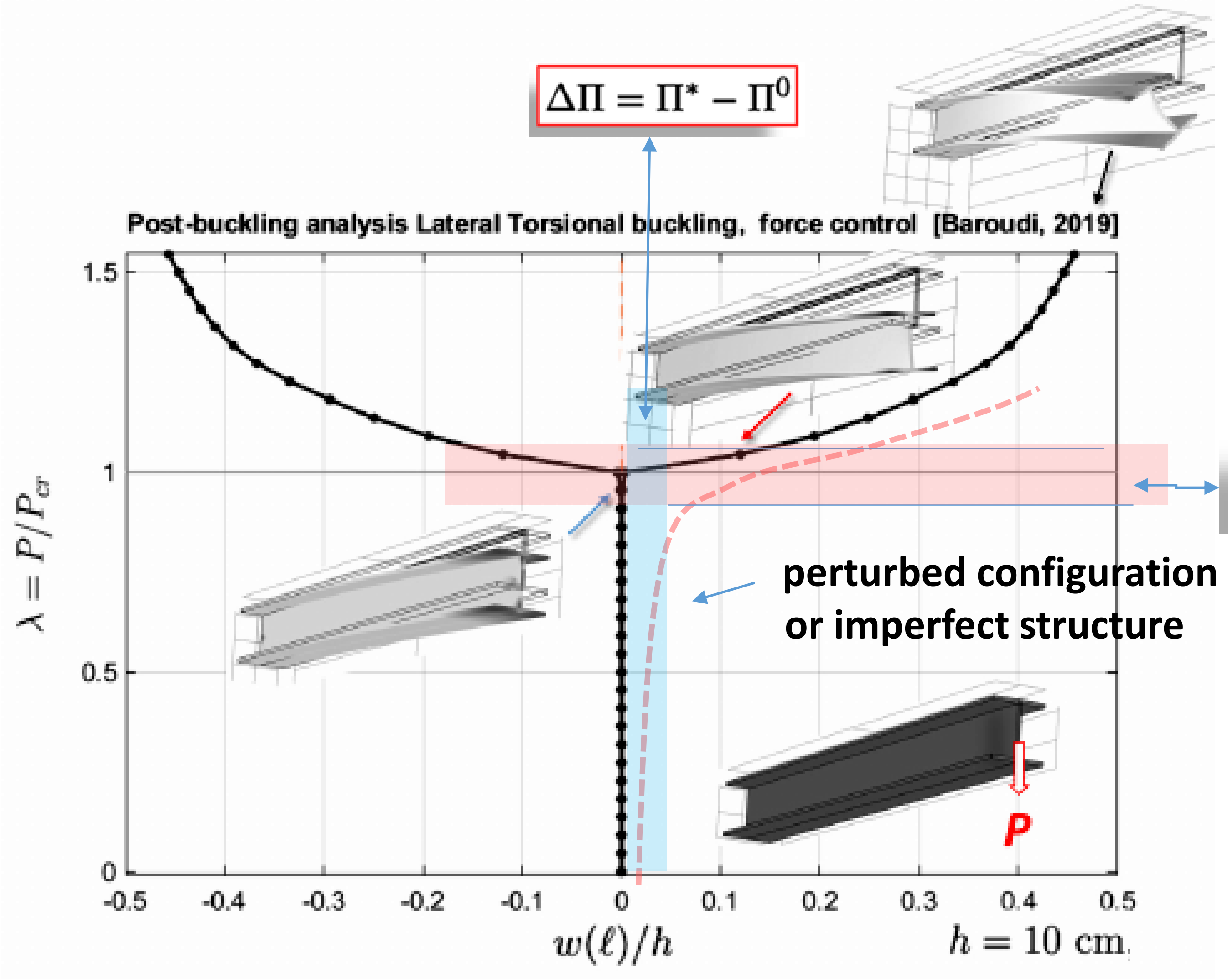
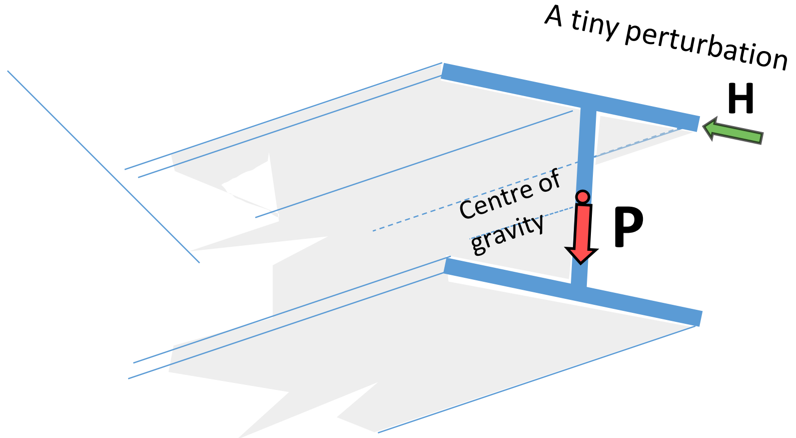


Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

Change of total potential energy **between which** two **states**?

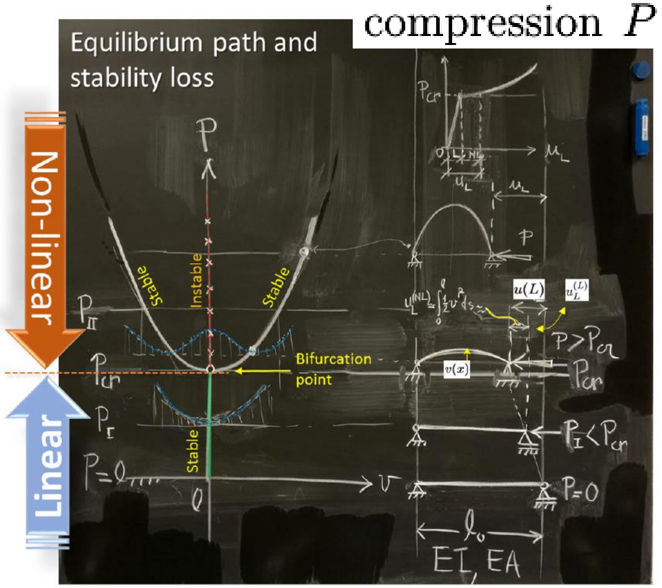
$$\Delta\Pi = \Pi^* - \Pi^0 \implies \delta(\Delta\Pi) = 0$$

H **- a tiny perturbation**



Example of use of stability criteria in the form $\delta(\Delta\Pi) = 0$

compression $P > 0$



Stability (loss) energy criterion

$$\Delta\Pi[v] = \frac{1}{2}\int_0^\ell EIv''^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2 \mathrm{d}x$$

$$\delta(\Delta\Pi[v]) = 0, \forall\delta u$$

$$\Downarrow$$

$$\delta\left(\tfrac{1}{2}\int_0^\ell EIv''^2 \mathrm{d}x - P\int_0^\ell \tfrac{1}{2}v'^2 \mathrm{d}x\right) = 0, \forall\delta u$$

$$= \int_0^\ell EIv''\delta v'' \mathrm{d}x - P\int_0^\ell v'\delta v' \mathrm{d}x = 0$$

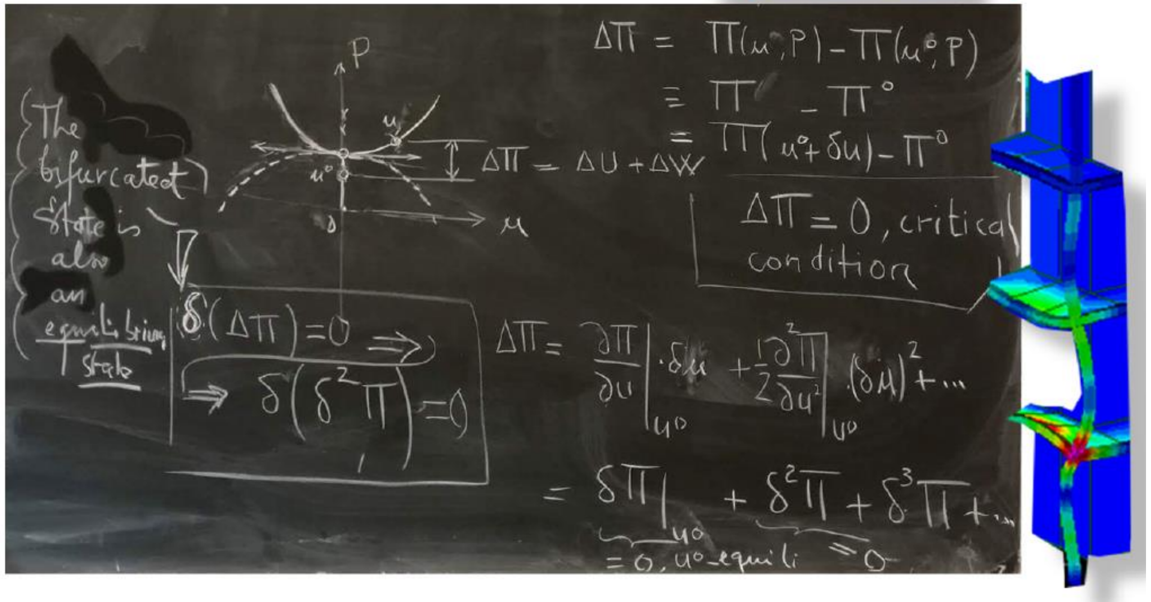
$$\Downarrow$$

Euler-Lagrange equations stability of a column

$$(EIv'')'' + Pv'' = 0 \quad \& \quad 4 \quad \text{BCs.}$$

The above homogeneous differential equation describes the stability problem and its solution provides us the critical buckling load together with the associated buckling-modes once the relevant four boundary conditions are specified.

Critical condition for loss: $\delta(\Delta\Pi) = 0$



Stability of an equilibrium.

Energy criterion of loss of stability (**Bryan** form)

The homogeneous equations of the *elastic-stability* can be derived based on the following three basic methods[73]:

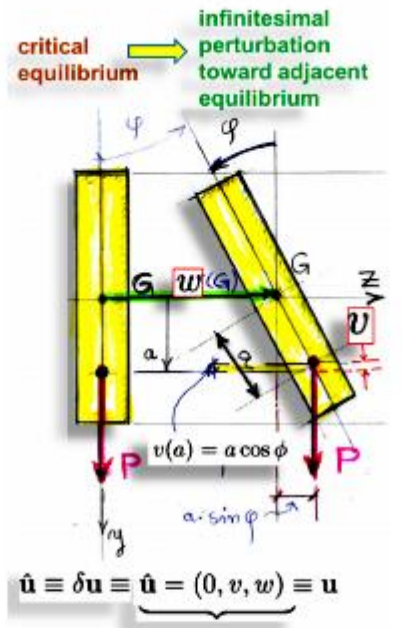
1. applying, systematically, the **energy criteria**[74] for bifurcation stability loss; $\delta(\Delta\Pi) = 0$ at the critical (equilibrium) point. Note that the increment of the total potential energy $\Delta\Pi$ should be, at least, expanded to the accuracy up-to second[75] order (the squares[76].
2. directly writing the **equilibrium equations in the deformed configuration** which stability we are investigating and adjacent to the initial equilibrium state.
3. of course, one can derive first the full (geometrically) **non-linear equations** in the vicinity of the critical point and then **linearise** them near the initial equilibrium point.

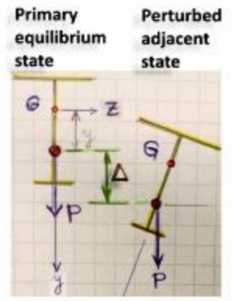
As seen previously, the linear strain-displacement relation is not sufficient for stability analysis. It come out that non-linear effect up to second order should be accounted for.

We use systematically this condition

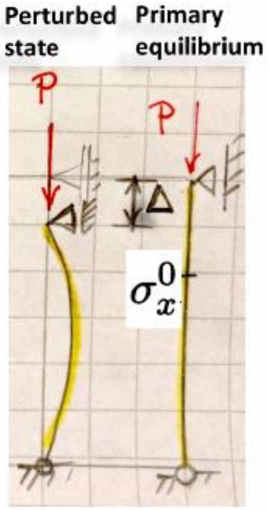
$$\Delta\Pi = \frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V + \int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V.$$

+ should also include increment of work of external work not already accounted in by the work of initial stresses





Additional work $\Delta W_{\mathrm{ext}} = P \cdot \Delta$ (*Lateral torsional buckling*)

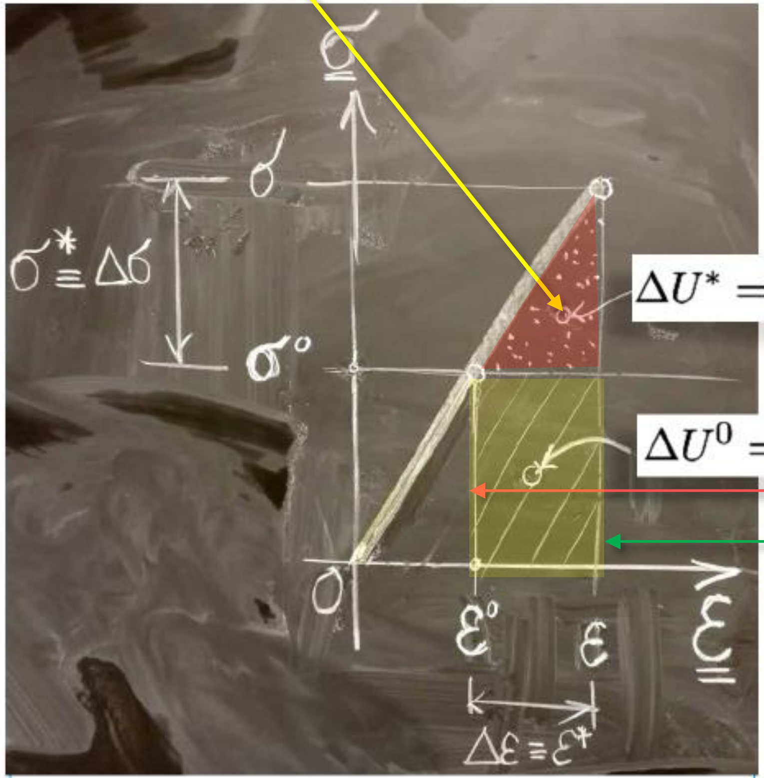


Additional work $\Delta W_{\mathrm{ext}} = P \cdot \Delta$ (*Flexural buckling*)

example in next slide ...

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments in } \Delta W(\sigma^0)}$$

- Additional work of external force not included in the pre-stress

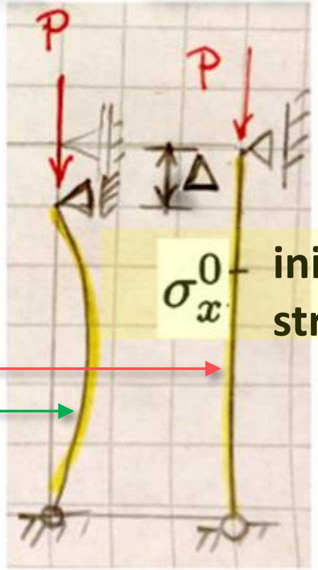


$\Delta U^* = \frac{1}{2}\sigma^*\epsilon^*$

$\Delta U^0 = \sigma^0\epsilon^*$

$\Delta U = \Delta U^0 + \Delta U^* = \sigma^0\epsilon^* + \frac{1}{2}\sigma^*\epsilon^*$

Perturbed state | **Primary equilibrium**



σ_x^0 **initial axial stress**

Example: Buckling of a column

end-thrust $-P = N^0(x) < 0$

The total potential energy increment in Bryan form was

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell N^0(x)(v')^2 \mathrm{d}x,$$

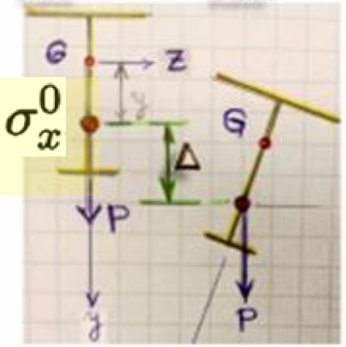
$\epsilon_1 = -yv''(x)$ — Linear part of the strain

$\sigma_x^0 A = N^0(x)$ — Initial stress

$\epsilon_2 = \frac{1}{2}(v')^2$ — Quadratic part of the strain

Primary equilibrium state | **Perturbed adjacent state**

initial bending σ_x^0 **stress**



The strain energy change between reference equilibrium state $\mathbf{u}^0$ and a perturbed neighbouring (equilibrium) state $\mathbf{u}$. The change in strains being $\epsilon^* = \Delta\epsilon = \epsilon - \epsilon^0$ and in stresses $\sigma^* = \Delta\sigma = \sigma - \sigma^0$

Finite deformation (strains)

$$\epsilon_{ij}^* = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

After order of magnitude analysis for the strain increment, and keeping only up-to second order terms (the non-linear (quadratic) part can expressed in terms of rotations) one finally obtains

$$\epsilon_x = e_x + \frac{1}{2}\left(\omega_z^2 + \omega_y^2\right)$$
$$\epsilon_y = e_y + \frac{1}{2}\left(\omega_x^2 + \omega_z^2\right)$$
$$\epsilon_z = e_z + \frac{1}{2}\left(\omega_y^2 + \omega_x^2\right)$$
$$\gamma_{xy} = 2e_{xy} - \omega_x\omega_y$$
$$\gamma_{yz} = 2e_{yz} - \omega_y\omega_z,$$
$$\gamma_{zx} = 2e_{zx} - \omega_z\omega_x.$$

The rotation component

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$
$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

quadratic part

the linear part

$$e_x = \frac{\partial u}{\partial x}, \qquad e_y = \frac{\partial v}{\partial y}, \qquad e_z = \frac{\partial v}{\partial z},$$
$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$
$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$
$$e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

Ex. Plate: **quadratic** part of strains

$$\epsilon_{xx}^* = \frac{1}{2}[\underbrace{u_{,x}^2 + v_{,x}^2}_{\approx 0 \ll w_{,x}^2} + w_{,x}^2] \approx \frac{1}{2}w_{,x}^2,$$
$$\epsilon_{yy}^* = \frac{1}{2}[\underbrace{u_{,y}^2 + v_{,y}^2}_{\approx 0} + w_{,y}^2] \approx \frac{1}{2}w_{,y}^2,$$
$$\gamma_{xy}^* = 2\epsilon_{xy}^* = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}.$$

What deformations are significant in buckling?

- In stability analysis while deriving the linear stability loss equations (the linear Eigen-value problem) the amplitude of the linear part e_i of the strains, during the infinitesimal perturbation of the initial equilibrium to the (bifurcated) adjacent one, remains small[a] as compared to changes in the rotation components of ω_i. they will work with initial stresss
- Consequently, the quadratic terms in terms in strains e_i^2 and $\omega_i e_j$ are of second order increments as compared to changes in the rotation components, and for that reason will be dropped (ignored). In the above strain increments expressions, only terms shown in the above strains are retained for stability analysis.
- In addition to that, (*Cf.* Alfutov), terms containing the derivatives of initial primary displacements can be neglected (this, *their contribution to the increment of total potential energy* $\Delta\Pi$ *can be neglected*) too.

[a] As a consequence of the choice of the initial primary equilibrium and the close neighbouring adjacent (bifurcated) equilibrium. These two states are infinitesimally close.

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V \epsilon_1{}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V \epsilon_2{}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments in } \Delta W(\sigma^0)}$$

Advanced reading - these may be too technical details which however are needed when deriving buckling equations

Flexural buckling

$$\delta(\Delta\Pi) = 0 \implies$$ Equations (of loss) of stability

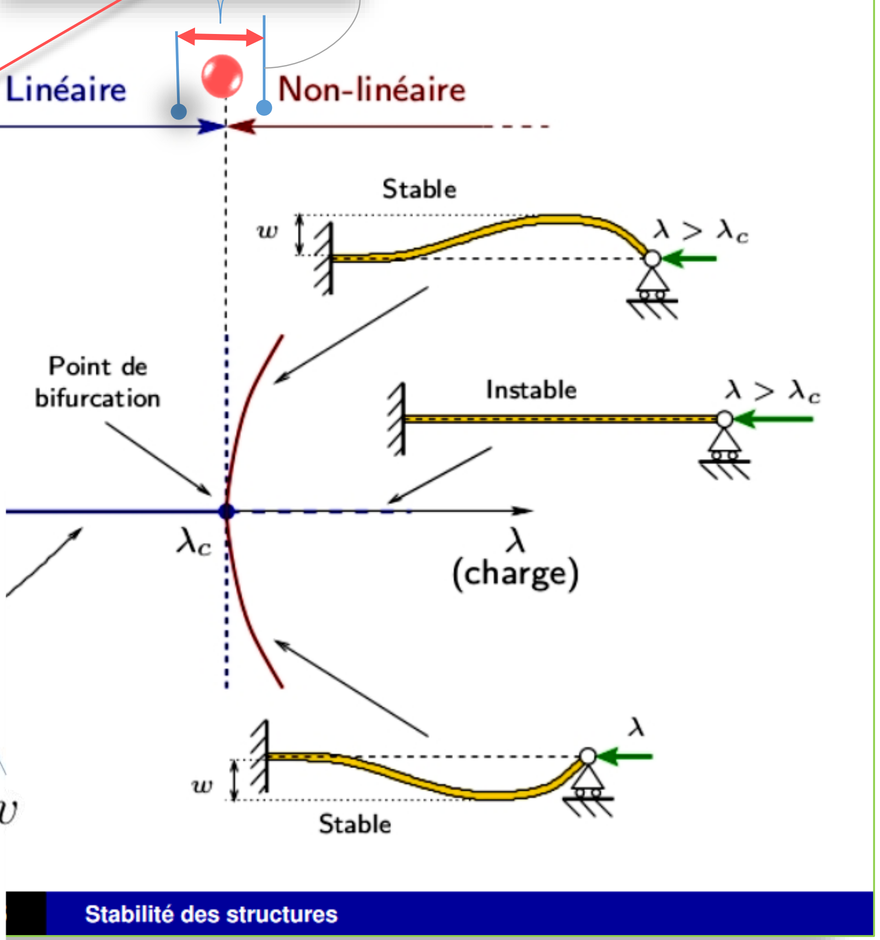
FLAMBEMENT D'UNE POUTRE DROITE

$$\Delta\Pi = \Pi^* - \Pi^0$$

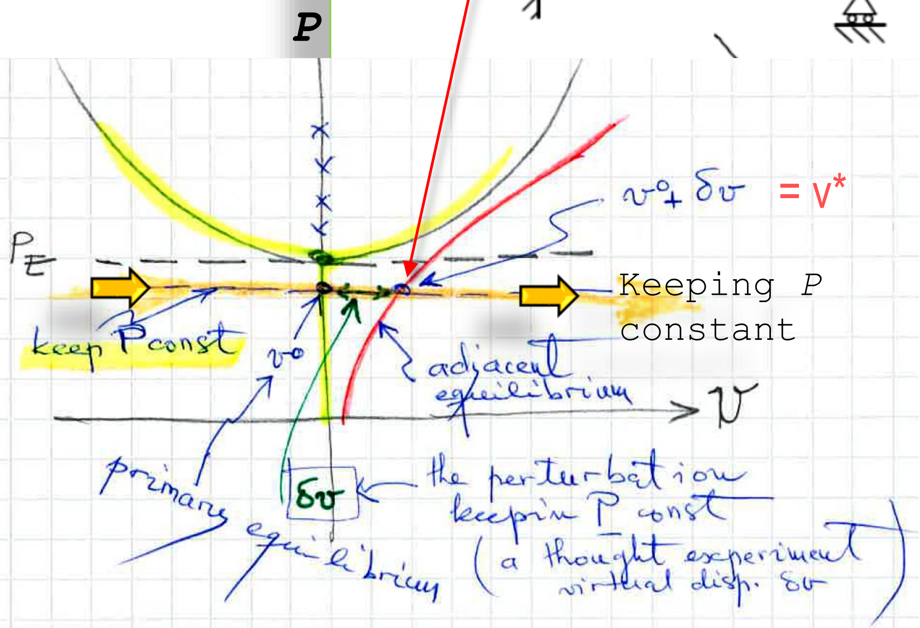
This difference does to zero at buckling

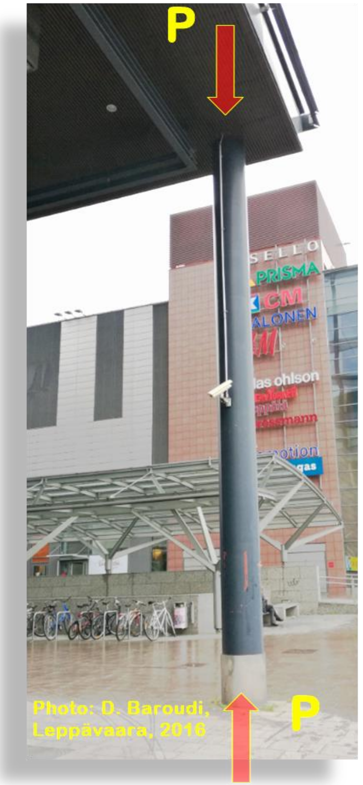
A thought experiment

N.B. The perturbed configuration [.]* can be (also) **thought** achieved keeping the load constant and for instance, giving a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v*



erso.ensta-paristech.fr/~touze/PDF/MS205/amphi1_ms205.pdf





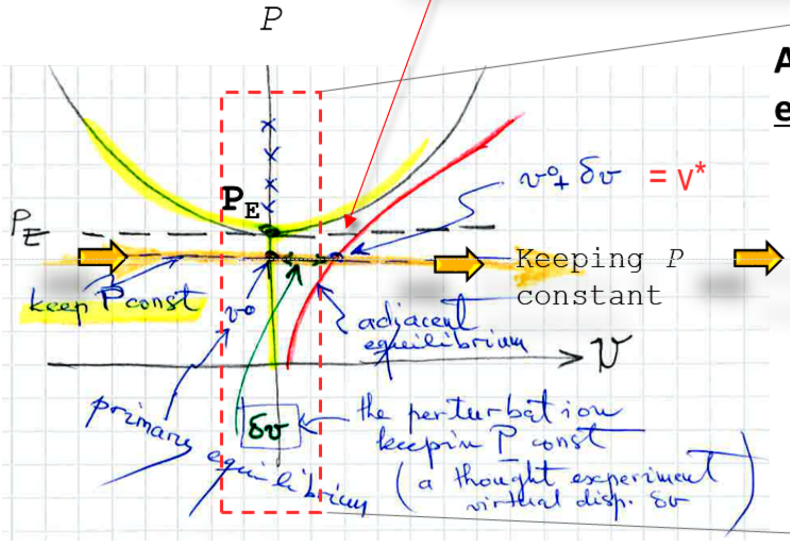
Estimate the critical load!

1) One way to think how form the increment of total potential energy is through a real loading sequence where the load increases quasi-statically and monotonically from zero to the buckling load $P_E^+ = P_E + \varepsilon$ where it buckles where ε being infinitesimally small > 0. The primary non-buckled configuration (primary equilibrium) corresponds to $P_E^- = P_E - \varepsilon$. Now one can form the increment of the total potential energy between these two real states and takes the limit when $\varepsilon \to 0$ to say that we are at the bifurcation or limit-point where now the critical load being P_E.

$$\Delta\Pi = \Pi^* - \Pi^0 \implies \delta(\Delta\Pi) = 0 \implies$$ Equations (of loss) of stability

N.B. The perturbed configuration [.]* can be (also) **thought** achieved keeping the load constant and for instance, giving the primary equilibrium configuration v_0 a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v^*

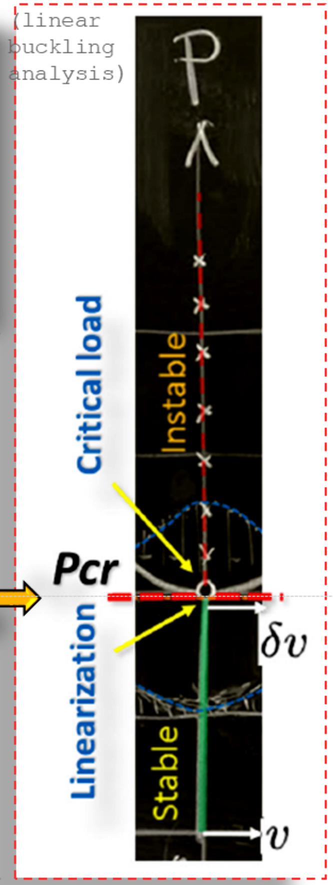
$$\Delta\Pi = \Pi^* - \Pi^0$$



A thought experiment

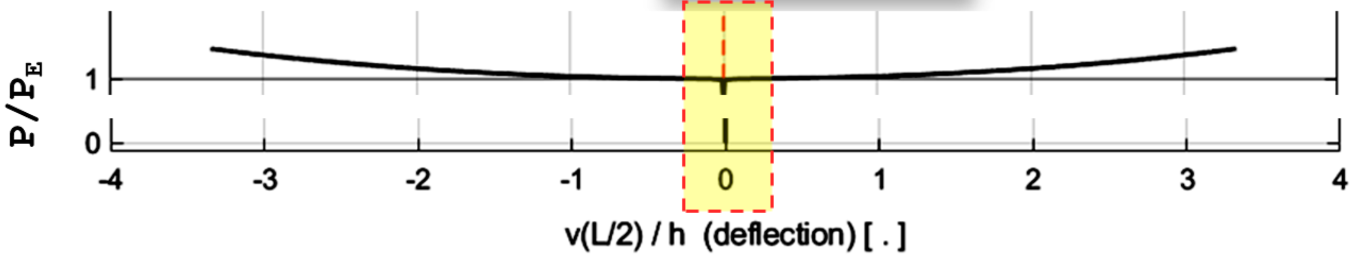
Keeping P constant

Zoom (linear buckling analysis)



2) the other more classical way how form the increment of total potential energy is by a *thought experiment where* we give an infinitesimal virtual perturbation to the primary equilibrium configuration to an adjacent neighbor equilibrium configuration while keeping all the loads unchanged. Then we write the increment od total potential energy between these to states of equilibrium.

A Finite element post-buckling analysis of a simply supported column under axial thrust. *This shows how 'sallow' is the critical point infinitesimal neighborhood*



Buckling of a beam-column

Solutions for some classical cases

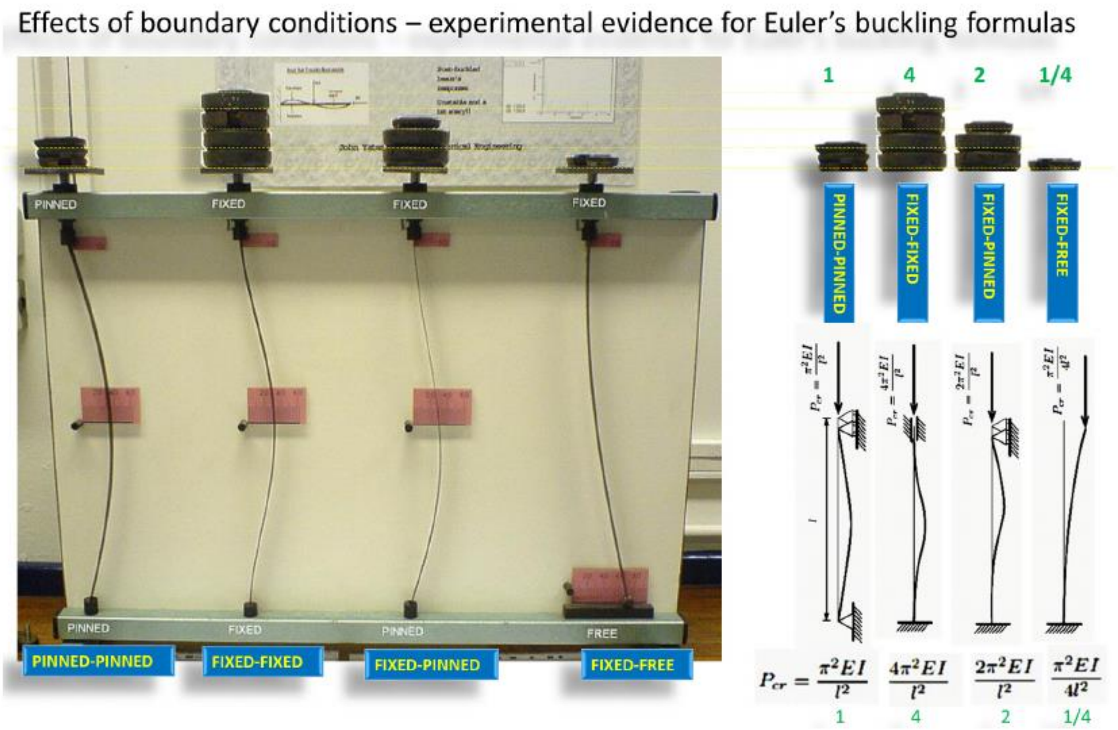
$$P_{cr} = \mu\pi^2 \frac{EI}{\ell^2} \equiv P_E$$

$$\sigma_{cr} \equiv \sigma_E = \frac{P_E}{A} = \mu\pi^2 \frac{EI}{A\ell^2} = \mu\pi^2 E \left(\frac{r_{min}}{\ell}\right)^2 = \mu\pi^2 E/\lambda^2_{min},$$

Critical strain

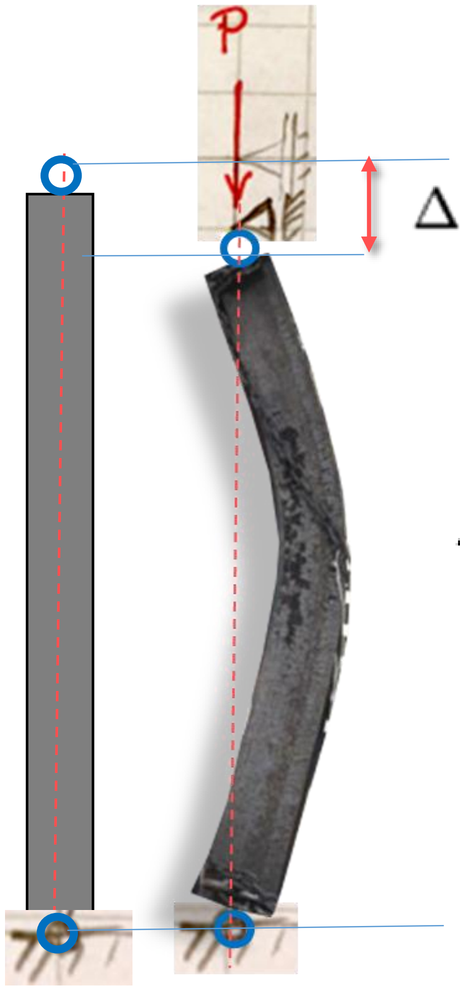
$$\epsilon^0_{cr} \equiv \epsilon_E = \frac{\sigma_E}{E} = \mu\pi^2 \left(\frac{r_{min}}{\ell}\right)^2 = \mu\pi^2/\lambda^2_{min}$$

! does not depend on material properties



Rudimentary experimental evidence for Euler's basic buckling formulas and the effect of boundary conditions on the buckling load.

Buckling of a beam-column



Buckling just started
Π^*

$$\Delta\Pi = \Pi^* - \Pi^0$$

No buckling

$$\Downarrow$$

$$\Delta\Pi = \frac{1}{2}\int_0^{\ell} EI(v'')^2 \mathrm{d}x - \frac{1}{2}P\int_0^{\ell}(v')^2 \mathrm{d}x.$$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

This criticality condition for bifurcation provides the Buckling Equations

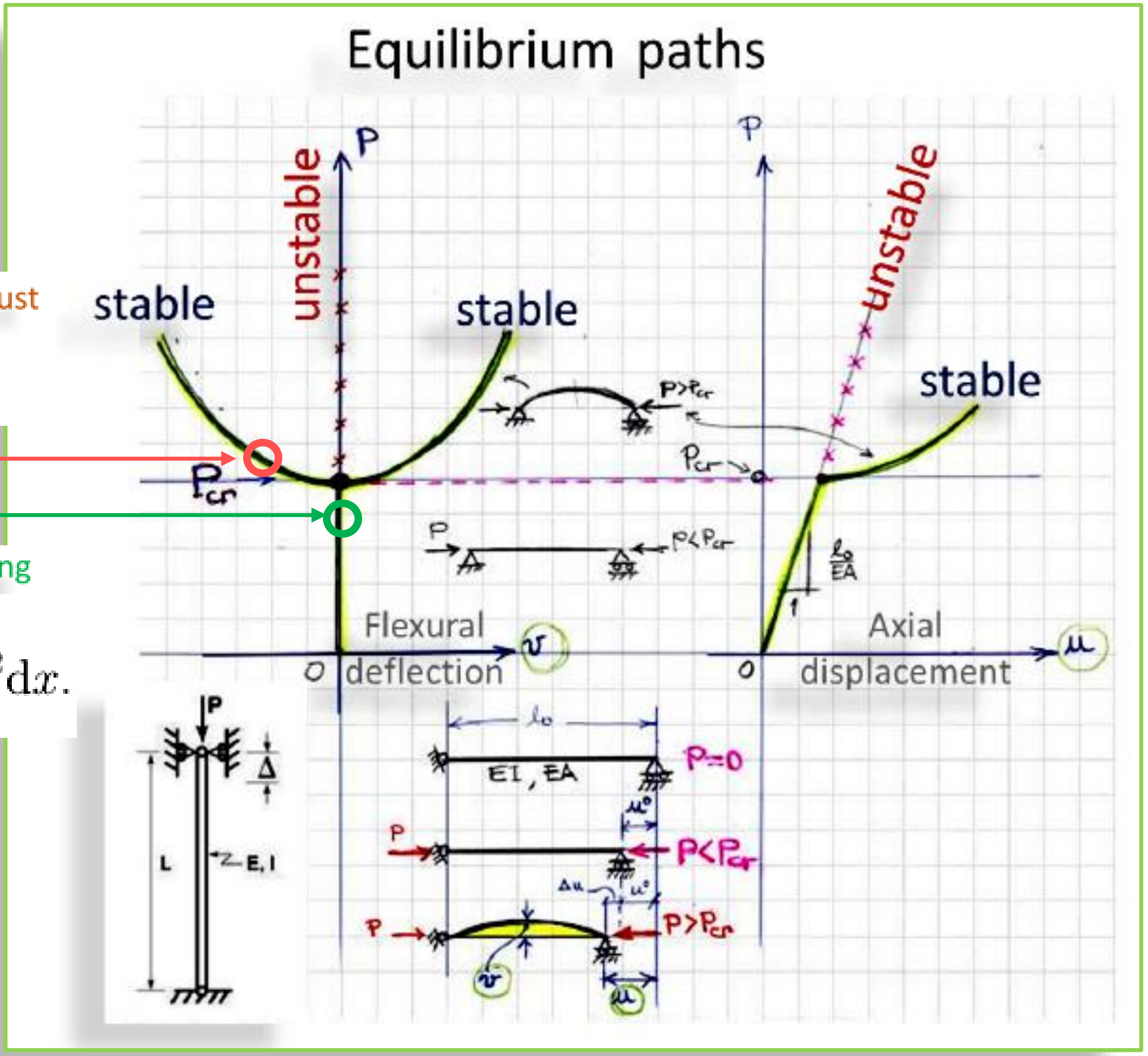


Illustration for equilibrium paths and bifurcation points for perfect structure (no imperfections).

The linearised eigenvalue problem:

$$(EIv'')'' + Pv'' = 0$$

& four boundary conditions.

Linearised theory of buckling

Combined compression and bending

The transition from the straight stretched beam-column equilibrium initial configuration to the neighbour adjacent buckled (flexural) equilibrium state occurs with no additional stretching for very small bifurcational deflection v. Therefore, it is assumed that the changes in length are of higher order. Consequently, the axial force does not changes $N \approx N_0$ from the axial force obtained in the straight state of equilibrium.

assumption needed to determine buckling load and modes as the solutions of a *linearised eigenvalue problem*

σ_x^0

Linear means that $\delta(\Delta\Pi) = 0$ (or generically $\Pi' = 0 \rightarrow$ equations of equilibrium) is linearised with respect to the generalised displacements and rotations

Generally, this means that in the Taylor series one should keep only terms up-to quadratic. For instance

$$du/dx = 1 - \sqrt{1-(v')^2} \approx 1 - [1 - \frac{1}{2}v'^2] = \frac{1}{2}v'^2$$

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 \approx 1 - \frac{1}{2}v'^2$$

the shortening $u(\ell)$ at the point of application of the load P

$$u(\ell) = \int_0^\ell du = \int_0^\ell [1 - \sqrt{1 - v'^2}]dx.$$

Moderate rotations assumption

work increment of the load P during buckling

Linearised theory of buckling

$$\Delta W_e = Pu(\ell) = P\int_0^\ell du = P\int_0^\ell [1 - \sqrt{1 - v'^2}]dx. \approx \frac{1}{2}P\int_0^\ell (v')^2 dx$$

Deriving buckling equations from energy principle

Buckling of a beam-column

The total potential energy increment in Bryan form was

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x + \frac{1}{2}\int_0^\ell N^0(x)(v')^2\mathrm{d}x,$$

$\epsilon_1 = -yv''(x)$ Linear part of the strain

$\sigma_x^0 A = N^0(x)$ Initial stress

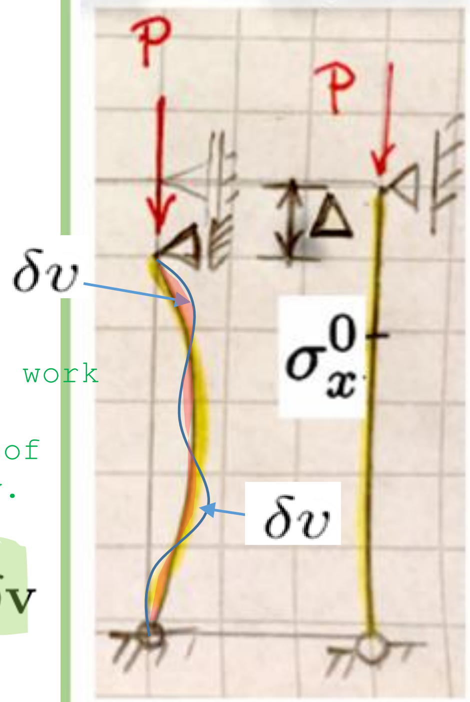
$\epsilon_2 = \frac{1}{2}(v')^2$ Quadratic part of the strain

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x.$$

end-thrust $-P = N^0(x) < 0$

N.B. compare to the virtual work principle which the more general than the principle of stationary potential energy.

$$\delta W_{int} + \delta W_{ext} = \delta W_{acc.}, \forall\delta\mathbf{v}$$



Additional work $\Delta W_{\text{ext}} = P \cdot \Delta$ (*Flexural buckling*)

Stability loss criteria

Taking the variation $\delta(\Delta\Pi) = 0 \Longrightarrow \underbrace{\int_0^\ell EIv''\delta v''}_{-\delta(\Delta W_{\text{int}})} - \underbrace{P\int_0^\ell v'\delta v'\mathrm{d}x}_{+\delta(\Delta W_{\text{ext}})} = 0, \forall\delta v$

$\Downarrow$

which gives after twice integration by parts

$$\Longrightarrow \int_0^\ell \underbrace{[EIv^{(4)} + Pv'']}_{=0}\delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0, \forall\delta v$$

Field equation — BCs — BCs

The linearised buckling equation

$$\Longrightarrow \quad (EIv'')'' + Pv'' = 0$$

& four boundary conditions.

Buckling of a beam-column

The differential approach - general solution

(loss of) Stability equations

$$(EIv'')'' + Pv'' = 0$$

& four boundary conditions.

general solution $v(x)$ for the buckling of such column-beam :

$$\begin{cases} v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x), & P > 0 \text{ compression} \\ v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + v_0(x), & P < 0 \text{ tension} \end{cases}$$

where $k^2 = P/EI$

We now, for a while, make a break and go to energy principles and then come back and recall shortly the differential approach

The few following slides are a recall form *Beams and Frames course (2018)* Related to how the *stability equations* are derived by considering equilibrium of a deformed differential beam element

1.11.2 Energy criteria in ’the full form’

This subsection is reproduced here as a very short answer to a student question that was asked today. The student was wondering, and he is completely right, a question about why the initial pre-buckled equilibrium state u (initial trivial equilibrium state) has disappeared from the expression of the *change* (or the increment of) total potential energy that was used in the previous subsection (1.11.1)?

We used previously this incremental form:

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2 \mathrm{d}x.$$

What if, insted one uses the 'full' total potential energy at the tiny buckled state

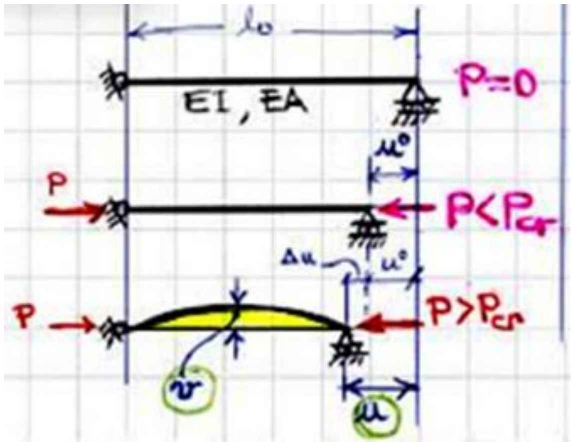
$$\Pi^*[v,u] = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EA(u')^2 \mathrm{d}x - P\cdot u(\ell)$$

Deriving buckling equations from energy principle

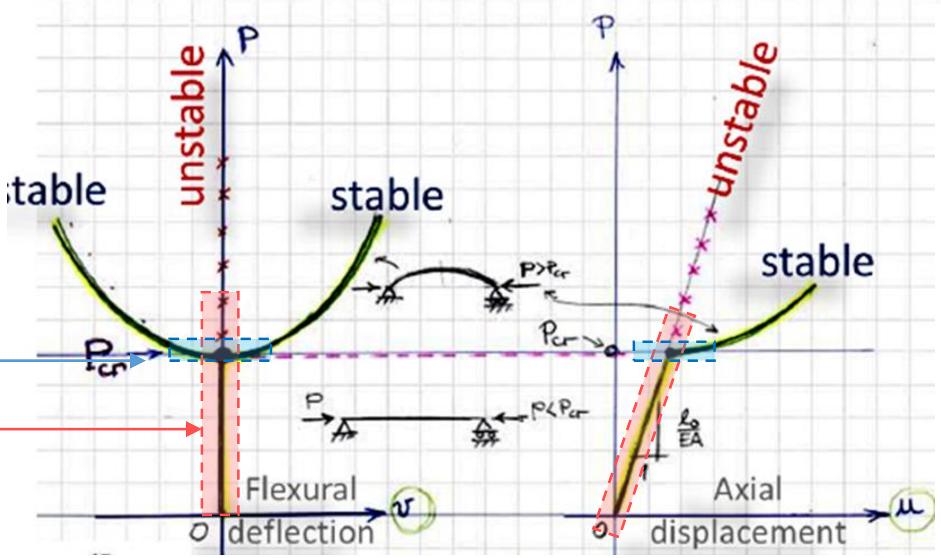
What if, insted one uses the 'full' total potential energy at the tiny buckled state

$$\Pi^*[v,u] = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EA(u')^2 \mathrm{d}x - P\cdot u(\ell)$$

Buckling of a beam-column



$$\delta\Pi^* = \int_0^\ell EIv''\cdot\delta v''\mathrm{d}x - P\int_0^\ell v'\cdot\delta v'\mathrm{d}x + \int_0^\ell EAu'\cdot\delta u'\mathrm{d}x - P\delta u(\ell) = 0, \quad \forall\delta u,\ \forall\delta v$$



Again, integration by part[102] gives now two equilibrium equations with respective boundary terms;

$$\int_0^\ell \underbrace{\left[EIv^{(4)} + Pv''\right]}_{=0,\,\text{buckled equilibrium}} \delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell +$$

$$-\int_0^\ell \underbrace{\left[EAu''\right]}_{=0,\,\text{initial equilibrium}} \cdot\delta u\mathrm{d}x - [(\underbrace{EAu'}_{=N} - P)\delta u]_0^\ell = 0, \quad \forall\delta u,\ \forall\delta v$$

The following two energy principles are used to

1: Stationary total potential energy

NB. this principle holds for conservative systems

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

At buckling $\delta(\Delta\Pi) = 0, \quad \forall \delta v$

this stationarity condition follows (also) from the VWP

2: Virtual work principle (VWP)

NB. this principle is universal and holds for all systems (conservative and non-conservative, linear and non-linear)

$$\delta W_{int} + \delta W_{ext} = \delta W_{acc.}, \quad \forall \delta \mathbf{v}$$

- to derive, in general, equilibium or motion equations
- to obtain good hand approximations for dynamics, buckling laod and corresponding modes
- to derive good numerical methods and in particular the FEM discrete equations

Some examples of direct application these principles will be shown

In these course we foccus on stability aspects

Taking the variation $\delta(\Delta\Pi) = 0$

Energy principle can be used to obtain good approximations for the buckling laod and modes

Displacement method

We go back to the cantilever buckling problem. We will use the energy principle in its weak form ready for computations as the eigenvalue problem

= virtual work principle

$$\sum_{i=1}^{n}\left[\underbrace{\int_0^\ell \phi_i'' EI \phi_j'' \mathrm{d}x}_{K_{ij}} - P \underbrace{\int_0^\ell \phi_i'(x)\cdot\phi_j'(x)\mathrm{d}x}_{S_{ij}}\right]\cdot a_i = 0,\ \forall j = 1,2,\ldots,n \quad \Longleftarrow \quad \delta(\Delta\Pi) = 0$$

$$[\mathbf{K} - P\cdot\mathbf{S}]\mathbf{a} = \mathbf{0},\ \mathbf{a} \neq \mathbf{0}$$

Critical load =

= smallest eigenvalue

The basis functions are ϕ_i and the test functions are the same ϕ_j (the Galerkin method, attend the FEM-course by prof. Jarko N. for standard notations). In this analysis, we use global approximations[174] for the buckling deflection

$$\hat{v}(x) = \sum_i a_i\phi_i(x),\ a_i \in R$$

where the basis, for this example, is

$$\phi = [1,\ x,\ x^2,\ x^3, x^4,\ \ldots, x^n].$$

... or trigonometric series

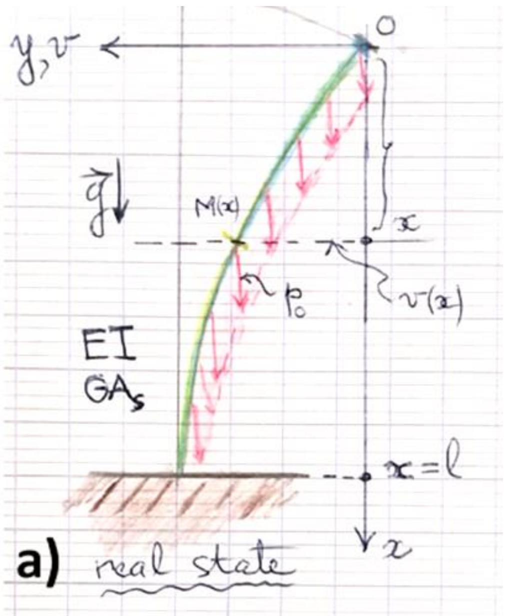
Application examples of stability study using energy principles

Cantilever column under self-weight

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x.$$

The energy principle can be used to obtain good approximations for the buckling laod

$$\underbrace{\int_0^\ell v''EI\delta v''\mathrm{d}x}_{\delta(\Delta W_{\text{int}})} - \underbrace{\int_0^\ell \left[\int_0^{\xi=x} p_0\cdot v'(\xi)\cdot\delta v'(\xi)\mathrm{d}\xi\right]\mathrm{d}x}_{\delta(\Delta W_{\text{ext}})} = 0,\ \forall\delta v\in V_{ad}.$$



The displacement approximation $v(x) = v_0 x^2/\ell^2.$

kinematically admissible $\hat{v}(0) = 0$ and $\hat{v}'(0) = 0.$

$\phi := x^2/\ell^2.$

$$\mathrm{d}u(x) = \int_0^{\xi=x}\frac{1}{2}[v'(\xi)]^2\mathrm{d}\xi$$

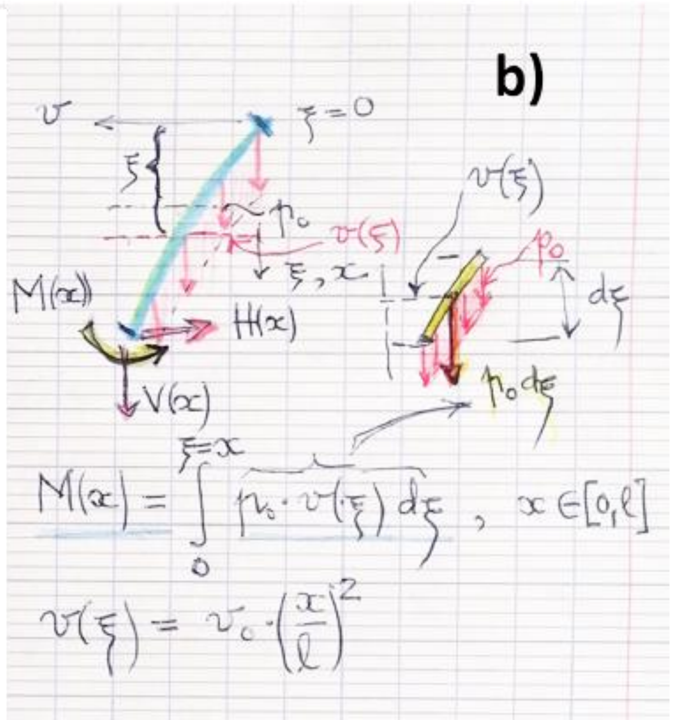
differential shortening

$$\Delta W_e = \int_0^\ell\left[\int_0^{\xi=x} p_0\cdot\frac{1}{2}[v'(\xi)]^2\mathrm{d}\xi\right]\mathrm{d}x$$

work done by the distributed constant self-weight p_0

more easy to use computationally

$$\int_0^\ell \phi''EI\phi''\mathrm{d}x - \int_0^{x=\ell}\left[\int_0^{\xi=x} p_0\cdot\phi'(\xi)\cdot\phi'(\xi)\mathrm{d}\xi\right]\mathrm{d}x = 0,\ \forall\delta v\in V_{ad}.$$



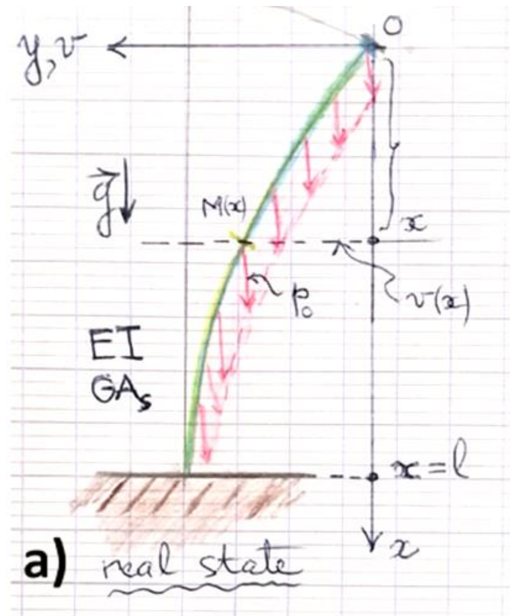
Application examples of stability study using energy principles

Cantilever column under self-weight

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x.$$

Taking the variation $\delta(\Delta\Pi) = 0$

$$\underbrace{\int_0^\ell v'' EI\delta v''\mathrm{d}x}_{\delta(\Delta W_{\text{int}})} - \underbrace{\int_0^\ell \left[\int_0^{\xi=x} p_0 \cdot v'(\xi)\cdot \delta v'(\xi)\mathrm{d}\xi\right]\mathrm{d}x}_{\delta(\Delta W_{\text{ext}})} = 0,\ \forall \delta v \in V_{ad}.$$

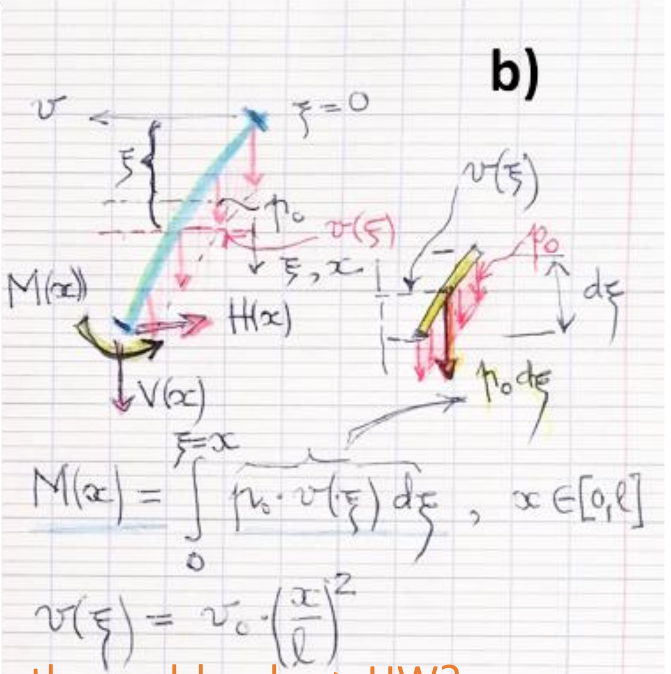


The displacement approximation $v(x) = v_0 x^2/\ell^2$. $\phi = x^2/\ell^2,\ \phi' = 2x/\ell^2,\ \phi'' = 2/\ell^2.$

$$\int_0^\ell \phi'' EI\phi''\mathrm{d}x - \int_0^{x=\ell}\left[\int_0^{\xi=x} p_0\cdot\phi'(\xi)\cdot\phi'(\xi)\mathrm{d}\xi\right]\mathrm{d}x = 0,\ \forall\delta v \in V_{ad}.$$

$$\left(\int_{\ell^e}\frac{2}{\ell^2}\cdot EI\cdot\frac{2}{\ell^2}\mathrm{d}x - p_0\int_0^{x=\ell}\left[\int_0^{\xi=x}[2\xi/\ell^2]\cdot[2\xi/\ell^2]\mathrm{d}\xi\right]\mathrm{d}x.\right)\cdot v_0 = 0,$$

$$(p_0\ell)_{cr} \approx 12\frac{EI}{\ell^2} \approx 1.2\pi^2\frac{EI}{\ell^2} > \underbrace{7.84\frac{EI}{\ell^2} \approx 0.79\pi^2\frac{EI}{\ell^2}}_{\text{analytical}}$$



Use a better mode approximation: for instance the analytical exact mode for buckling under the end-load --> HW?

Application examples of stability study using energy principles

approximation $\hat{v}(x) = \sum_i a_i \phi_i(x),\ a_i \in R$

basis: $\{x^2, x^3, x^4\}$

$\phi_1 = x^2,\ \phi_1' = 2x,\ \phi_1'' = 2$

$\phi_2 = x^3,\ \phi_2' = 3x^2,\ \phi_2'' = 6x$

$\phi_3 = x^4,\ \phi_3' = 4x^3,\ \phi_3'' = 12x^2$

$\forall j = 1, 2, \dots, n$

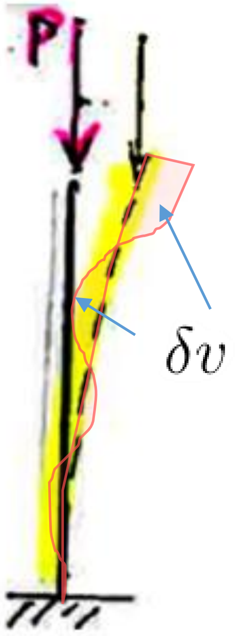
Hand-version of FEM

$$\sum_{i=1}^{n} \left[\underbrace{\int_0^\ell \phi_i'' EI \phi_j'' \mathrm{d}x}_{K_{ij}} - P \underbrace{\int_0^\ell \phi_i'(x) \cdot \phi_j'(x) \mathrm{d}x}_{S_{ij}} \right] \cdot a_i = 0,$$

Linearised stiffness matrix

Geoemetric . stiffness matrix

$$[\mathbf{K} - P \cdot \mathbf{S}]\mathbf{a} = \mathbf{0}$$

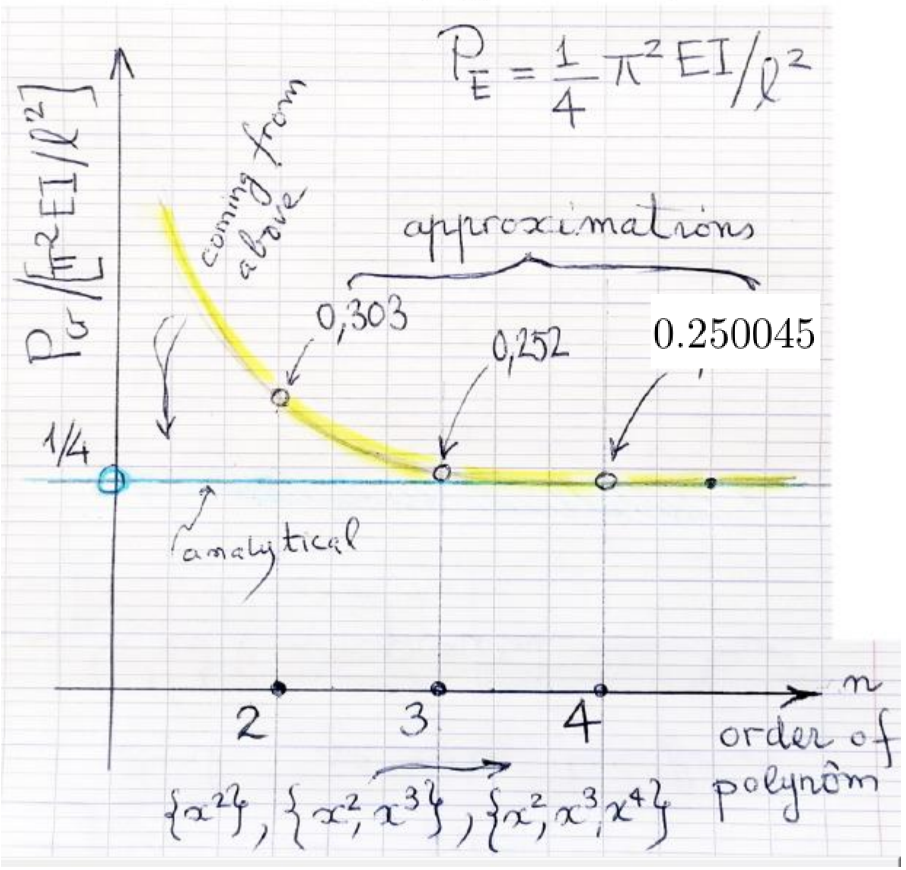


$$\left(\begin{bmatrix} 4 & 6 & 8 \\ 6 & 12 & 18 \\ 8 & 18 & 144/5 \end{bmatrix} - \frac{P\ell^2}{EI} \begin{bmatrix} 4/3 & 3/2 & 8/5 \\ 3/2 & 9/5 & 2 \\ 8/5 & 2 & 16/7 \end{bmatrix} \right) \begin{bmatrix} a_1 \ell \\ a_2 \ell^2 \\ a_3 \ell^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The smallest eigenvalue gives the critical load

$$\Longrightarrow P_{cr} = 2.4677 \frac{EI}{\ell^2} \approx 0.250045 \cdot \pi^2 \frac{EI}{\ell^2} > \underbrace{\frac{1}{4} \pi^2 \frac{EI}{\ell^2}}_{\text{analytical, } \approx 0.250000}$$

Convergence: the displacement method gives estimates from above (upper-bounds)



Stability loss criteria

Taking the variation $\delta(\Delta\Pi) = 0$

$$\Downarrow \int_0^\ell \underbrace{EI v'' \delta v''}_{-\,\delta(\Delta W_{\text{int}})} - \underbrace{P \int_0^\ell v' \delta v' \mathrm{d}x}_{+\,\delta(\Delta W_{\text{ext}})} = 0,\ \forall \delta v$$

Effect of initial imperfection

The initial shape imperfection

$$v_0(x) = v_0 \cdot \sin(\pi x/\ell) \qquad v_P(x) = v_1 \sin(\pi x/\ell)$$

$$v(x) = v_0(x) + v_P(x).$$ total deflection

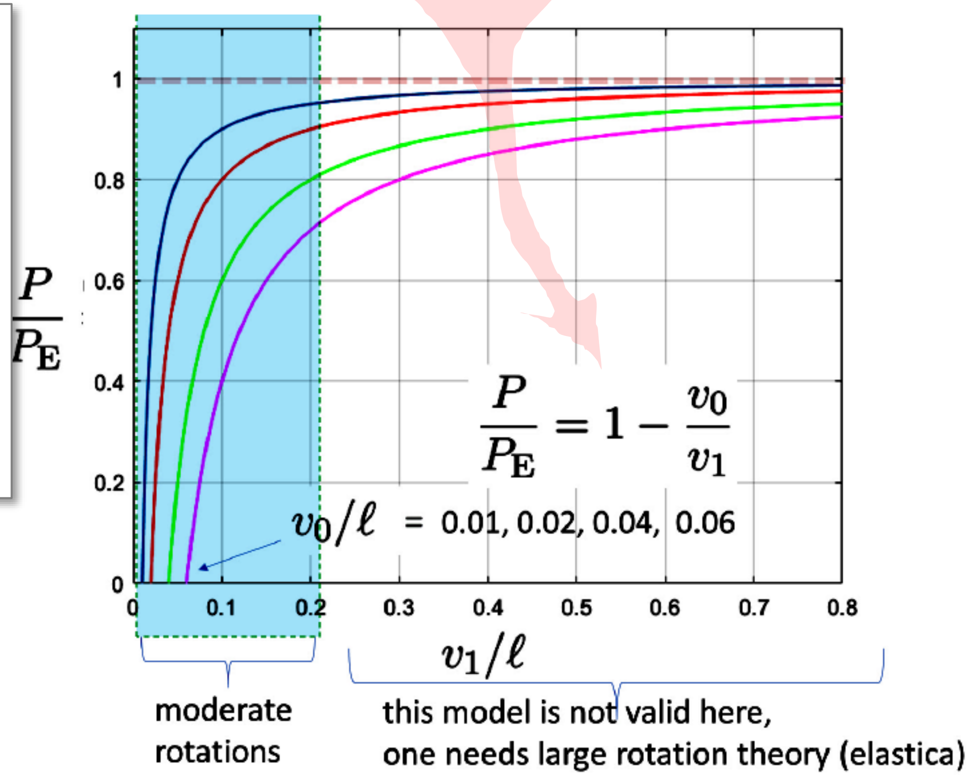
$$\delta(\Delta W_{int.}) + \delta(\Delta W_{ext.}) = 0, \ \forall \delta v$$

$$\delta(\Delta W_{int.}) = -\int_0^\ell EI\kappa \cdot \delta\kappa \mathrm{d}x$$
$$= -\int_0^\ell EI[v''(x) - v_0''(x)] \cdot \delta v''(x) \mathrm{d}x$$
$$= -EI\left[\frac{\pi}{\ell}\right]^4 \int_0^\ell (v_1 - v_0)\sin^2(\pi x/\ell)\,\mathrm{d}x \cdot \delta v_1$$
$$= -EI\left[\frac{\pi}{\ell}\right]^4 \frac{\ell}{2}(v_1 - v_0) \cdot \delta v_1$$

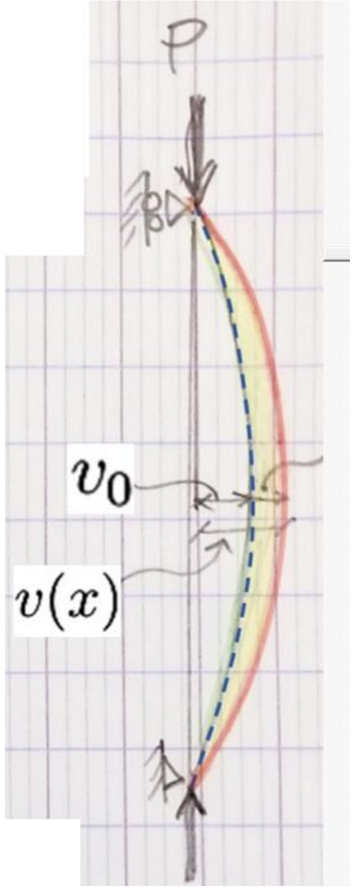
$$\delta(\Delta W_{ext.}) = P\int_0^\ell v' \cdot \delta v' \mathrm{d}x$$
$$= P v_1 \left[\frac{\pi}{\ell}\right]^2 \int_0^\ell \cos^2(\pi x/\ell)\,\mathrm{d}x \cdot \delta v_1$$
$$= P v_1 \left[\frac{\pi}{\ell}\right]^2 \frac{\ell}{2} \cdot \delta v_1$$

Therefore, at equilibrium, we must have $\delta(\Delta W) = 0$, thus

$$\underbrace{\left(-\underbrace{EI\left[\frac{\pi}{\ell}\right]^2}_{\equiv P_E}(v_1 - v_0) + P v_1\right)}_{=0}\left[\frac{\pi}{\ell}\right]^2 \frac{\ell}{2} \cdot \delta v_1 = 0, \quad \forall \delta v_1$$



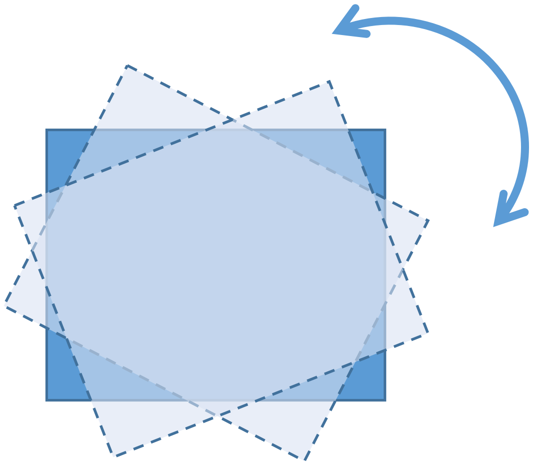
Relation between load and lateral displacement for a simply supported column having initial curvature and axially loaded.



Dynamic stability loss

Failure under wind excitation during construction phase under combined very slow torsional free vibrational mode (*less* ~1 Hz) and flexural mode resulting in excessive displacements (**resonnance**) and <u>finally joints failure</u>. Additional remarks: there is practically no torsional rigidity at all, to cite only one error. (note that there was no temporary supports!)

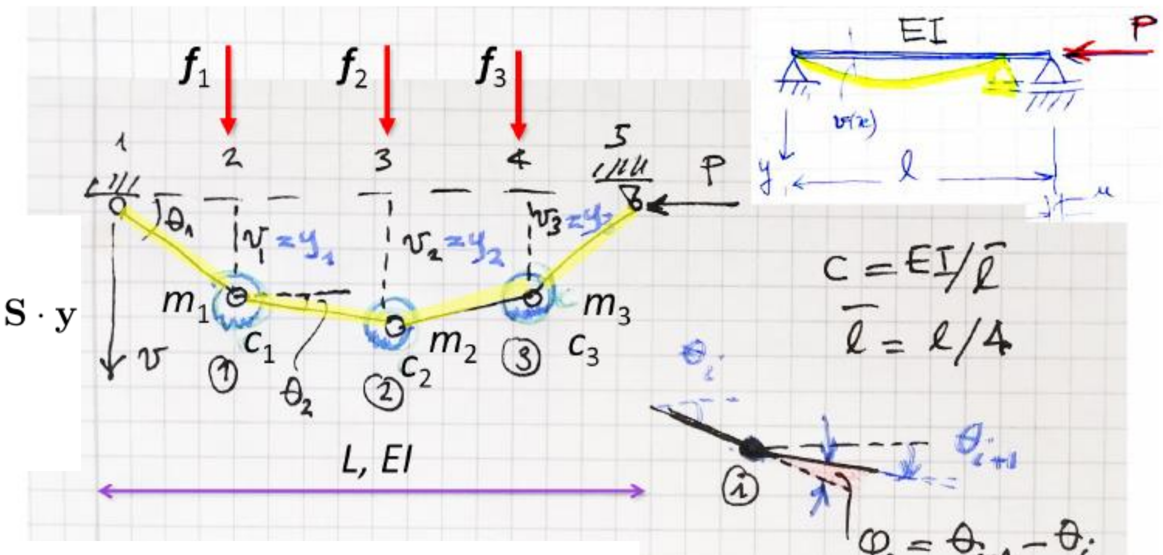
Reference: extracted from video Youtube 2021 (link sent by Dr. Athanasios M.)



torsional free vibrational mode

Dynamics and stablity

Application examples of stability using energy principles



Physical discrete model.

A discrete model of a pin-ended column

The discrete kinematics $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\quad \Delta \mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{\equiv \mathbf{S}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \equiv \mathbf{S} \cdot \mathbf{y}$

Discrete curvature:

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \theta_1 - \theta_2 \\ \theta_2 - \theta_3 \\ \theta_3 - \theta_4 \end{bmatrix} = \frac{1}{\bar{\ell}} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \equiv \mathbf{R} \cdot \mathbf{y}$$

total shortening $\Delta u(0) = \sum_{i=1}^{4} \Delta u^{(i)} = \sum_{i=1}^{4} \ell_i (1 - \cos\theta_i) \approx \sum_{i=1}^{4} \ell_i \cdot \frac{1}{2}\theta_i^2$ $\quad c_i = EI/\ell_i, \quad \ell_i = \ell/4 \equiv \bar{\ell} \quad m_i = \rho A_i \ell_i, \ \ell_i = \ell/4$

The discrete equations of motion

$$\delta W_{int} + \delta W_{ext} = \delta W_{acc.}, \ \forall \delta \mathbf{v}$$

$$\Downarrow$$

$$\delta \mathbf{y}^{\mathrm{T}} \cdot \left[\mathbf{M}\ddot{\mathbf{y}} + \left(\mathbf{K} - \frac{P}{\bar{\ell}} \mathbf{K}_{\mathrm{G}} \right) \mathbf{y} - \mathbf{f} \right] = 0, \ \forall \delta \mathbf{y}$$

This is the equation of motion

Virtual internal work:

$$\delta W_{int} = -\sum_i M_i \delta\phi_i = -\sum_i (c_i \phi_i) \cdot \delta\phi_i$$

$$= -\delta \mathbf{y}^{\mathrm{T}} \underbrace{\left[\mathbf{R}^{\mathrm{T}} \cdot \mathbf{c} \cdot \mathbf{R} \right]}_{\equiv \mathbf{K}} \mathbf{y}$$

$$\mathbf{y} = A_i \sin(\omega_i t + \phi_0) \implies \left[-\omega^2 \mathbf{M} + \underbrace{\left(\mathbf{K} - \frac{P}{\bar{\ell}} \mathbf{K}_{\mathrm{G}} \right)}_{\equiv \mathbf{K}(P)} \right] \cdot \mathbf{y} = \mathbf{0}$$

Free vibration harmonic assumption

effect of axial loading (compression/tension) on natural frequencies of the transversal vibrations

$$\omega = \sqrt{k/m}$$

Increasing compression, decreases frequency, and inversely.

Dynamics and stablity

to demonstrate the power of virtual power

Application examples of stability using energy principles

A discrete model of a pin-ended column

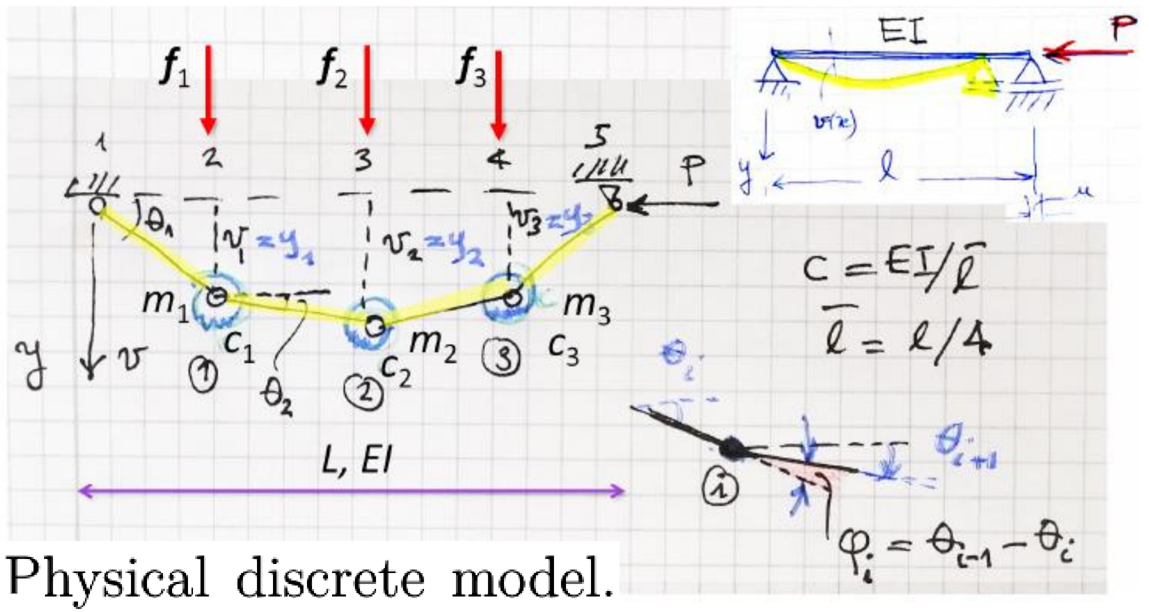
The discrete equations of motion

$$\delta W_{int} + \delta W_{ext} = \delta W_{acc.}, \quad \forall \delta \mathbf{v}$$

= runsauden sarvi

$$\delta \mathbf{y}^{\mathrm{T}} \cdot \left[\mathbf{M}\ddot{\mathbf{y}} + \left(\mathbf{K} - \frac{P}{\bar{\ell}} \mathbf{K}_G \right) \mathbf{y} - \mathbf{f} \right] = 0, \ \forall \delta \mathbf{y}$$

This is the equation of motion



Physical discrete model.

$$c_i = EI/\ell_i, \quad \ell_i = \ell/4 \equiv \bar{\ell} \qquad m_i = \rho A_i \ell_i, \ \ell_i = \ell/4$$

$$\mathbf{y} = A_i \sin(\omega_i t + \phi_0) \Longrightarrow \left[-\omega^2 \mathbf{M} + \underbrace{\left(\mathbf{K} - \frac{P}{\bar{\ell}} \mathbf{K}_G \right)}_{\equiv \mathbf{K}(P)} \right] \cdot \mathbf{y} = \mathbf{0}$$

Free vibration harmonic assumption

$$\delta W_{int} = -\delta \mathbf{y}^{\mathrm{T}} \underbrace{\left[\mathbf{R}^{\mathrm{T}} \cdot \mathbf{c} \cdot \mathbf{R} \right]}_{\equiv \mathbf{K}} \mathbf{y}$$

1, 2, 3

1) Vibrating string $P = 0$

$(c_i = 0 \rightarrow \mathbf{K} = \mathbf{0})$

natural frequency

$$\omega = \frac{3.06}{\ell} \cdot \sqrt{\frac{T}{\mu}} \approx \underbrace{\frac{\pi}{\ell} \sqrt{\frac{T}{\mu}}}_{\text{analytical}}$$

$$\mu \equiv \rho A$$

2) Static buckling

$\omega = 0$ no transversal load

$$\left(\begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} - \underbrace{\frac{P\bar{\ell}^2}{EI}}_{\equiv \lambda^2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right) \cdot \mathbf{y} = \mathbf{0},$$

```
Matlab: [v, D] = eig(K, KG)
```

$$\lambda^2 = \frac{P\bar{\ell}^2}{EI} = 0.5858 \qquad \approx 0.95\pi^2 \frac{EI}{\ell^2} < \underbrace{1 \cdot \pi^2 \frac{EI}{\ell^2}}_{\text{theorithical}}$$

4) Transversal free vibrations of beams

$$\left(\begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} - \omega^2 \cdot \underbrace{\frac{\bar{m}\bar{\ell}^3}{EI}}_{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \mathbf{y} = \mathbf{0}$$

$$\omega_1 = \underbrace{9.37\sqrt{EI/(\rho A \ell^4)}}_{\text{3-dofs discrete model}} \approx \underbrace{\pi^2 \sqrt{EI/(\rho A \ell^4)}}_{\approx \text{continuous analytical}}$$

$$\omega = \sqrt{k/m}.$$

$$k \rightarrow \mathbf{K} - P/\ell \cdot \mathbf{K}_G$$

$$m \rightarrow \mathbf{M}$$

4) variable natural frequency with changes in axial force (tension/compression)

effect of axial loading (compression/tension) on natural frequencies of the transversal vibrations

$$\omega(P) = \omega = \sqrt{k/m}.$$

Increasing compression, decreases frequency, inversely.

crutial for structural design of (stayed bridges) cables: parametric vibrations lead to structural failue and fatigue failure

Dynamics and stablity

Vibrating column - frequency-compression interaction diagrams

$$EIv^{(4)}(x,t) - \underbrace{N^{(0)}(x,t)}_{=-P<0} v''(x,t) + \rho A\ddot{v}(x,t) = 0$$

$$\Longrightarrow \quad EIv^{(4)} + Pv'' + \rho A\ddot{v} = 0$$

compression p > 0

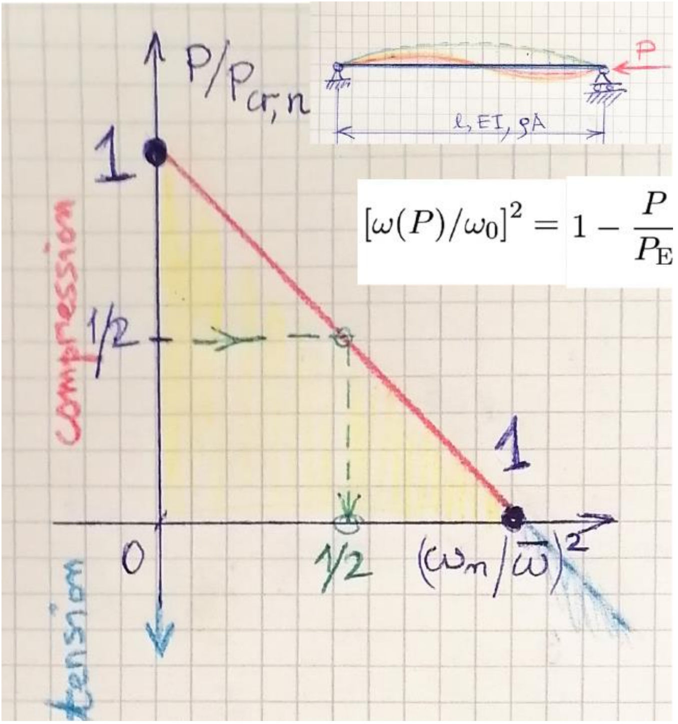
tension P < 0

$$\omega_n^2 = \frac{n^4\pi^4 EI}{\rho A \ell^4}\left(1 - \frac{P\ell^2}{n^2\pi^2 EI}\right)$$

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t)X_n(t) = \sum_{n=1}^{\infty} V_n(t)\sin\left(n\pi\cdot\frac{x}{\ell}\right)$$

$$\sum_{n=1}^{\infty} \underbrace{\left[\left(EI[\frac{n\pi}{\ell}]^2 - P\right)[\frac{n\pi}{\ell}]^2 - \omega_n^2\rho A\right]}_{=0,\ \text{since}\ V_n(t)\not\equiv 0} V_n(t) = 0.$$

$$\left(\frac{\omega_n}{\bar{\omega}}\right)^2 = 1 - \frac{P}{P_{cr,n}}.$$



Now we have showed how and how much a constant applied axial compressive force reduces the natural frequencies of the system. In structural design, we should account for this reduction of the fundamental frequencies (those close to the wind excitation frequencies) to avoid resonance of columns in high-rise buildings when, for instance, under dynamical wind load excitation. Just remember that compression reduces effective bending stiffness k, and, at its turn, this reduces the frequency according to the very well-known high-school formula $\omega = \sqrt{k/m}$ when the mass is not changing[204] (as in closed systems).

$$P_{cr,n} = \underbrace{n^2\cdot\frac{\pi^2 EI}{\ell^2}}_{\text{static Euler buckling load}}$$

$$\bar{\omega}_n^2 = \underbrace{n^4\cdot\frac{\pi^4 EI}{\rho A\ell^4}}_{\text{natural frequncy with zero axial force}}.$$

Dynamical stablity loss

Application examples of stability study using energy principles

Loss of stability of a rotating axis

Consider a cantilever slender elastic beam which is rotating with constant angular velocity ω about its length-axis. One interesting design and operational question is to estimate what will be the lowest critical angular velocity ω to ensure that the beam will not loss its stability (buckles)? Here the transverse perturbation can be for instance, the shape imperfection (non-uniform mass distribution) around the centre of of gravity of the cross-section. Such mass eccentricity results in tiny centrifuge forces which are enough to perturb the initial straight configuration and leads motion into the neighbouring bended state (=buckling).

The centrifuge forces $\mathrm{d}f_c = \mathrm{d}m \cdot \omega^2 v(x)$, distributed mass
$f_c = m \cdot \omega^2 v(\ell)$, end-mass

$$\delta W_{int} + \delta W_{ext} - \delta W_{acc.} = 0, \ \forall \delta \mathbf{u} \implies \text{the equations of motion}$$

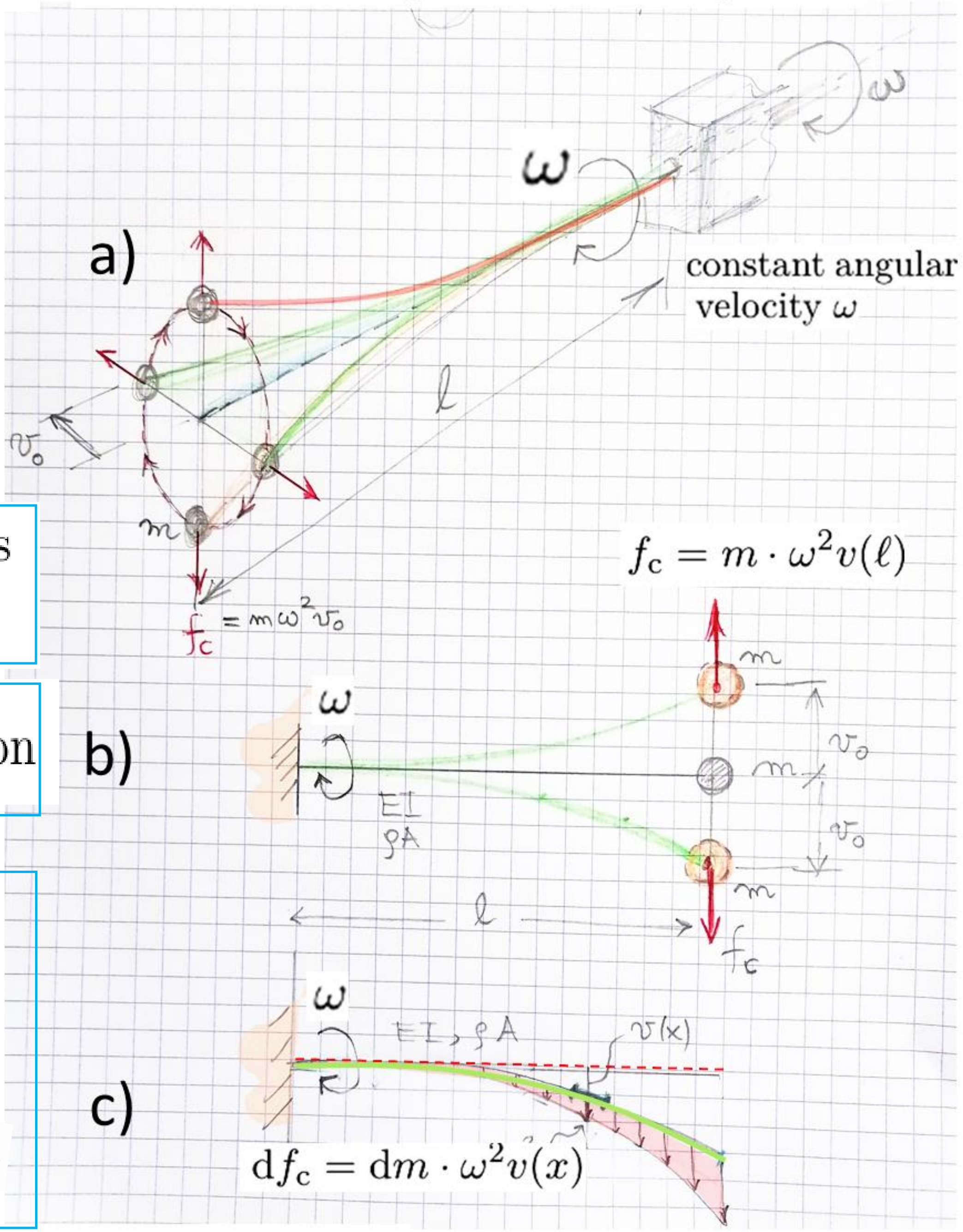
use approximated buckling modes

$$v(x) \approx v_0/\ell^2 \cdot x^2$$

$$v(x) \approx v_0 \cdot [1 - \cos\left(\frac{\pi x}{2\ell}\right)]$$

estimates for the critical rotation angular velocity ω_{cr},

Loss of stability of a rotating axis



Dynamical stablity loss

Application examples of stability study using energy principles

Loss of stability of a rotating axis

The centrifuge forces $\mathrm{d}f_c = \mathrm{d}m \cdot \omega^2 v(x)$, distributed mass

$f_c = m \cdot \omega^2 v(\ell)$, end-mass

$$\delta W_{int} + \delta W_{ext} - \delta W_{acc.} = 0, \ \forall \delta \mathbf{u} \implies \text{the equations of motion}$$

use approximated buckling modes

$$v(x) \approx v_0/\ell^2 \cdot x^2$$

$$v(x) \approx v_0 \cdot [1 - \cos\left(\frac{\pi x}{2\ell}\right)]$$

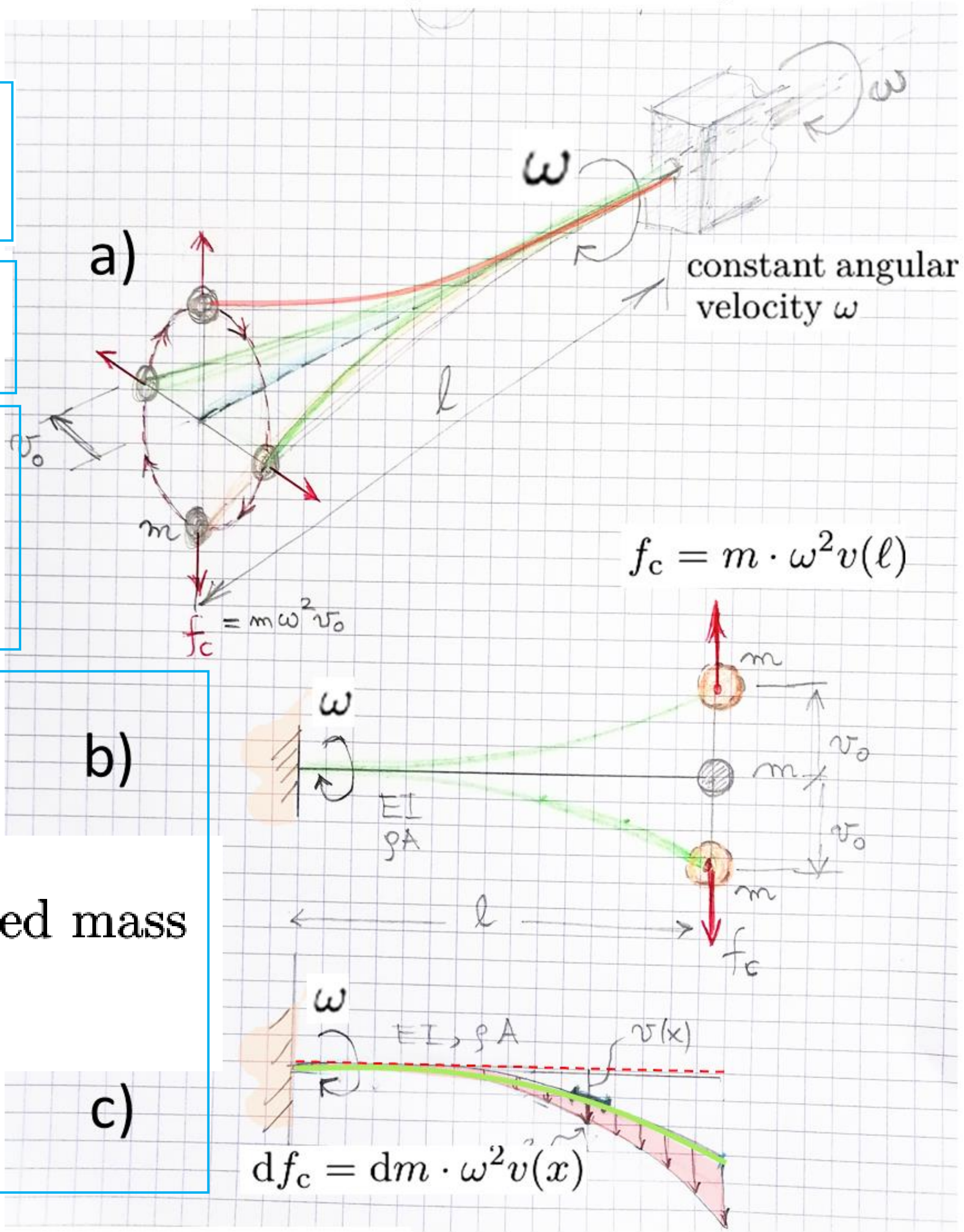
$$\delta(\Delta W_{\text{int}}) = -\int_0^\ell M \cdot \delta\kappa = -\int_0^\ell v'' EI \delta v'' \mathrm{d}x$$

$$\delta(\Delta W_{\text{acc}}) = \int_0^\ell f_c \cdot \delta v \mathrm{d}x = \int_0^\ell \rho A \omega^2 v \cdot \delta v \mathrm{d}x, \quad \text{distributed mass}$$

$$= m\omega^2 v(\ell) \cdot \delta v(\ell), \quad \text{end-mass}$$

now $\delta(\Delta W_{\text{ext}}) \approx 0$ is neglected (we assume so, in this example)

stability of a rotating axis



Dynamics and stablity

Loss of stability of a rotating axis

=buckles = loss of stability

Let a cantilever rod be rotating about its axis at a fixed angular velocity ω. If, for some accidental reason, the rod becomes bent, centrifugal loadings are set up—a concentrated force for a weightless rod with a point mass of the rod is continuously distributed along its length

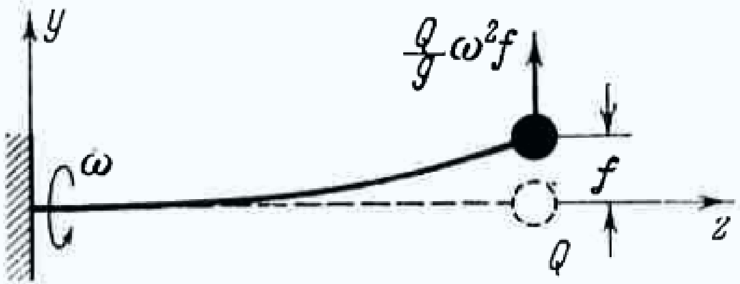
How we show this result? Propositions?

critical values for the angular velocity ω:

$$\omega_{cr} = \sqrt{\frac{cg}{Q}}, \quad \omega_{cr} = \frac{3.51}{l^2}\sqrt{\frac{gEJ}{q}}.$$

c = ?

the centrifugal force

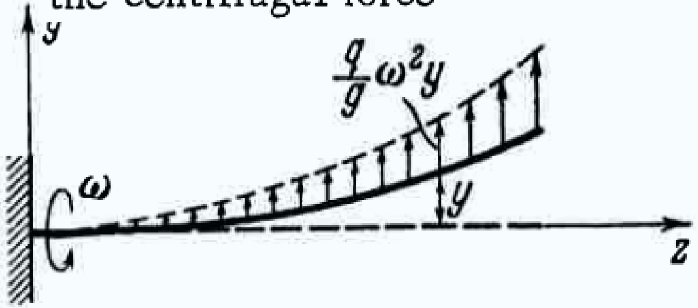


Centrifugal force proportional to the deflection f of the end of the beam.

Q is the weight of the load at the end,

$$r = \frac{q}{g}\omega^2 y,$$

the centrifugal force



Loading intensity from the centrifugal force proportional to deflection y.

q is the weight of the rod per unit length.

Reference:

STABILITY AND OSCILLATIONS OF ELASTIC SYSTEMS
PARADOXES, FALLACIES, AND NEW CONCEPTS

USTOICHIVOST' I KOLEBANIYA UPRUGIKH SISTEM
SOVREMENNYE KONTSEPTSII, PARADOKSY I OSHIBKI

УСТОЙЧИВОСТЬ И КОЛЕБАНИЯ УПРУГИХ СИСТЕМ
СОВРЕМЕННЫЕ КОНЦЕПЦИИ, ПАРАДОКСЫ И ОШИБКИ

Yakov Gilelevich Panovko

Iskra Ivanovna Gubanova

Foreword by W. Flügge
Stanford University

CONSULTANTS BUREAU · NEW YORK · 1965

approximated buckling mode $v(x) \approx v_0 \cdot [1 - \cos\left(\frac{\pi x}{2\ell}\right)]$
(first mode)

$$\delta(\Delta W_{\text{int}}) = -\int_0^\ell v'' EI \cdot \delta v'' \mathrm{d}x = \\ = -\left(\frac{\pi}{2\ell}\right)^4 EI \int_0^\ell \cos^2\left(\frac{\pi x}{2\ell}\right) \mathrm{d}x \cdot v_0 \cdot \delta v_0 \\ = -\left(\frac{\pi}{2\ell}\right)^4 EI \cdot \frac{\ell}{2} \cdot v_0 \cdot \delta v_0$$

$$\delta(\Delta W_{\text{acc}}) = -\int_0^\ell \rho A \omega^2 v \cdot \delta v \mathrm{d}x = \\ = -\rho A \omega^2 \int_0^\ell [1 - \cos\left(\frac{\pi x}{2\ell}\right)]^2 \mathrm{d}x \cdot v_0 \cdot \delta v_0 \\ \approx -0.2267\, \rho A \omega^2 \cdot \ell \cdot v_0 \cdot \delta v_0$$

$$\mathrm{d}\vec{f}_{\text{acc}} = -\mathrm{d}\vec{f}_{\text{c}} = -\mathrm{d}m \cdot \omega^2 \vec{v}(x) = -\rho A \omega^2 \cdot \vec{v}(x)\mathrm{d}x, \qquad \vec{f}_{\text{acc}} = -\vec{f}_{\text{c}} = -m\omega^2 \cdot \vec{v}(\ell),$$

The virtual total work should vanish, for any δv, therefore

$$\delta(\Delta W_{ext}) \approx 0$$

$$\delta(\Delta W_{\text{int}}) - \delta(\Delta W_{\text{acc}}) = 0 \implies \underbrace{[-\frac{1}{2}\left(\frac{\pi}{2\ell}\right)^4 EI + 0.2267\, \rho A \omega^2]}_{=0 \implies \omega_{cr} = \dots} \ell \cdot \underbrace{v_0}_{\neq 0} \cdot \underbrace{\delta v_0}_{\forall \delta v_0} = 0$$

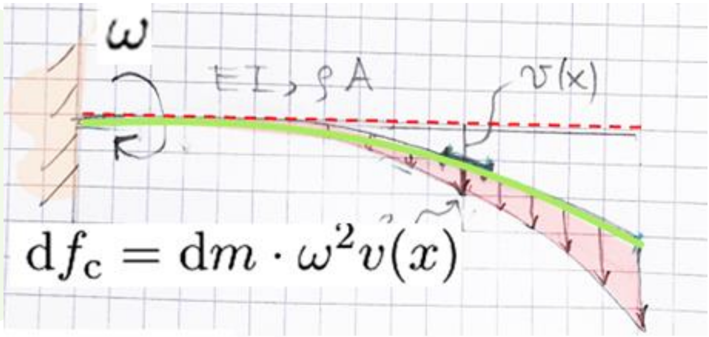
$$\Downarrow$$

$$\omega_{cr} \approx \underbrace{0.371 \left(\frac{\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho A}}}_{\text{using approximated mode}} > \underbrace{0.356 \left(\frac{\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho A}}}_{\text{analytical, ref. Panovko \& Gubanova}}$$

Yakov Gilelevich Panovko

Iskra Ivanovna Gubanova

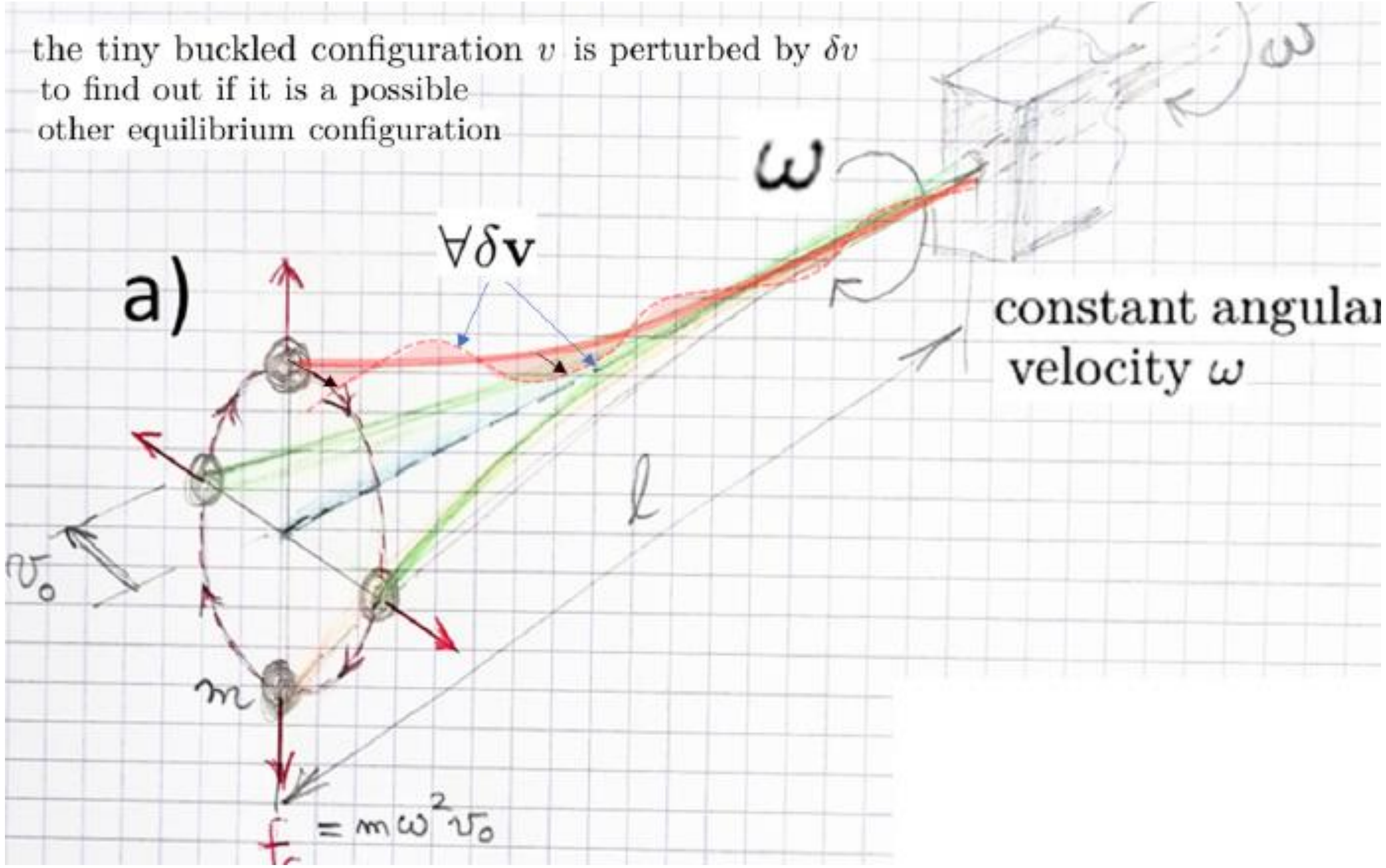
Foreword by W. Flügge
Stanford University



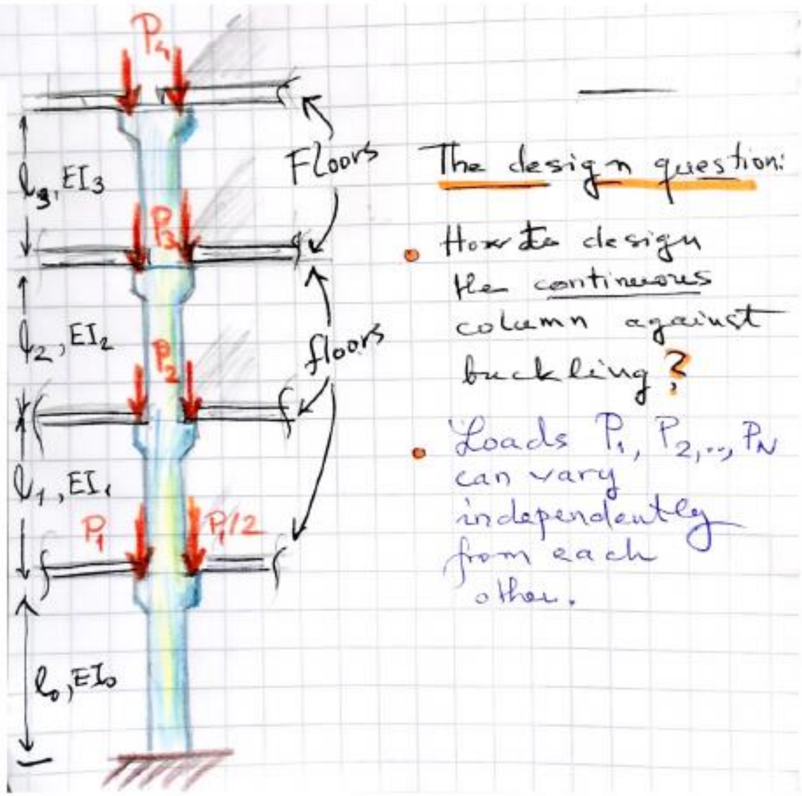
Loss of stability of a rotating axis

[187]It is very important, for understanding, to notice that we are testing the *nature of dynamic equilibrium of the perturbed configuration* and not the equilibrium of the initial pre-buckled configuration. This last one is already in primary equilibrium. That is why it the tiny buckled configuration v which is perturbed by δv to find out if it is a possible other equilibrium configuration. If yes, then system moves from its initial (dynamic) equilibrium to the bended (buckled) neighbouring one, under tiny perturbations (Fig. 1.87). This motion is the one associated with loss of stability.

Readings



Load combination - interaction buckling diagrams

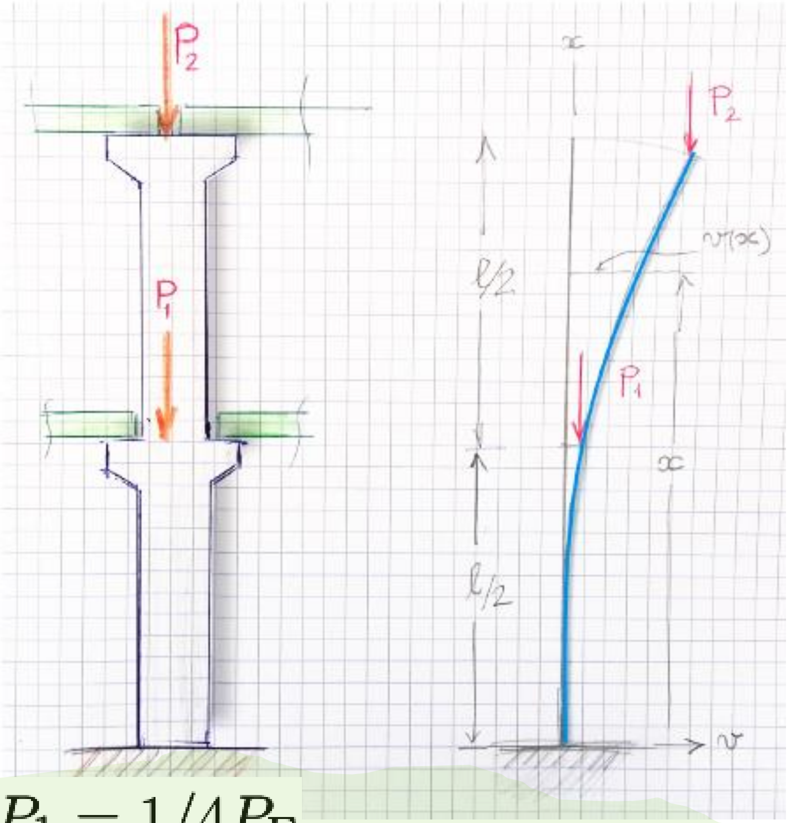


The design question:
and analysis question
How to design the column against *buckling* when the loads can be *combined independently* of each other?

A loaded column with a set of independent axial loads P_i.

Analysis: Thanks to energy method, we can have good estimates before diving into computer simulations/analysis

Application examples of stability study using energy principles



$P_1 = 1/4 P_\mathrm{E}$

how much load P_2 can one put ?

When only P_2 acts: $P_\mathrm{E} = 1/4\,\pi^2 EI/\ell^2$

Problem:

Non-proportional loading: Now first floor is already loading the column with $P_1 = 1/4 P_\mathrm{E}$. So, how much load P_2 can one put, at maximum, in the second floor before buckling?

Proportional loading: $P_1 = P_2 = P$; acting simultaneously

First, let's solve for proportional loading to have a reference

Approximation $v(x) \approx a_1\left[1 - \cos\left(\frac{\pi x}{2\ell}\right)\right] = a_1 \phi_1(x),\; a_1 \neq 0$

this is the exact buckling mode when only P_2 is acting alone

$$\delta(\Delta W_{\text{int}}) + \delta(\Delta W_{\text{ext}}) = 0, \quad \forall \text{ tiny virt. perturbation } \delta a_1$$

Taking the variation $\delta(\Delta\Pi) = 0$

$$P_{\text{cr}} = \int_0^\ell EI\phi_1''\phi_1''\mathrm{d}x \Big/ \left(\int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x + \int_0^\ell \phi_1'\phi_1'\mathrm{d}x.\right)$$

$$\int_0^\ell EI\phi_1''\phi_1''\mathrm{d}x = \left(\frac{\pi}{2\ell}\right)^4 \cdot EI\,\frac{\ell}{2}$$

$$I_{\ell/2} = \int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x = \left(\frac{\pi}{2\ell}\right)^2 \cdot \frac{\ell}{4}\left(1 - \frac{2}{\pi}\right)$$

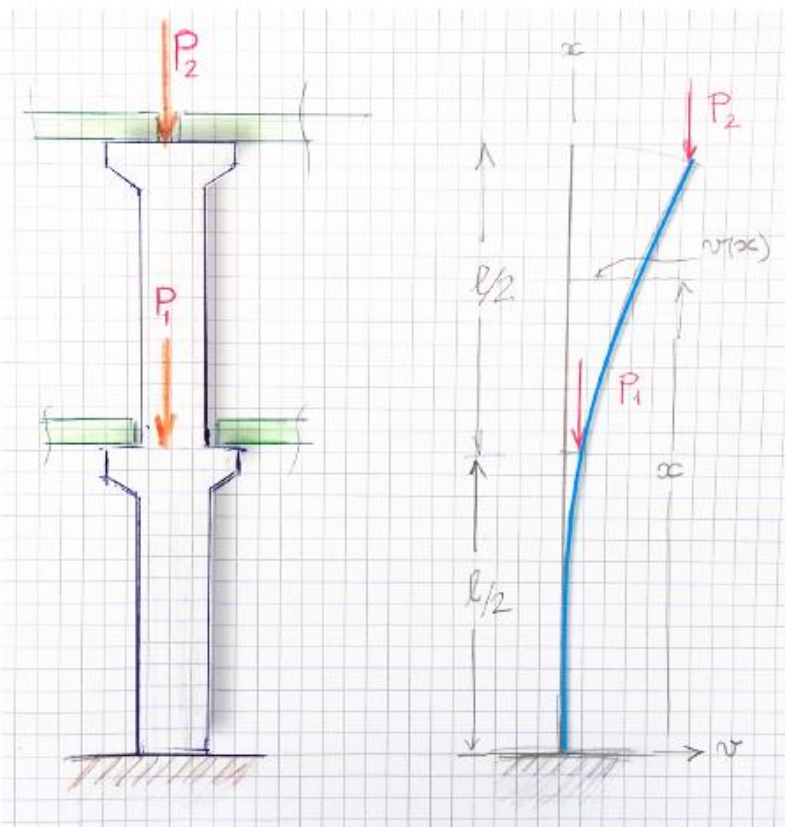
$$I_\ell = \int_0^\ell \phi_1'\phi_1'\mathrm{d}x = \left(\frac{\pi}{2\ell}\right)^2 \cdot \frac{\ell}{2}$$

$$\phi_1'(x) = -\frac{\pi}{2\ell}\sin\left(\frac{\pi x}{2\ell}\right)$$

$$\phi_1''(x) = \left(\frac{\pi}{2\ell}\right)^2\cos\left(\frac{\pi x}{2\ell}\right)$$

$$(P_1)_{\text{cr}} = (P_2)_{\text{cr}} \equiv (P)_{\text{cr}} \geq 0.846 \cdot \frac{\pi^2 EI}{4\ell^2} = 0.846 P_{\text{E}}.$$

Application examples of stability study using energy principles



When only P_2 acts: $P_{\text{E}} = 1/4\,\pi^2 EI/\ell^2$

reference load

Notice that the column supports now both loads $P_1 = P_2 = P$

total loading is $2 \cdot P_{\text{cr}} = 1.69\pi^2 EI/[4\ell^2]$

How to cross-check?

Non-proportional loading: Now first floor is already loading the column with $P_1 = 1/4 P_E$. So, how much load P_2 can one put, at maximum, in the second floor before buckling?

exact buckling mode when only P_2 is acting alone

Approximation $v(x) \approx a_1[1 - \cos\left(\frac{\pi x}{2\ell}\right)] = a_1\phi_1(x),\ a_1 \neq 0$

$$\delta(\Delta W_{\text{int}}) + \delta(\Delta W_{\text{ext}}) = 0, \quad \forall \text{ tiny virt. perturbation } \delta a_1$$

$$\Downarrow$$

$$-\int_0^\ell EI\phi_1''\phi_1''\mathrm{d}x + \underbrace{P_1}_{\text{known}} \cdot \int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x + \underbrace{P_2}_{\text{unknown}} \int_0^\ell \phi_1'\phi_1'\mathrm{d}x = 0,$$

$$\Downarrow$$

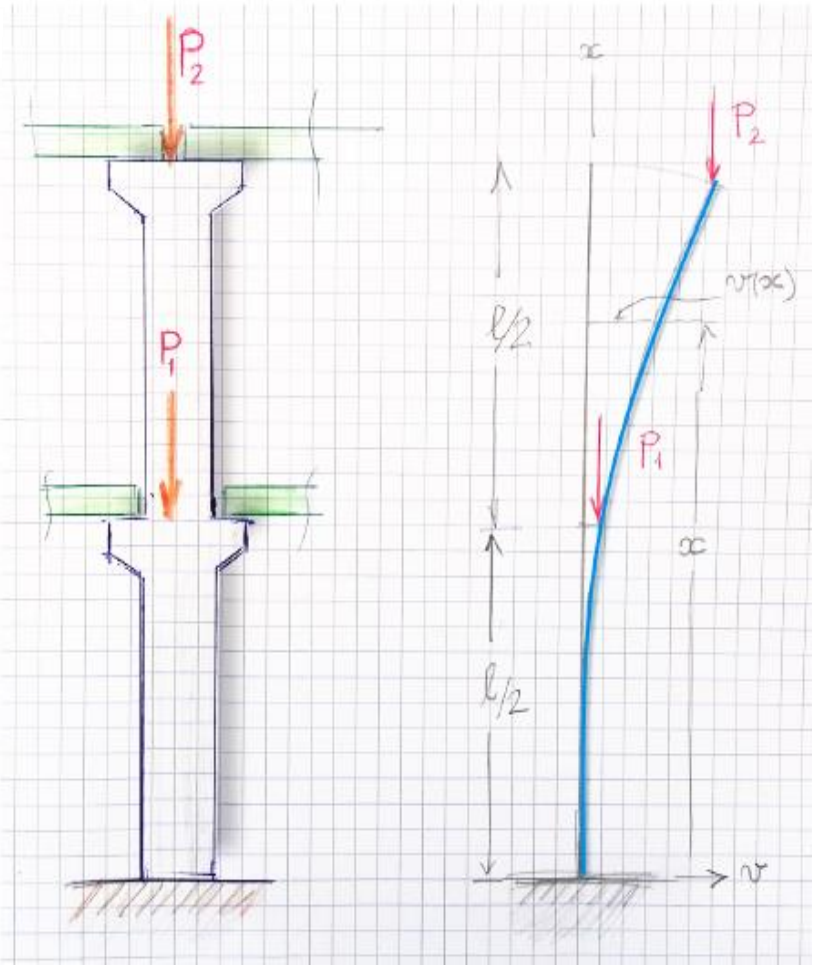
$$P_{2,\text{cr}} = \frac{\int_0^\ell EI\phi_1''\phi_1''\mathrm{d}x - P_1 \cdot \int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x}{\int_0^\ell \phi_1'\phi_1'\mathrm{d}x}$$

$$= \frac{\int_0^\ell EI\phi_1''\phi_1''\mathrm{d}x}{\int_0^\ell \phi_1'\phi_1'\mathrm{d}x} - P_1 \cdot \frac{\int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x}{\int_0^\ell \phi_1'\phi_1'\mathrm{d}x}$$

$$= \frac{\pi^2 EI}{4\ell^2} - \underbrace{P_1}_{=1/2P_E} \cdot \underbrace{\frac{\int_0^{\ell/2} \phi_1'\phi_1'\mathrm{d}x}{\int_0^\ell \phi_1'\phi_1'\mathrm{d}x}}_{=I_{\ell/2}/I_\ell} \Longrightarrow$$

approximate solution

$$= 0.91 P_E$$



reference load $P_E = 1/4\,\pi^2 EI/\ell^2$

$P_1 = 1/4 P_E$

how much load P_2 can one put

How to cross-check?

Buckling of a beam-column

The differential approach - general solution

$$\delta W_{int} + \delta W_{ext} = \delta W_{acc.}, \forall \delta \mathbf{v}$$

Stability loss criteria

Taking the variation $\delta(\Delta\Pi) = 0 \Longrightarrow \underbrace{\int_0^\ell EIv''\delta v''}_{-\,\delta(\Delta W_{\text{int}})} - \underbrace{P\int_0^\ell v'\delta v' \mathrm{d}x}_{+\,\delta(\Delta W_{\text{ext}})} = 0, \forall \delta v$

$\Downarrow$

which gives after twice integration by parts

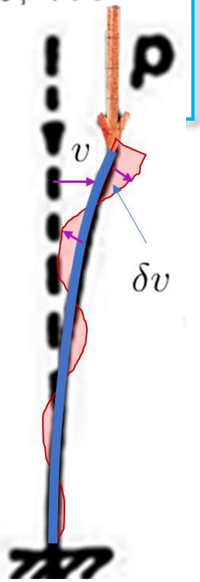
$$\Longrightarrow \int_0^\ell \underbrace{\left[EIv^{(4)} + Pv''\right]}_{=0} \delta v \mathrm{d}x + [\underbrace{EIv''}_{-M}\,\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0, \forall \delta v$$

Field equation — BCs — BCs

(loss of) Stability equations

$$(EIv'')'' + Pv'' = 0$$

& four boundary conditions.



general solution $v(x)$ for the buckling of such column-beam :

$$\begin{cases} v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x), & P > 0 \text{ compression} \\ v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + v_0(x), & P < 0 \text{ tension} \end{cases}$$

where $k^2 = P/EI$

Recall the following slides if you need it:

The few following slides are recalled form *Beams and Frames course (2018)*
Related to how the *stability equations* are derived by considering equilibrium of a deformed differential beam element

We jump directly to the slide:
Geometrically non-linear analysis of frames by the Slope-deflection method

Combined compression and bending

Linearised theory of buckling

Writing the equilibrium equations (both vertical and horizontal resultant vanish - `FBD and equilibrium as during our` 1^{st} `lecture` for a differential material element ds one obtains the *basic equation of stability theory* for a straight beam-column as

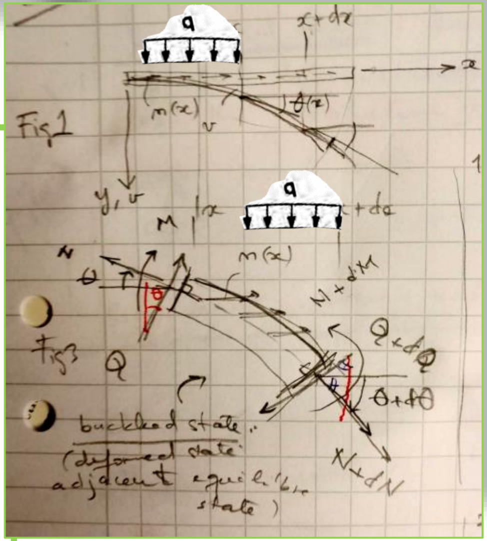
$$(EIv'')'' - (Nv')' = q \qquad (38)$$

Accounting for the linearisation around the initial equilibrium, we have $N \approx N_0$ and in our case only external compressive load $P > 0$ at the tip

$$(EIv'')'' - (N_0 v')' = q \qquad (39)$$

Assuming $N \approx N_0$ and for external compressive load $P > 0$, $N_0 = -P_0$ at one end of the column-beam is acting, and accounting for $M' = Q$ together with the constitutive relation $M = -EIv''$ we obtain

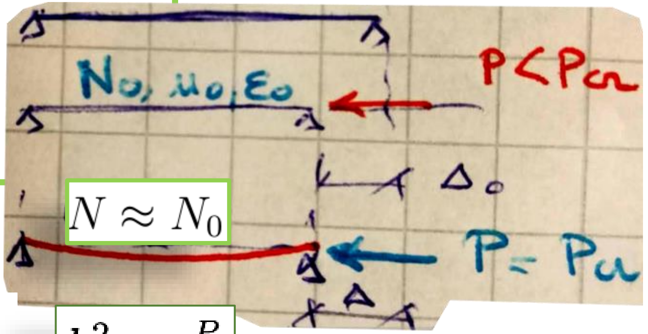
$$(EIv'')'' + (Pv')' = q \quad \& \quad 4 \quad \text{Bcs}$$



$$(EIv'')'' - (Nv')' = q$$

(compression $P > 0$) $\quad v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$

tension $P < 0$ $\quad v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + \bar{v}(x)$

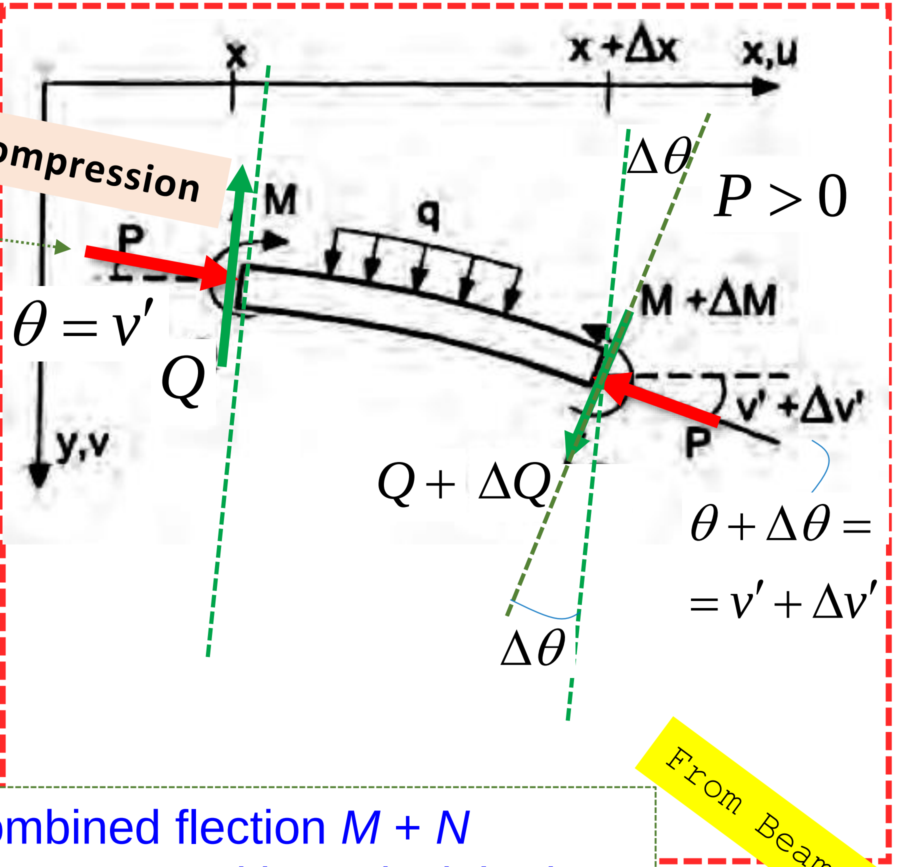


$N \approx N_0$

$k^2 = \frac{P}{EI}$

Combined compression and bending

To account for the second order effects, the idea is to write the **equilibrium equation in the deformed configuration** **/geometrical nonlinearity/** (account for the nonlinear part of the strain tensor)



Assumptions:

- **Large displacements**
- **Moderate rotations**
- **Linear elastic material (Hooke's law)**

'Moderate' rotations

$$\tan\theta = v', \ |\theta| << 1 \Rightarrow \tan\theta \approx \theta,$$

$$\sin\theta \approx \theta, \ \cos\theta \approx 1$$

Combined flection *M* + *N*
The superposition principle does not hold anymore

From Beams and Frames course

$$\Downarrow \quad \Delta Q\cos(\Delta\theta) \approx \Delta Q \qquad P\sin\theta \approx P\theta = Pv'$$

$$(Q + \Delta Q)\cos(\Delta\theta) \approx Q + \Delta Q$$

Equilibrium

$$-Q + (Q + dQ) + Pv' - P(v' + dv') + q\,dx = 0$$

0

Combined compression and bending

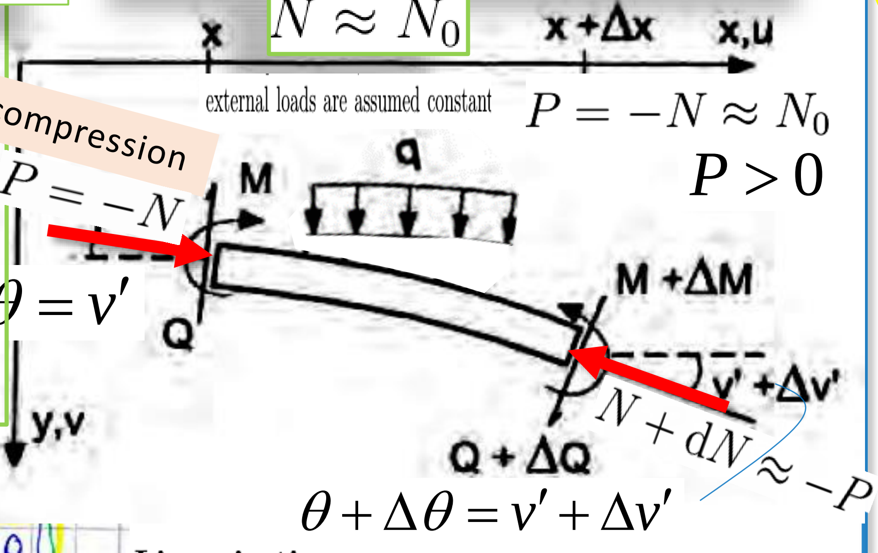
To account for the second order effects, the idea is to write the **equilibrium equation in the deformed configuration**

/geometrical nonlinearity/ (accounts for the nonlinear part of the strain tensor) and membrane forces $N \approx N_0$ from the undeformed

$$\epsilon^{\circ} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

Linearised theory of buckling

$N \approx N_0$



$P = -N \approx N_0$

$P > 0$

$\theta + \Delta\theta = v' + \Delta v'$

$-Q + Q + dQ + Pv' - P(v' + dv') + q\,dx = 0,$

$\Rightarrow \frac{dQ}{dx} - P\frac{dv'}{dx} + q = 0. \Rightarrow M'' - Pv'' + q = 0$

$Q = M'$

$M = EI\kappa = -EIv''$

$\Rightarrow$ Saadaan vihdoin:

$$\begin{cases} (EIv'')'' + Pv'' = q \\ + \text{Reunaehdot} \end{cases} \quad (1)$$

geometrinen epälineaarisuus tai nk. toisen asteen efektit

$P > 0$

Tapauksessa $EI = $ vakio (1) $\Rightarrow$

$$\begin{cases} v^{(4)} + k^2 v'' = \frac{q}{EI}, \quad \text{jossa} \quad k = \sqrt{P/EI} \\ \text{4 kpl. reunaehtoja} \end{cases}$$

Linearisation:

$\theta = v', \sin(\theta) \approx \theta, \sin(\theta + d\theta) \approx \theta + d\theta,$

$\cos(\theta) \approx 1, \cos(\theta + d\theta) \approx 1$

$$\begin{cases} (EIv'')'' + Pv'' = q \\ + \text{Reunaehdot} \end{cases}$$

ovat 4.:nnen kertaluvun tavallisia diff yhtälöitä.

ratkaisu on ($P > 0$)

$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x)$

homogeenisen diff. ylt. yleinen ratk.

jokin yksityisratkaisu (q:sta riippuva)

+ 4 kpl reunaehtoja

pilareiden nurjahduskuorma tutkimalla (3) kun $q \equiv 0$ erityyppisellä reunaehdoilla.

The General solution (for compression case)

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$$

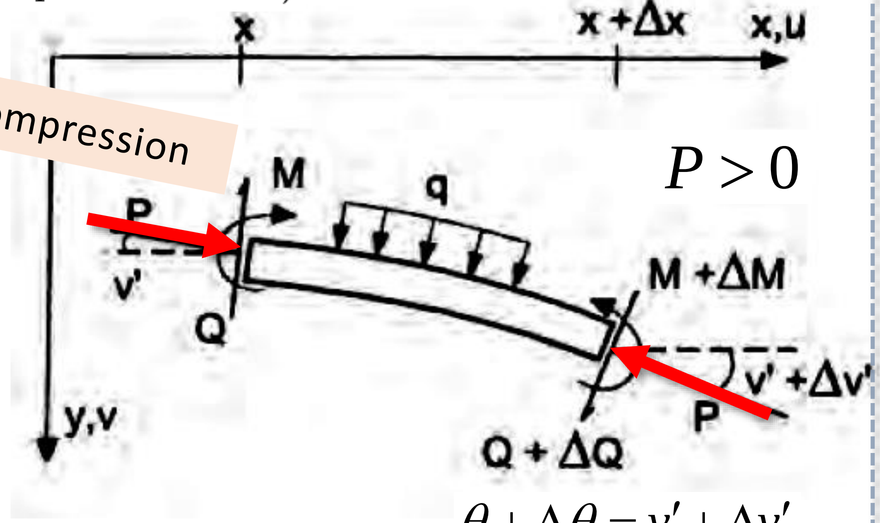
From Beams and Frames course

Combined compression/tension and bending

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$$

$$v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + \bar{v}(x)$$

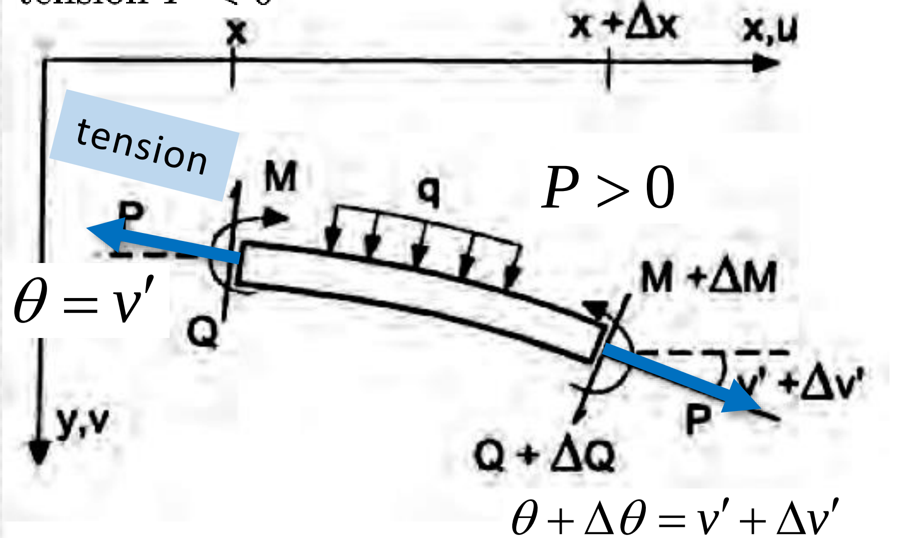
(compression $P > 0$)



$$\begin{cases} (EIv'')'' + Pv'' = q \\ + \text{Randbedingungen} \end{cases}$$

$$k^2 = \frac{P}{EI}$$

tension $P < 0$



$$\begin{cases} (EIv'')'' - Pv'' = q \\ + \text{Randbedingungen} \end{cases}$$

The General solution (for compression case)

NB. The compression have a <u>softening</u> (of the $P > 0$ effective bending rigidity) effect on bending

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x)$$

The General solution (for tension case)

NB. The tension have a <u>stiffening</u> effect on bending

$$v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + v_0(x)$$

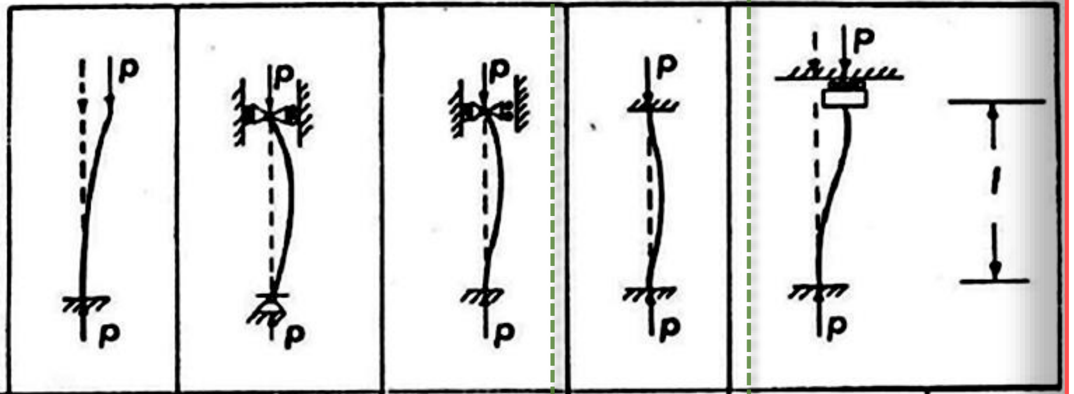
***N.B.* for** $P = 0 \rightarrow \; v(x) = A + Bx + Cx^2 + Dx^3 + v_0(x)$

Euler's basic buckling cases

From Beams and Frames course

Eulerin perusnurjahdustapaukset

$$P_{cr} = \mu \frac{\pi^2 EI}{\ell^2}$$

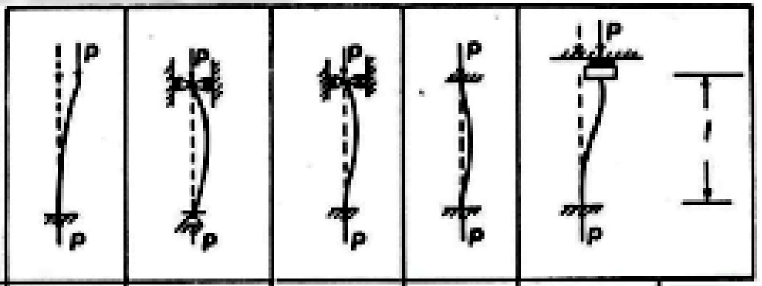


Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{cr} = 4\frac{\pi^2 EI}{\ell^2}$$

Euler's basic buckling cases

Eulerin perusnurjahdustapaukset

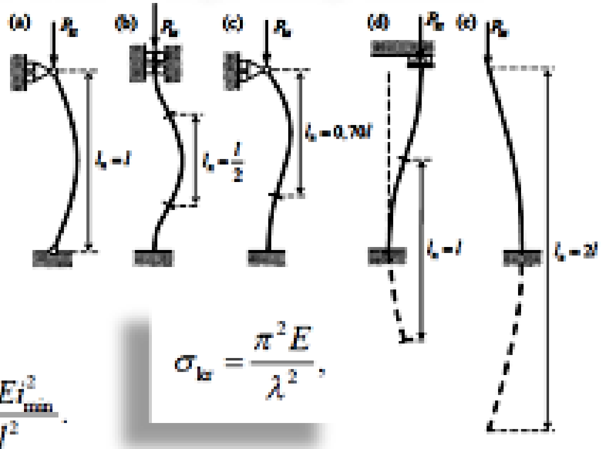


Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{cr} = \mu \frac{\pi^2 EI}{\ell^2}$$

Buckling length - Pilareiden nurjahduspituudet

$$P_{cr} = \frac{\pi^2 EI}{\ell_n^2}$$



$$\lambda = \frac{l_n}{i_{min}} = \frac{l}{\sqrt{\mu} \cdot i_{min}}$$

λ := slenderness
hoikkuusluku

Buckling stress:
Nurjahdusjännitys:

$$\sigma_{kr} = \frac{P_{kr}}{A} = \frac{\pi^2 E i_{min}^2}{l_n^2} = \mu \frac{\pi^2 E i_{min}^2}{l^2}.$$

$$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2},$$

$$i_{min}^2 = I_{min} / A$$

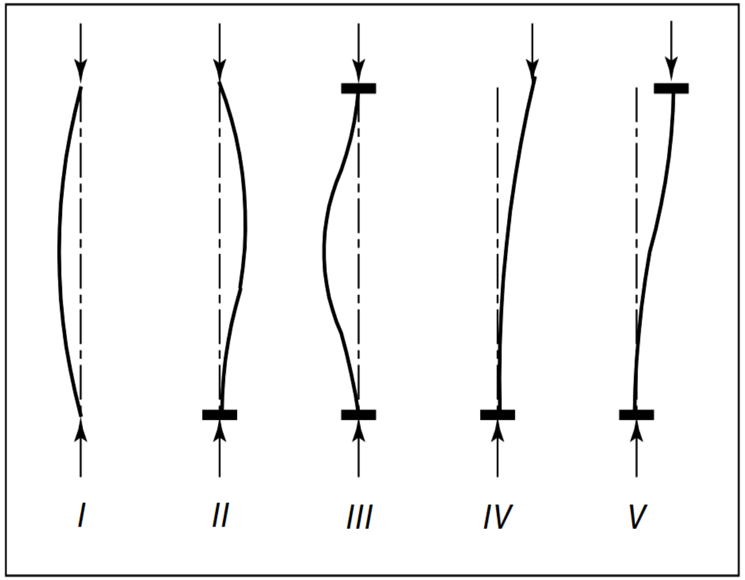
Five Fundamental Cases of Column Buckling

Case	Boundary Conditions	Buckling Determinant	Eigenfunction Eigenvalue Buckling Load	Effective Length Factor
I	$v(0)=v''(0)=0$ $v(L)=v''(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2\sin kL & -k^2\cos kL \end{vmatrix}$	$\sin kL=0$ $kL=\pi$ $P_{cr}=P_E$	1.0
II	$v(0)=v''(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\tan kl = kl$ $kl=4.493$ $P_{cr}=2.045\,P_E$	0.7
III	$v(0)=v'(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & k & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\sin\frac{kL}{2}=0$ $kL=2\pi$ $P_{cr}=4\,P_E$	0.5
IV	$v'''(0)+k^2v'=v''(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 0 & 0 & 0 & -k^2 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\cos kL=0$ $kL=\frac{\pi}{2}$ $P_{cr}=\frac{P_E}{4}$	2.0
V	$v'''(0)+k^2v'=v'(0)=0$ $v(L)=v'(L)=0$	$\begin{vmatrix} 0 & 1 & k & 0 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$	$\sin kL=0$ $kL=\pi$ $P_{cr}=P_E$	1.0

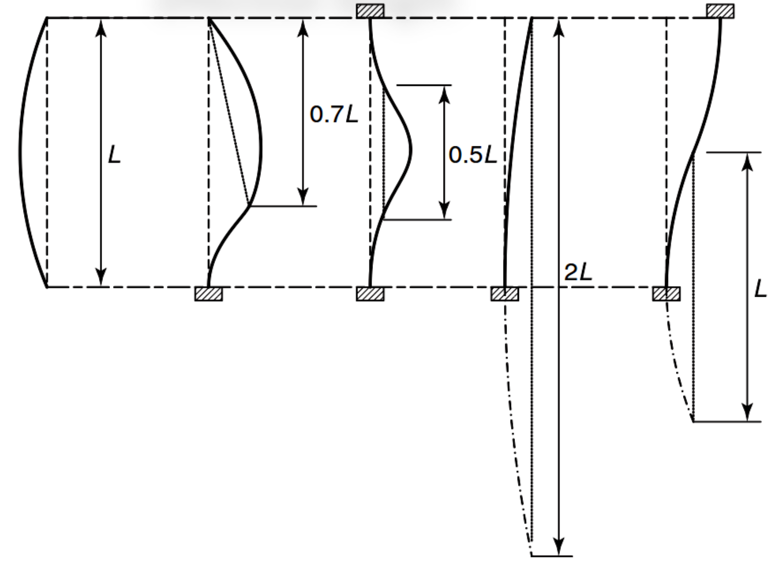
Adapted from the reference:
STRUCTURAL STABILITY OF STEEL: CONCEPTS AND APPLICATIONS FOR STRUCTURAL ENGINEERS. THEODORE V. GALAMBOS ANDREA E. SUROVEK
JOHN WILEY & SONS, INC.

From Beams and Frames course

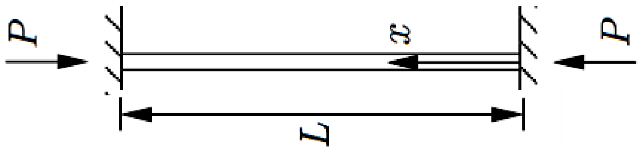
Elementary buckling cases



Geometric interpretation of the effective length



Example – rigidly fixed ends column



$v(0) = v'(0) = v(L) = v'(L) = 0.$

$$v(x) = A\sin kx + B\cos kx + Cx + D$$

$$v'(x) = Ak\cos kx - Bk\sin kx + C.$$

$$\begin{aligned} B + D &= 0, \\ kA + C &= 0, \\ A\sin kL + B\cos kL + CL + D &= 0, \\ kA\cos kL - kB\sin kL + C &= 0. \end{aligned}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & L & 1 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Cf.

$$\Longrightarrow \mathbf{Hq} = \mathbf{0},$$

$$\det\{\mathbf{H}\} = 0$$

Non-trivial solution: the determinant vanishes: $\det\{\mathbf{H}\} = 0$

$$4k\sin\frac{kL}{2}\left(\sin\frac{kL}{2} - \frac{kL}{2}\cos\frac{kL}{2}\right) = 0.$$

$\Longrightarrow$ Criticality: $\sin\frac{kL}{2} = 0$ or $\tan\frac{kL}{2} = \frac{kL}{2},$

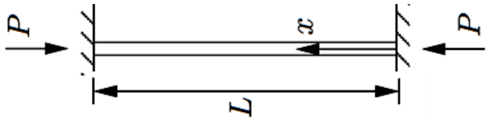
The zeros of the determinant: $\frac{kL}{2} = n\pi\ ,\quad n = 1, 2, \ldots\ ,$ $\qquad \frac{kL}{2} \approx 4.493.$

The critical load is the smallest: $k_1 = \frac{2\pi}{L},\quad (n = 1),$ $\qquad P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.$

The critical load from the Euler's 'Table' : $P_{cr} = 4\frac{\pi^2 EI}{\ell^2}$

Adapted from ref: prof. Tuomala M.

Examples – rigidly fixed ends column



Four BCs: $v(0) = v'(0) = v(L) = v'(L) = 0.$

$$v'(x) = Ak\cos kx - Bk\sin kx + C.$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & L & 1 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From Beams and Frames course

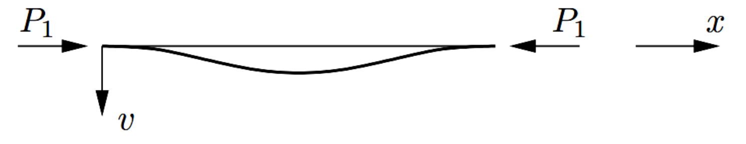
Non-trivial solution: the determinant vanishes: $4k\sin\frac{kL}{2}\left(\sin\frac{kL}{2} - \frac{kL}{2}\cos\frac{kL}{2}\right) = 0.$

(Stability loss criterion) Criticality: $\sin\frac{kL}{2} = 0$ or $\tan\frac{kL}{2} = \frac{kL}{2},$

The zeros of the determinant: $\frac{kL}{2} = n\pi\ , \quad n = 1, 2, \ldots\ ,$ $\qquad \frac{kL}{2} \approx 4.493.$

The critical load is the smallest: $k_1 = \frac{2\pi}{L}, \quad (n = 1),$ $\quad P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.$

The corresponding Eigen- (buckling) mode:
(insert the solution back and solve the integration constants…up to a constant)



$$\begin{aligned} B + D &= 0, \\ k_1 A + C &= 0, \\ A\sin k_1 L + B\cos k_1 L + CL + D &= 0, \\ k_1 A\cos k_1 L - k_1 B\sin k_1 L + C &= 0, \end{aligned}$$

$$A = C = 0, \quad D = -B,$$

$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$

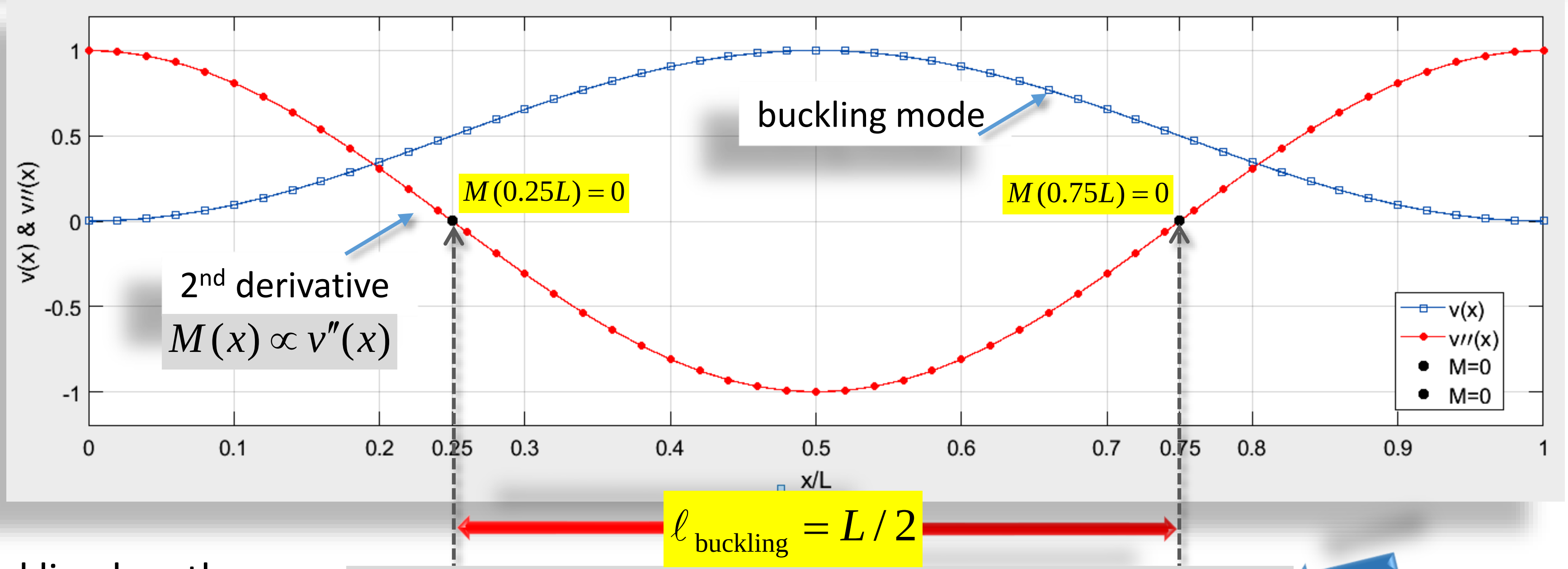
$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$

Examples – what is the buckling length?

corresponding buckling mode:

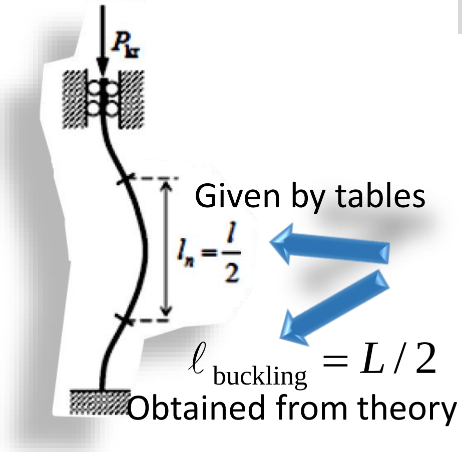
critical load: $P_{cr} = 4\frac{\pi^2 EI}{L^2}$ $P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.$ $v(x) = B(\cos\frac{2\pi x}{L} - 1).$

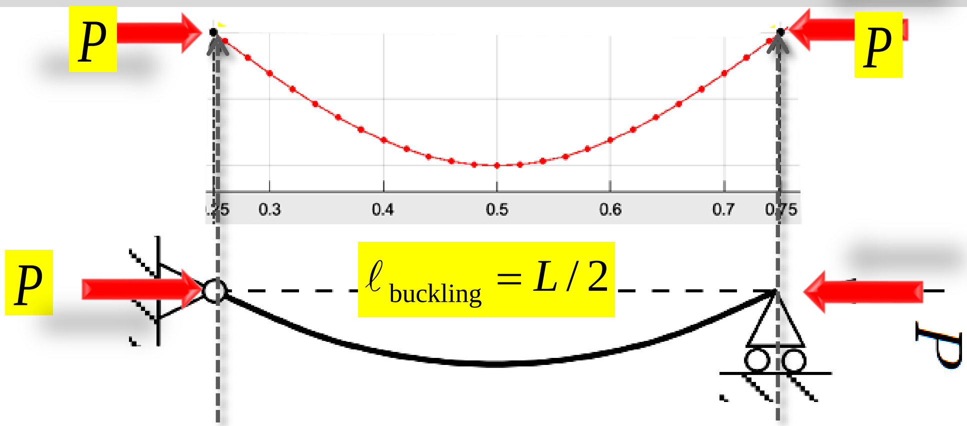
From Beams and Frames course



buckling length

$$P_{cr} = 4\pi^2 EI / L^2 = \pi^2 EI / \ell_{buckling}^2 \Rightarrow \ell_{buckling} = L/2$$





$$\ell_{buckling} = \frac{1}{\sqrt{\mu}} L$$

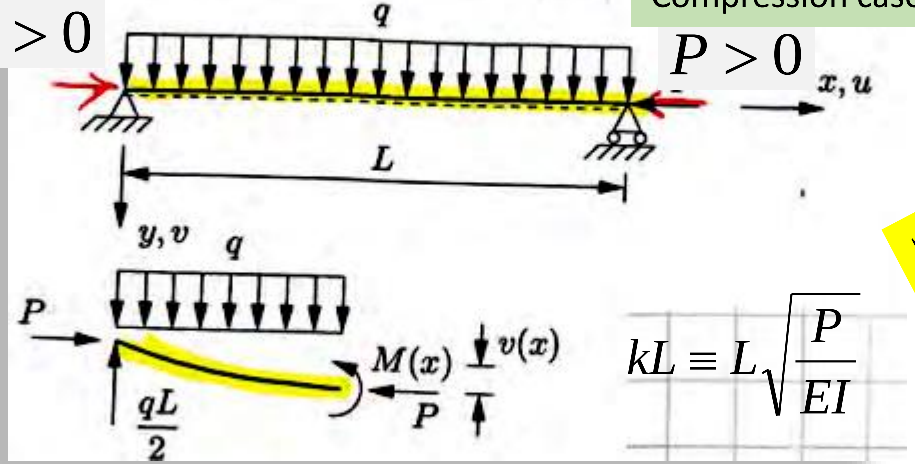
Second-order effects

The stress-problem:
Solve the deflection *v*(*x*) as function of the axial load *P* (the loading parameter)

$P > 0$

Compression case

$P > 0$



$$kL \equiv L\sqrt{\frac{P}{EI}}$$

$$EIv''(x) + Pv(x) = -\frac{qL^2}{2}\frac{x}{L}\left(1-\frac{x}{L}\right).$$

$$v(x) = \frac{qL^2}{P}\left[\frac{\sin k(L-x)+\sin kx}{(kL)^2\sin kL} - \frac{1}{(kL)^2} - \frac{1}{2}\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right],$$

$\Downarrow$

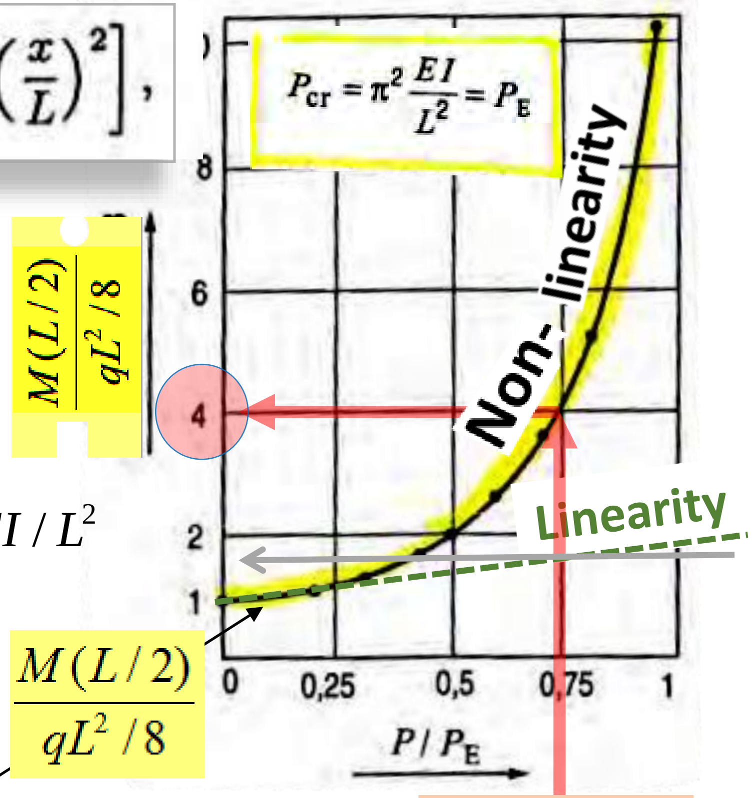
Homework: Show* that max. bending moment reduces to:

$$\Rightarrow \frac{M(L/2)}{qL^2/8} = \frac{8}{\pi^2 P_E}\left(1/\cos\left(\frac{\pi}{2}\sqrt{P/P_E}\right)-1\right)$$

$$q = 0 \rightarrow P_E = \pi^2 EI/L^2$$

and that maximum deflection is:

$$\Rightarrow v(L/2) = \frac{q}{\pi k^2}\left[1/\cos(kL/2)-1\right] - \frac{q(L/2)^2}{2P}$$



$$\frac{M(L/2)}{qL^2/8}$$

$P = 0.75 \times P_E$

From Beams and Frames course

Second-order effects = <u>non-linear</u> effects

Slope-deflection method – Stiffness-equation

Frames – recall from 'beams and frames course for the **slope-deflection method** with Berry's stability functions

The stiffness equations of the slope-deflection method with axial load

$$M_{ij} = A_{ij}(P)\varphi_{ij} + B_{ij}(P)\varphi_{ij} - C_{ij}(P)\psi_{ij} + MK_{ij}(P)$$

$$ij = \{\, 12,\ 21 \}$$

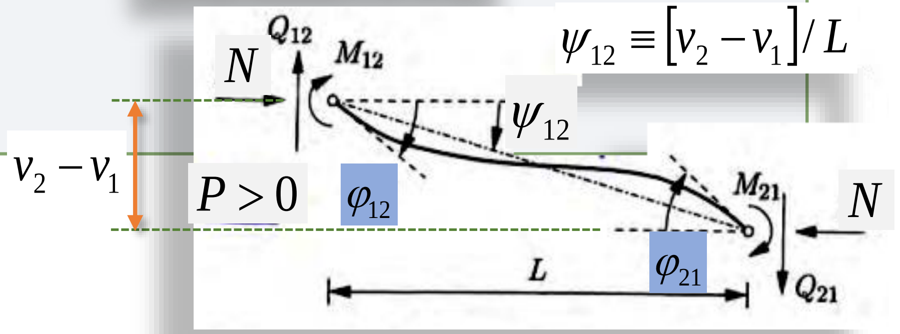
Stiffness-coefficients and loading terms depend on the member axial force

$$\lambda \equiv kL \equiv L\sqrt{\frac{P}{EI}}$$

$$M_{12} = A^0_{12}(kL)\varphi_{12} - C^0_{12}(kL)\psi_{21} + MK^0_{12}(kL)$$

Stiffness-coefficients are symmetric with respect to *i* and *j*

Compression : P > 0



Member axial force can be compressive or tensile. The stiffness-coefficients are different in compression and in tension.

$N \equiv -N_{12} = P > 0$ Case of compression : P > 0

Frames – recall from 'beams and frames course for the slope-deflection method with Berry's stability functions

The stiffness coefficients – axial compression and bending

Compression : P > 0 $\psi_{12} \equiv [v_2 - v_1]/\ell$

NB. Notation: $y \equiv v$, $\theta \equiv \varphi$

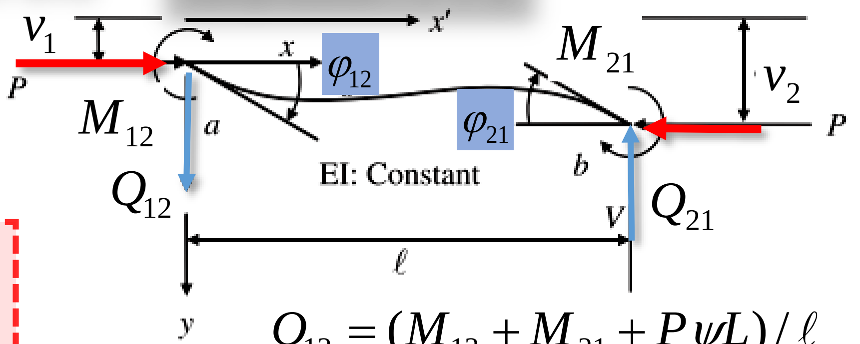
$$v^{(4)}(x) + k^2 v''(x) = 0$$

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Boundary conditions:

$v(0) = v_1 = 0$ $v(\ell) = v_2 \equiv \psi_{12}\ell = v_2 - v_1 \equiv \Delta$

$v'(0) = \varphi_{12}$ and $v'(\ell) = \varphi_{21}$



$$Q_{12} = (M_{12} + M_{21} + P\psi L)/\ell$$

$$\Delta = \psi_{12}\ell = v_2 - v_1 = v_2 - 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin\beta & \cos\beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k\cos\beta & -k\sin\beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ \varphi_{12} \\ \varphi_{21} \end{Bmatrix}$$

However, it is more practical to express the stiffness coefficients in terms of Berry's functions as we did till now.

$\Downarrow$

$$M_{12} = M(0) = -EIv''(0) = EIBk^2$$

$$\beta \equiv k\ell \equiv \lambda$$

$$= \left[\frac{EIk^2}{k(2\cos\beta + \beta\sin\beta - 2)}\right][(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (k - k\cos\beta)\Delta]$$

$$= \left[\frac{EI\beta}{\ell(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (\beta - \beta\cos\beta)\frac{\Delta}{\ell}\right]$$

$$A_{12}(k\ell) = M(0, k\ell) \downarrow$$

$$A_{12}(\lambda) = \frac{\lambda(\lambda\cos\lambda - \sin\lambda)}{2\cos\lambda + \lambda\sin\lambda - 2}$$

$$\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$$

```
(exp_BC1)  D+B
(exp_BC2)  A sin(L k)+B cos(L k)+C L+D
(exp_BC3)  A k+C
(exp_BC4)  -B k sin(L k)+A k cos(L k)+C
```

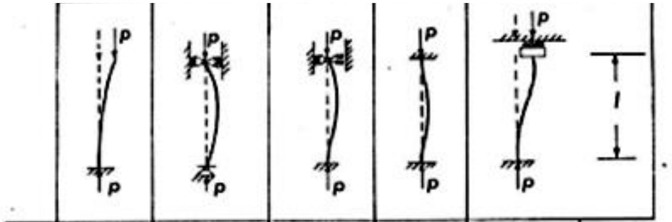
We have earlier established these eqs previously when using Maxima

$$\beta \equiv k\ell \equiv \lambda$$

Frames – recall from 'beams and frames course

Formulary

Eulerin peruskaavat nurjahdukselle: $P_{cr} = \mu \cdot \dfrac{\pi^2 EI}{l^2}$



Tapaus	1	2	3	4	5
μ	0.25	1	2.046	4	1

The stiffness coefficients (are symmetric)

Puristettu ja taivutettu sauva:

Kulmanmuutosmenetelmä

$$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ji} - C_{ij}\psi_{ij} + \overline{MK}_{ij}$$

$$M_{ij} = A^0_{ij}\varphi_{ij} - C^0_{ij}\psi_{ij} + \overline{MK}^0_{ij} \quad \text{(sauvan päässä } j \text{ on nivel)}$$

Tasajäykkä sauva :

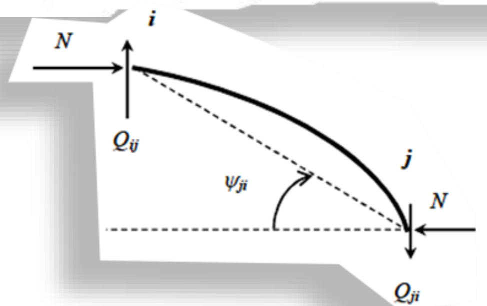
$$A_{ij} = A_{ji} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L}, \quad B_{ij} = B_{ji} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} \quad \text{ja} \quad C_{ij} = A_{ij} + B_{ij},$$

$$\overline{MK}_{ij} = -A_{ij}\overline{\alpha}^0_{ij} - B_{ji}\overline{\alpha}^0_{ji}, \qquad \overline{MK}_{ji} = -A_{ji}\overline{\alpha}^0_{ji} - B_{ji}\overline{\alpha}^0_{ij},$$

$$A^0_{ij} = C^0_{ij} = \frac{1}{\psi(kL)}\frac{3EI}{L}, \qquad \overline{MK}^0_{ij} = -A_{ij}\overline{\alpha}^0_{ij}$$

Leikkausvoima:

$$Q_{ij} = Q^0_{ij} - (M_{ij} + M_{ji})/L - N\psi_{ij} \quad (N \text{ positiivinen, kun sauva puristettu})$$



Berry's functions (stability function)

Berryn funktiot:

Olkoon $\lambda \equiv kL$,

Compression

Puristettu sauva:

$$\phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(\tan\frac{\lambda}{2} - \frac{\lambda}{2}\right),$$

Vedetty sauva:

$$\phi(\lambda) = \frac{6}{\lambda}\left(-\frac{1}{\sinh\lambda} + \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(-\frac{1}{\lambda} + \frac{1}{\tanh\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(-\tanh\frac{\lambda}{2} + \frac{\lambda}{2}\right),$$

Extension

$$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ij} - C_{ij}\psi_{ij} + \overline{M}_{ij}$$

EI constant

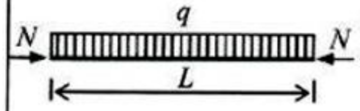
$$A_{12} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = A_{21},$$

$$B_{12} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = B_{21},$$

$$C_{12} = A_{12} + B_{12}, \quad C_{21} = A_{21} + B_{21}.$$

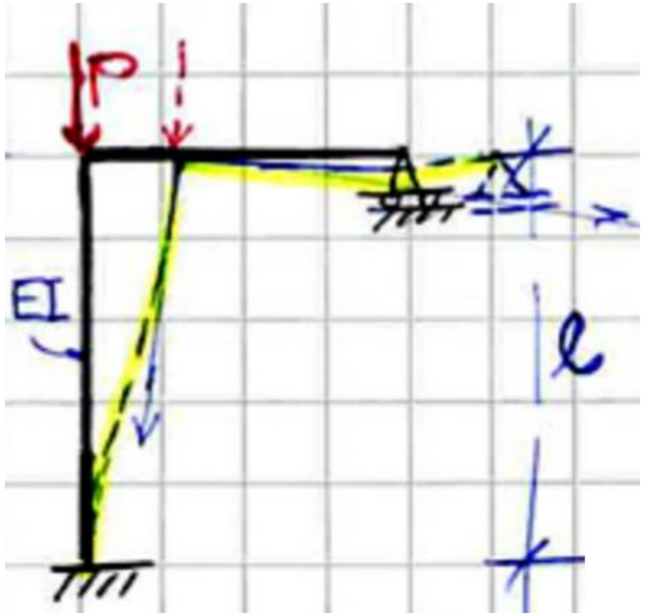
Loading terms

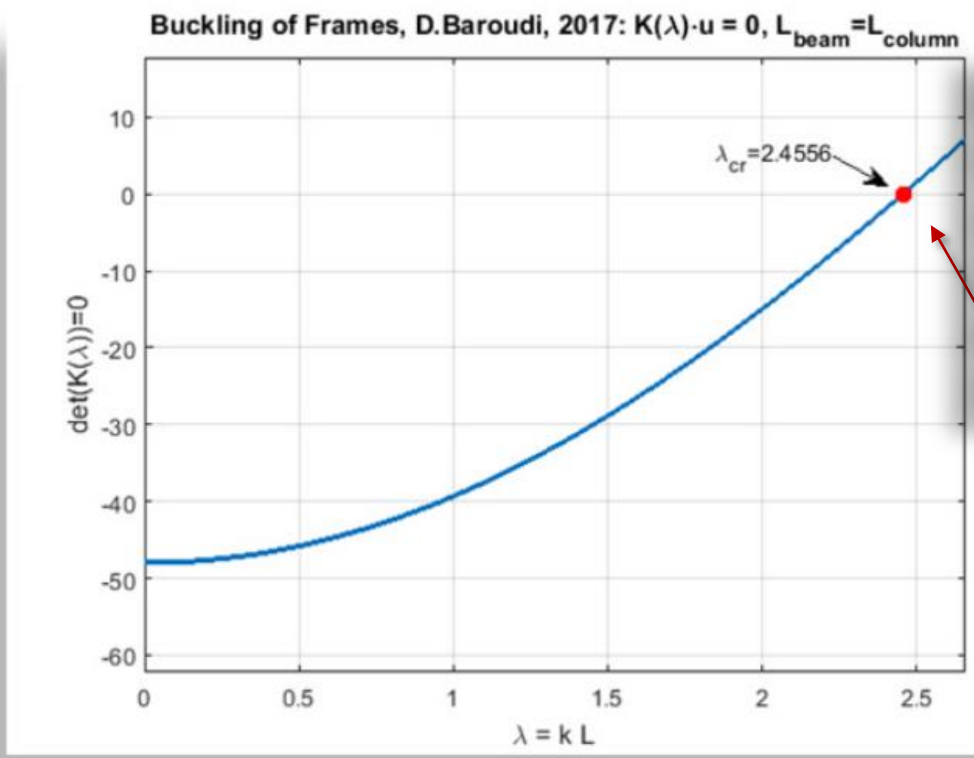
$$\overline{M}_{ij} = -\overline{M}_{ji}$$

N:o	Kuormitus	Kiinnitysmomentit: $\overline{M}_{ij}$, $\overline{M}_{ji}$
1		$\overline{MK}_1 = -\overline{MK}_2 = -\dfrac{qL^2}{12}\dfrac{\chi(kL)}{\tan(\frac{kL}{2})/(\frac{kL}{2})}$

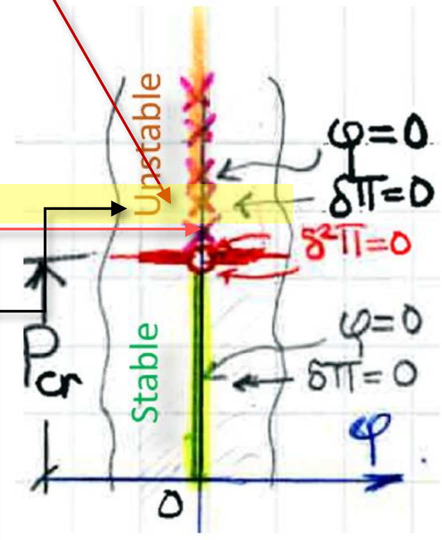
Application example of the **geometric non-linear theory** (for moderate rotations) for analysis of frames and continuous columns

Buckling analysis of side-sway frame





Buckling of frames – recall from 'beams and frames course for the **slope-deflection method** with Berry's stability functions



Linearised criterion

Geometric non-linear stiffness matrix

$$\begin{bmatrix} A_{21}(\lambda) + a_{23}^0 & -C_{21}(\lambda) \\ -C_{21}(\lambda) & +2C_{21}(\lambda)\psi - \lambda^2 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equilibrium eqautions in the tiny buckled configuration

$$\begin{cases} M_{21} + M_{23}^0 = 0, \\ Q_{21} = 0, \end{cases}$$

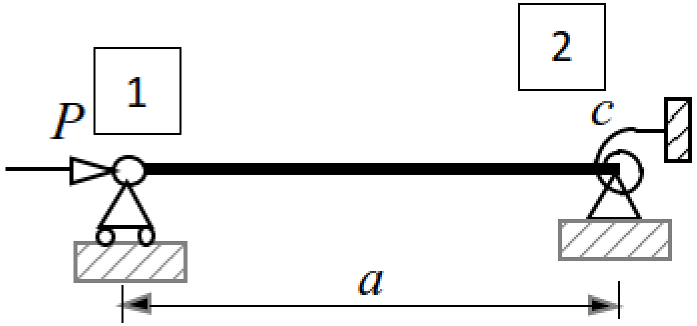
Eigenvalue problem:

$$\lambda_{cr} = \min .\text{sol.}\{\det(K(\lambda)) = 0\}.$$

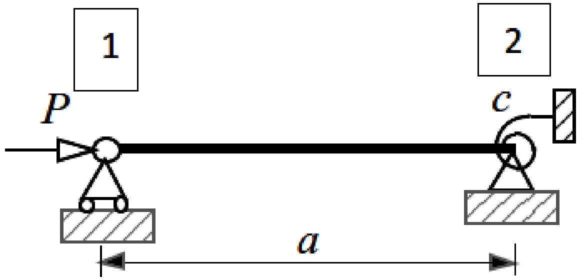
$$P_{cr} = \lambda_{cr}^2 EI/\ell^2$$

The stiffness coefficients – axial compression and bending

Example from exam 2018



A straight beam is simply supported at one end, and supported by a rotational spring, with spring constant $c = \alpha EI / a$, at the other. Its length is *a*, and bending stiffness *EI*. Determine the critical compressive load of the beam, when $\alpha = 1$. Show further that the result is covering the cases where the right hand end of the beam is simply supported and clamped by varying the coefficient α.



1. Easiest way is to apply the slope-deflection method. Thus the equilibrium equation is $M_{21} + M_{2s} = 0 \Rightarrow (A_{21}^{o} + c)\varphi_2 = 0$.

   $A_{21}^{o} + c = -\frac{1}{\Psi(ka)}\frac{3EI}{a} + \alpha\frac{EI}{a} = 0 \Rightarrow \Psi(ka) = \frac{3}{\alpha}$. Jos $\Psi(ka) = \frac{3}{ka}\left(\frac{1}{ka} - \frac{1}{\tan ka}\right)$

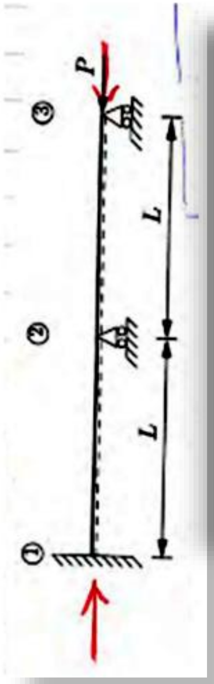
   $\Rightarrow \tan ka = \frac{\alpha ka}{\alpha + (ka)^2}$ If $\alpha = 1 \Rightarrow \tan ka = \frac{ka}{1+(ka)^2} \Rightarrow ka = 3.405 \Rightarrow P_{cr} = 1.175\frac{\pi^2 EI}{a^2}$

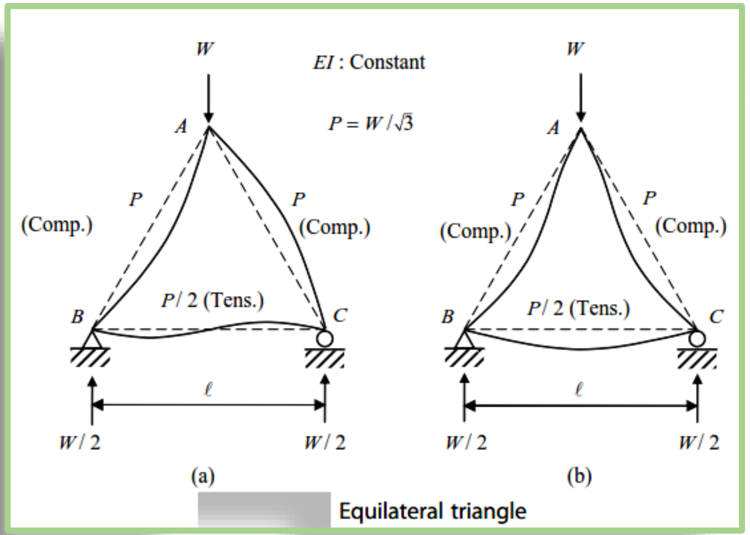
   If $\alpha = 0 \Rightarrow \tan ka = 0 \Rightarrow ka = n\pi \Rightarrow P_{cr} = \frac{\pi^2 EI}{a^2}$. If $\alpha = \infty \Rightarrow \tan ka = ka \Rightarrow P_{cr} = 2.046\frac{\pi^2 EI}{a^2}$.

   From differential equation, the solution is $v(x) = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$ where $k^2 = P / EI$ and the boundary conditions $v(0) = v''(0) = v(a) = 0,\ cv'(a) = -EIv''(a)$ yielding $C_2 = C_4 = 0$, $C_3 = -C_1 \sin ka / a$ and the condition $c(k\cos ka - \sin ka / a) = P\sin ka$, yielding the same result.

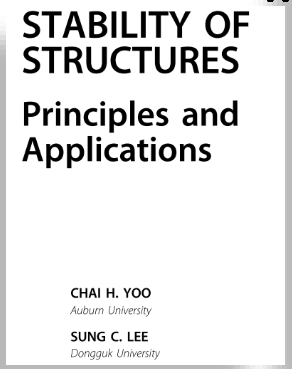
Buckling of Continuous Beam-Columns and Frames

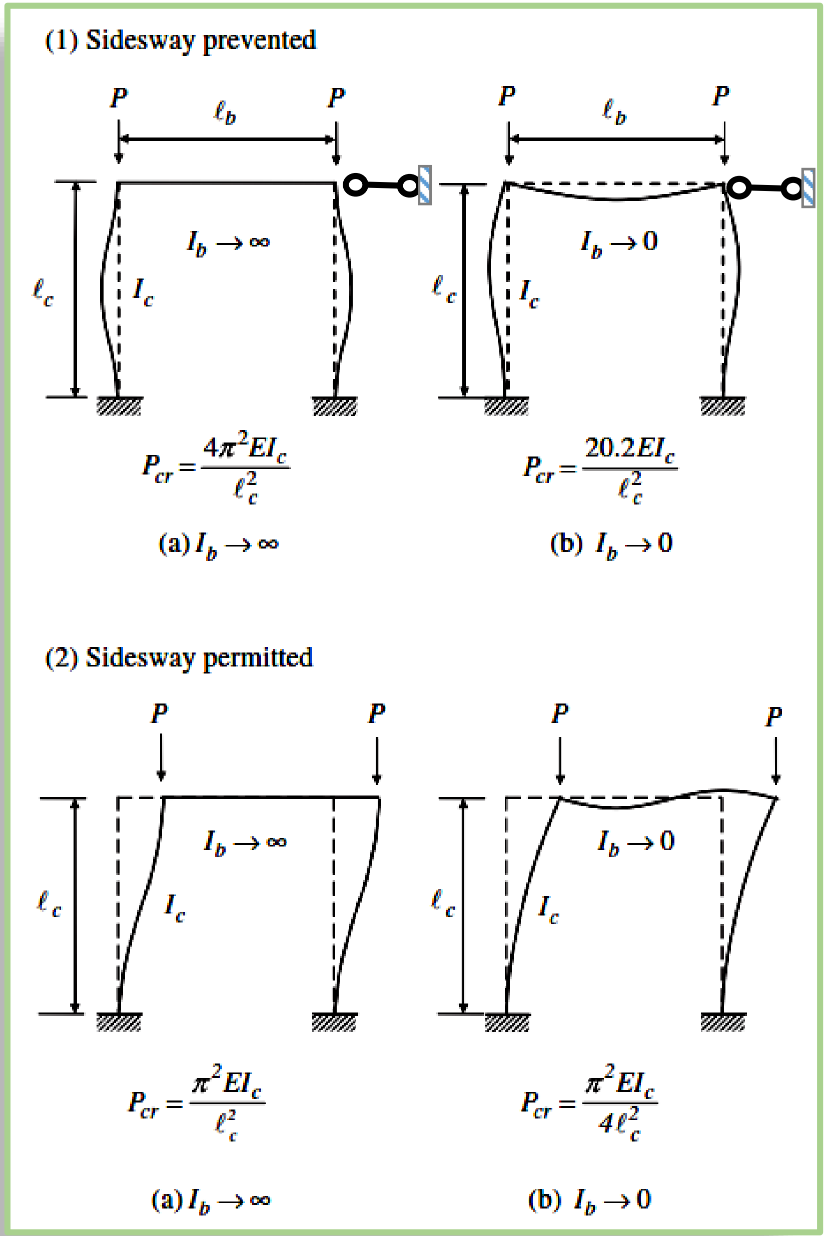
Frames – recall from 'beams and frames course





Examples from textbook:



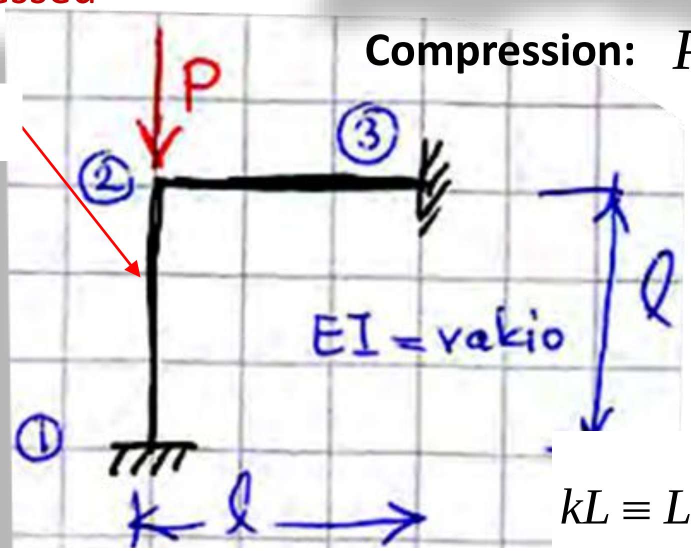


Buckling of frames - no side sway

Frames – recall from 'beams and frames course

only beam 1-2 is axially compressed

SOLUTION



Compression: $P > 0$

no side sway: $\psi_{ij} = 0$

$kL \equiv L\sqrt{\frac{P}{EI}}$

Solution: $\varphi_{21} = \varphi_{23} \equiv \varphi_2,$

$$M_{21} + M_{23} = 0 \Rightarrow (A_{21} + a_{23})\varphi_2 = 0,$$

compression $P > 0$ normal force = 0

$$\frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L}, \qquad \frac{4EI}{L}$$

Critical condition = non-trivial solution exists:

$$\phi_2 \neq 0 \Rightarrow A_{21}(kL) + a_{23} = 0 \Rightarrow kL = ?$$

$$\Downarrow$$

$$12\psi(kL) + 16\psi^2(kL) - 4\phi^2(kL) = 0$$

$$\Downarrow$$

$$kL = 5.33 \Rightarrow P_{cr} = 2.88\pi^2\frac{EI}{L^2}$$

Berry's stability functions: $\lambda \equiv kL$

$$\psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right), \qquad \phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right)$$

$$M_{ij} = A_{ij}(P)\varphi_{ij} + B_{ij}(P)\varphi_{ij} - C_{ij}(P)\psi_{ij} + MK_{ij}(P),$$

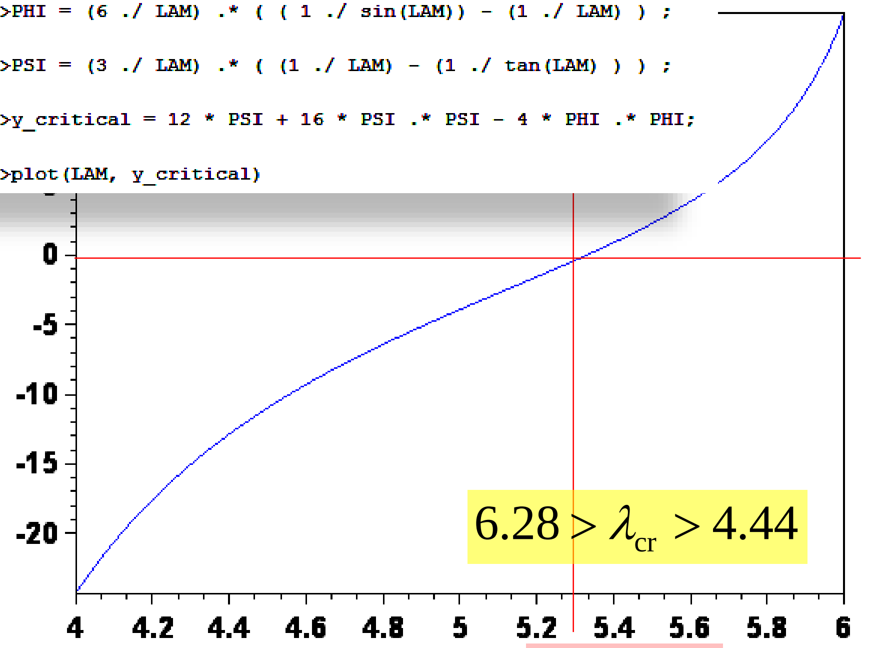
```
-->LAM = 4: 0.00001: 6;

-->PHI = (6 ./ LAM) .* ( ( 1 ./ sin(LAM)) - (1 ./ LAM) ) ;

-->PSI = (3 ./ LAM) .* ( (1 ./ LAM) - (1 ./ tan(LAM) ) ) ;

-->y_critical = 12 * PSI + 16 * PSI .* PSI - 4 * PHI .* PHI;

-->plot(LAM, y_critical)
```



$\lambda \equiv kL$ $kL = 5.33$

Geometrically non-linear analysis of frames by the Slope-deflection method

Moderate rotations and loads close to critical load but not over

Recall from previous course (beams and frames)

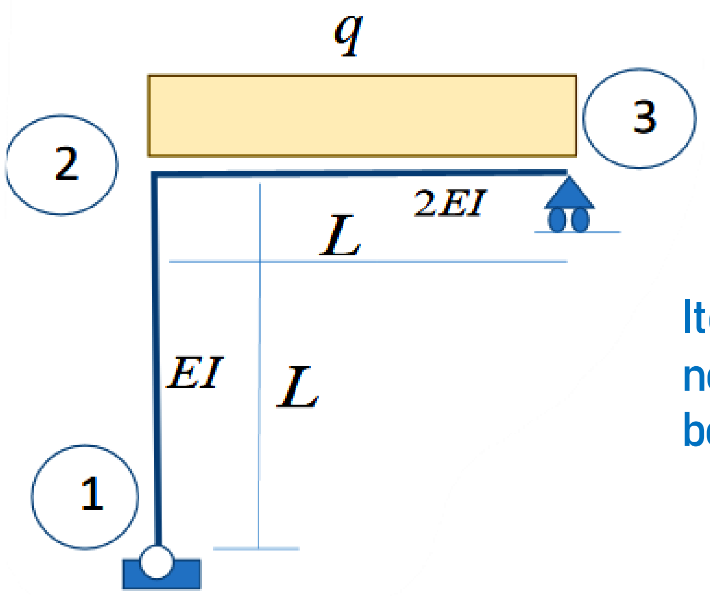
$$\varphi_{21}=\varphi_{23} \Rightarrow \frac{L}{3EI}\Psi(kL)M_{21}+\psi_{21}=\frac{L}{6EI}M_{23}+\frac{qL^3}{48EI}$$

$$(1+2\Psi(kL))M_2+\frac{6EI}{L}\psi_{21}=\frac{qL^2}{8}$$

Q: DETERMINE THE BENDING MOMENT AT RIGID JOINT #2

$$Q_{21}=0 \quad\Rightarrow\quad -\frac{M_2}{L}-P\psi_{21}=0 \Rightarrow \psi_{21}=-\frac{M_2}{PL}$$

$$(1+2\Psi(kL))M_2-\frac{3M_2}{k^2L^2}=\frac{qL^2}{8}$$



Iterations are needed to solve the bending moment:

$$\Rightarrow \begin{cases} M_2=\dfrac{qL^2}{8}\,\dfrac{PL^2}{PL^2(1+2\Psi(kL))-6EI} \\ N_{21}+Q_{32}=0 \end{cases}$$

Express Q_{32} in terms of end-moments

for $i \in 1..10$

$q \leftarrow 80\frac{kN}{m}$

$EI \leftarrow 2.1\cdot10^3\cdot kN\cdot m^2$

$L \leftarrow 6m$

$a \leftarrow q\cdot\frac{L^3}{48\cdot EI}$

$P_0 \leftarrow \frac{q\cdot L}{2}$

$M_0 \leftarrow 0$

$P_i \leftarrow P_0-\frac{M_{i-1}}{L}$

$kL_i \leftarrow \sqrt{\frac{P_i\cdot L^2}{EI}}$

$ps_i \leftarrow \left(\frac{3}{kL_i}\right)\cdot\left(\frac{1}{kL_i}-\frac{1}{\tan(kL)_i}\right)$

$M_i \leftarrow \frac{6\cdot P_i\cdot L\cdot EI\cdot a}{P_i\cdot L^2\cdot\left[1+2(ps)_i\right]-6\cdot EI}$

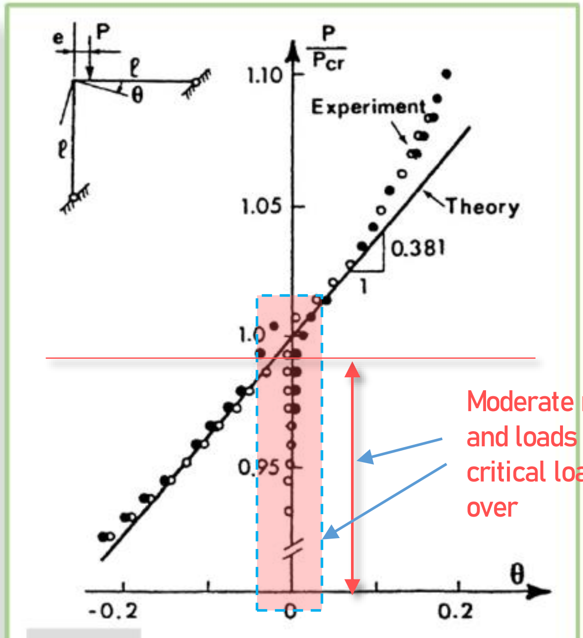
M

= (J)

	0
0	0
1	1.465·105
2	1.678·105
3	1.716·105
4	1.723·105
5	1.724·105
6	1.724·105
7	1.724·105
8	1.724·105
9	1.724·105
10	1.724·105

Geometrically non-linear analysis of frames by the Slope-deflection method

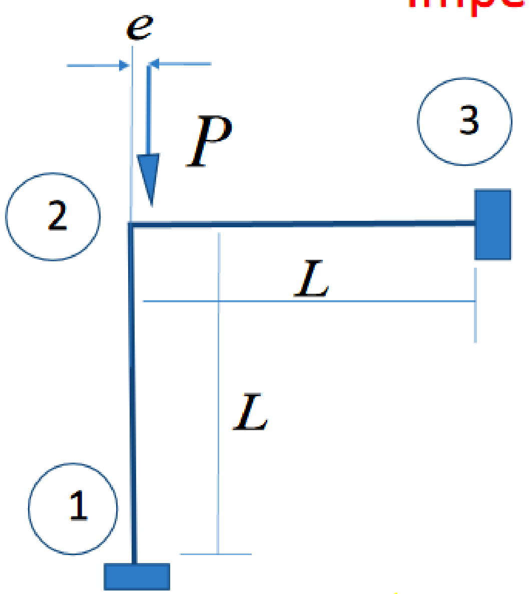
Moderate rotations and loads close to critical load but not over



Roorda's (1971) experimental verification of calculated postcritical response in asymmetric bifurcation of a Γ-frame.

Roorda, 1971, *An experience in equilibrium and stability*, Techn. Note No. 3, Solid Mech. Div., University of Waterloo, Canada.

Imperfection, eccentricity



$$M_{21} + M_{23} - Pe = 0 \Rightarrow \varphi_2 = \frac{Pe}{(A_{21} + a_{23})}$$

$$\begin{cases} M_{21} = A_{21}\varphi_2 = Pe\dfrac{A_{21}}{(A_{21} + a_{23})} \\ M_{23} = a_{23}\varphi_2 = Pe\dfrac{a_{23}}{(A_{21} + a_{23})} \\ M_{32} = b_{32}\varphi_2 = \dfrac{M_{23}}{2} \end{cases}$$

$$Q_{23} = -\frac{M_{23} + M_{32}}{L} = -\frac{3}{2}\frac{M_{23}}{L} = -Pe\frac{3a_{23}}{2(A_{21} + a_{23})}$$

NB. Q_{23} have the same sign as *e*

Recall from previous course (beams and frames)

$$N_{21} = P + Q_{23} = P(1 - e\frac{3a_{23}}{2(A_{21} + a_{23})}) = P(1 - e\frac{1}{\dfrac{\Psi(kL)}{4\Psi^2(kL) - \Phi^2(kL)} + \dfrac{2}{3}})$$

If now the eccentritcity *e* is negative, the value of the compressive load *P* is increasing all the time, and no convergence will be reached. If positive, the convergence is reached.

should be N_{21}

Computational Linear and non-linear buckling analysis

Use FEM softwares, at least for the questions 2) and 3)

Free Exercise - 20 extra-points for HW

1. Perform (elastic) linear buckling analysis for the perfect geometry and find the critical load and the respective buckling mode [4 points]

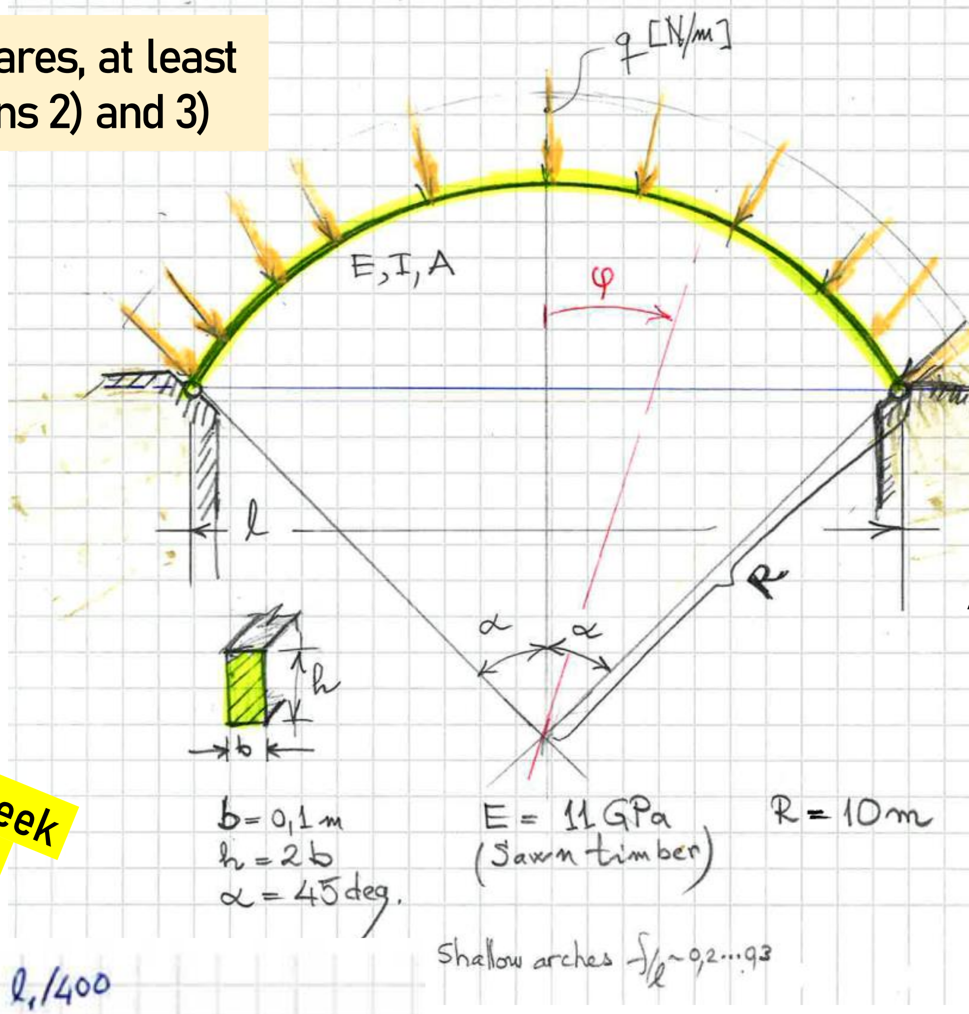
2. Find the second buckling load and the buckling mode [1 pnts]

3. Analysis the shape imperfection effect on the buckling load (GNA) [15 points]

Deadline: max. 1 week after the first exam.

For that do:

- Take a) the first buckling mode and then the second one (or their combination) multiplied by L/400 (L distance between mode nodes, as in Figs. on right) as a shape imperfection to add for the perfect geometry or b) instead of adding a tiny combination from buckling modes, just add the two shape imperfections given in the figures, separately or as a linear combination.
  - Determine the load-displacement curve at some characteristic points
  - What is the limit load? How much the buckling load of the perfect arch is reduced?

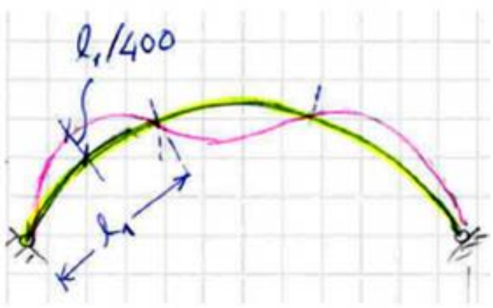


Anaytical solution:

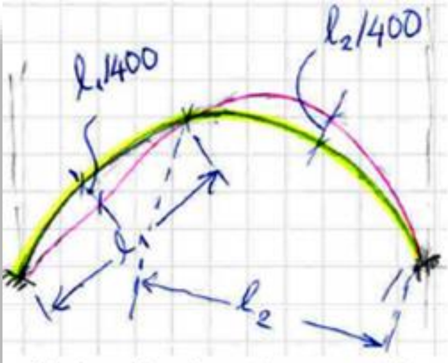
$$R\cdot q_{cr} = \frac{\pi^2 EI}{R^2}\left(\frac{1}{\alpha^2} - \frac{1}{\pi^2}\right)$$

$$\pi^2 \approx g \approx 10$$

$$R\cdot q_{cr} \approx \frac{\pi^2 EI}{R^2}\left(\frac{1}{\alpha^2} - 0.1\right)$$



a) loading is symmetric



b) loading is antisymmetric

Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Timoshenko column

Engesser (1891)

Timoshenko (1921)

There is cases when the effect of shear deformation should be considered.

$$\gamma = -\theta + v'.$$

$$\Delta\Pi = \frac{1}{2}\int_\ell EI\kappa^2 \mathrm{d}x + \frac{1}{2}\int_\ell k_s GA\gamma^2 \mathrm{d}x - \frac{1}{2}P\int_\ell (v')^2 \mathrm{d}x$$

the curvature

$$\kappa = -v''(1-\alpha P)$$

$$\gamma = \alpha P v',$$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

linearised buckling equation

$$(1-\alpha P)[EIv'']'' + Pv'' = 0$$

mean shear stress $\bar{\tau} = Q_y(x)/A$:

ξ being the *shear correction coefficient*

$$Q_y(x) = k_s GA\gamma = \frac{GA}{\xi}\gamma$$

$$\gamma \equiv \gamma_{xy} = \frac{\tau_{xy}}{G} = \xi\frac{Q_y}{GA} \equiv \alpha Q_y$$

$$\gamma(x) \equiv \gamma_{xy} = u_y + v_x = -\theta(x) + v'(x),$$

$$\gamma = \alpha P v',$$

$$M = EI\theta' = EI\kappa = EI(\gamma' - v'')$$

$$Q = GA\gamma/\xi = \gamma/\alpha$$

$$\begin{cases} Q - Pv' & = 0 \\ M'' - Pv'' & = 0, \end{cases}$$

Timoshenko column

Engesser (1891)

Timoshenko (1921)

There is cases when the effect of shear deformation should be considered.

Change of strain energy during buckling

$$\Delta\Pi = \frac{1}{2}\int_\ell EI\kappa^2 \mathrm{d}x + \frac{1}{2}\int_\ell k_s GA\gamma^2 \mathrm{d}x - \frac{1}{2}P\int_\ell (v')^2 \mathrm{d}x$$

Increment of work of external force during buckling

the curvature

$$\kappa = -v''(1-\alpha P)$$

$$\gamma = \alpha P v',$$

$$\gamma = -\theta + v'.$$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

linearised buckling equation

$$(1-\alpha P)[EIv'']'' + Pv'' = 0$$

linearised buckling equation

$$\begin{cases} v^{(4)} + k^2 v'' = 0 \\ k^2 = \dfrac{P}{EI}\dfrac{1}{1-\alpha P} \\ \alpha = \dfrac{\xi}{GA} \end{cases}$$

buckling of a cantilever column

$$\Downarrow$$

$$P^{\mathrm{T}} = P^{\mathrm{E}}\frac{1}{1+\alpha P^{\mathrm{E}}},$$

Timoshenko buckling load

Euler buckling load

Timoshenko column

Reduction coefficient of the Euler buckling load

Engesser (1891)

Timoshenko (1921)

Analysis of the results

$$\alpha = \frac{\xi}{GA}$$

all end-conditions excepts for fixed-pinned

$$P^{\mathrm{T}} = P^{\mathrm{E}} \frac{1}{1 + \frac{P^{\mathrm{E}}}{k_s GA}} = P^{\mathrm{E}} \frac{1}{1 + \alpha P^{\mathrm{E}}},$$

fixed-pinned ends

$$P^{\mathrm{T}} = P^{\mathrm{E}} \frac{1}{1 + 1.1 \frac{P^{\mathrm{E}}}{k_s GA}} = P^{\mathrm{E}} \frac{1}{1 + 1.1 \alpha P^{\mathrm{E}}}$$

$$1 + \frac{P^{\mathrm{E}}}{k_s GA} \ldots 1 + 1.1 \frac{P^{\mathrm{E}}}{k_s GA}$$

Reduction coefficient for the Euler buckling load

buckling of a cantilever column

$$\Downarrow$$

$$P^{\mathrm{T}} = P^{\mathrm{E}} \frac{1}{1 + \alpha P^{\mathrm{E}}},$$

Timoshenko buckling load

Euler buckling load

Reduction coefficient

$$\alpha P^{\mathrm{E}} = \frac{\xi}{GA} \cdot \mu \frac{\pi^2 EI}{\ell^2} = \xi \, \mu \, \pi^2 \frac{E}{G} \left[\frac{I/A}{\ell} \right]^2$$

Boundary conditions effects

Material effects
Linear effect

Cross-section geometry effects
Quadratic effect

Usually the decrease of the buckling load due to transverse shear effects is negligible for bars with solid cross-section. On the contrary, for some open-cross sections, the reduction may be of 50 % even.

Timoshenko column

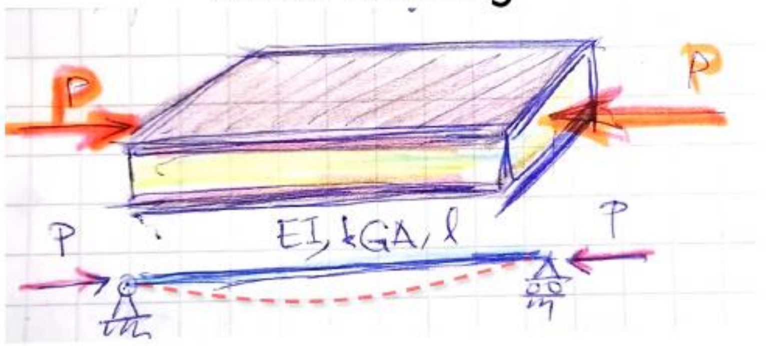
Application example - buckling of a sandwich beam

Reduction coefficient of the Euler buckling load

buckling of a cantilever column

$$\alpha = \frac{\xi}{GA} \qquad P^{\mathrm{T}} = P^{\mathrm{E}} \frac{1}{1+\alpha P^{\mathrm{E}}},$$

Global buckling



A generic sandwich beam

Buckling of a simple sandwich beam (model).

$$P_{\mathrm{T}} = \underbrace{\frac{\pi^2 EI}{\ell^2} \frac{1}{1+\pi^2 \frac{EI}{kGA\ell^2}}}_{\text{analytical}} \approx \underbrace{0.97 \frac{\pi^2 EI}{\ell^2} \frac{1}{1+\underbrace{\frac{48}{5}}_{\approx \pi^2} \frac{EI}{kGA\ell^2}}}_{\text{approx. by force method}}$$

(see the extended lecturer' notes)

$$P_{\mathrm{T}} = \underbrace{\frac{\pi^2 EI}{\ell^2}}_{\equiv P_{\mathrm{E}}} \frac{1}{1+\pi^2 \frac{EI}{kGA\ell^2}} = P_{\mathrm{E}} \frac{1}{1+\frac{P_{\mathrm{E}}}{S}} = \frac{\pi^2 B}{\ell^2} \frac{1}{1+\pi^2 \underbrace{\left[\frac{B}{S\ell^2}\right]}_{\equiv \Phi}} = P_{\mathrm{E}} \cdot \frac{1}{1+\pi^2 \Phi}$$

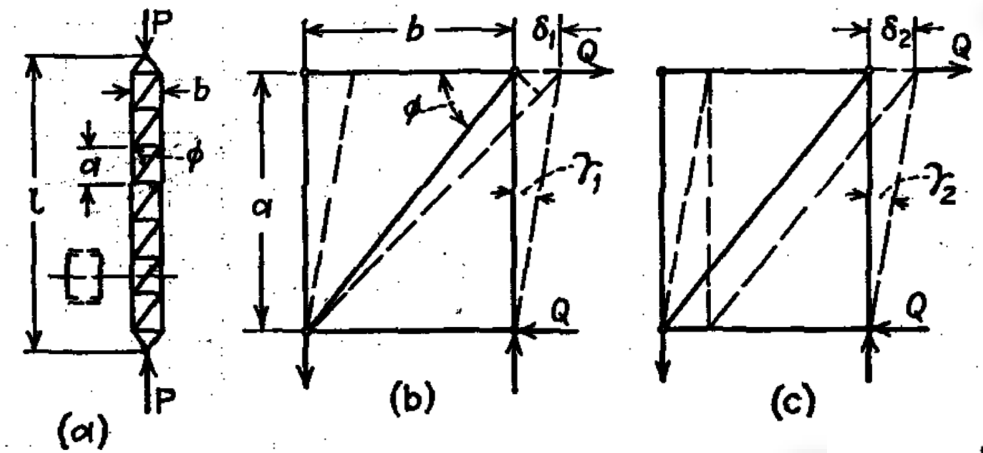
$EI \equiv B$ and $kGA \equiv S$ effective bending and shear rigidities

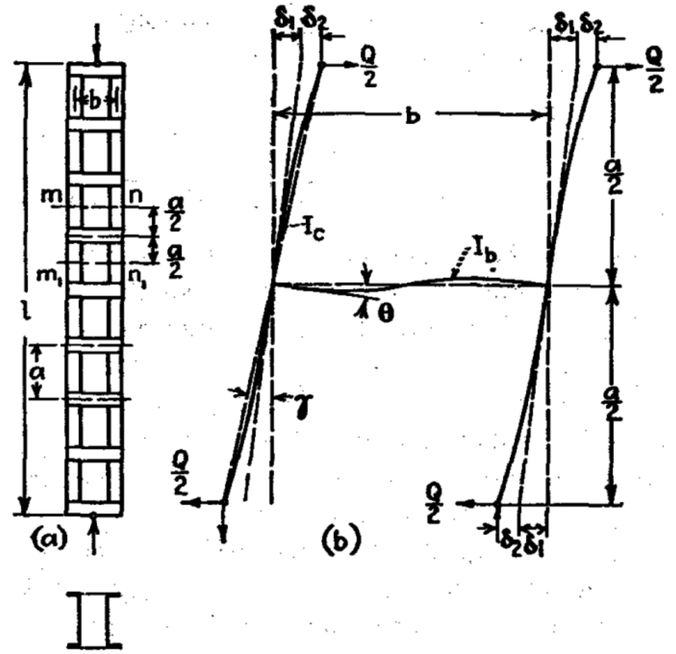
Timoshenko column

There is cases when the effect of shear deformation should be considered.

Built-in columns – 'ristikkopilari'

- Examples displayed for intellectual curiosity
- Ourdays, stability of such structures is analyzed computationally, especially because torsional stability loss is involved in addition to flexural modes which is quite complex when not impossible to analyze theoretically

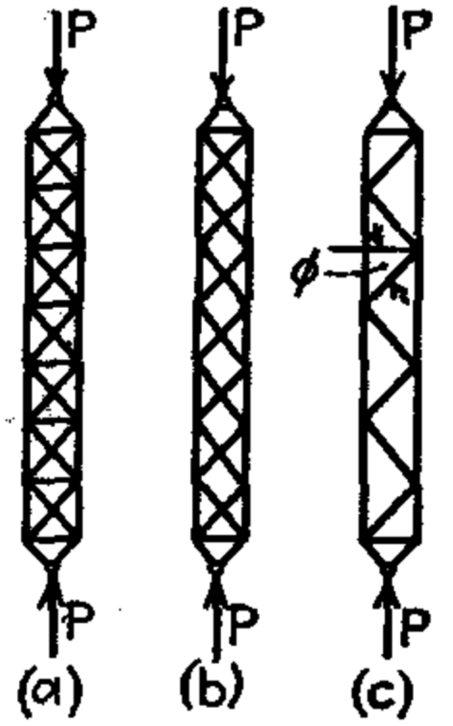




$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{1}{A_d E \sin\phi \cos^2\phi} + \frac{b}{a A_b E}\right)}$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{ab}{12EI_b} + \frac{a^2}{24EI_c}\right)}$$

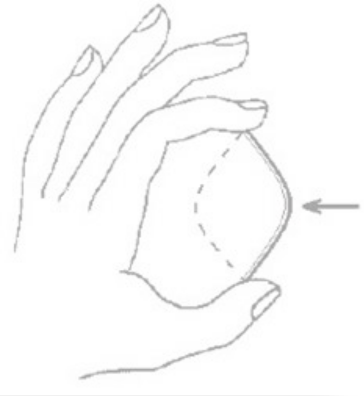
$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{ab}{12EI_b} + \frac{a^2}{24EI_c} + \frac{na}{bA_bG}\right)}$$

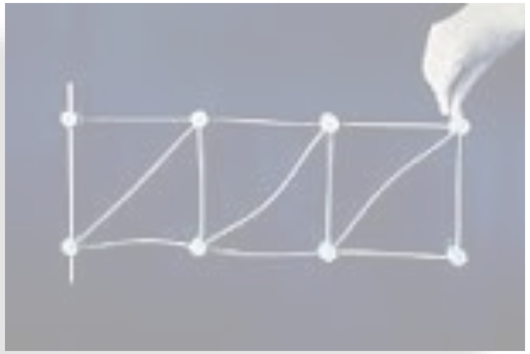


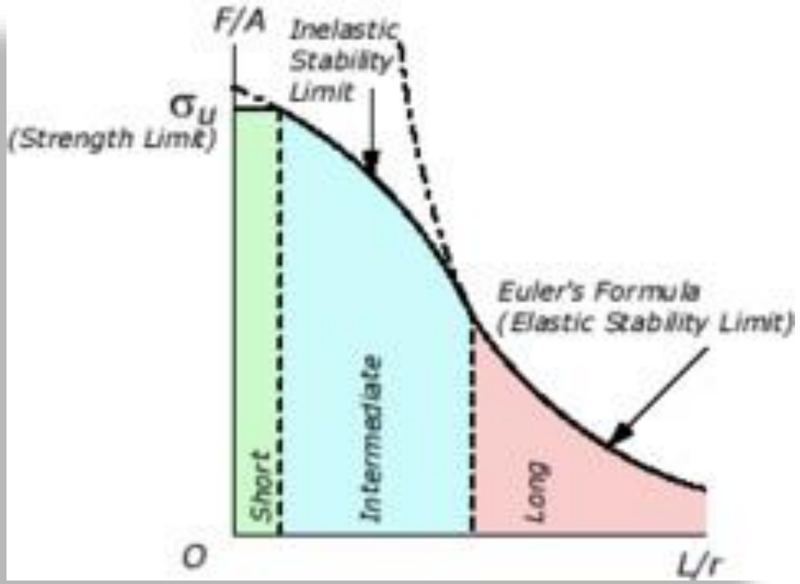
Ref. Timoshenko

Effects of imperfections

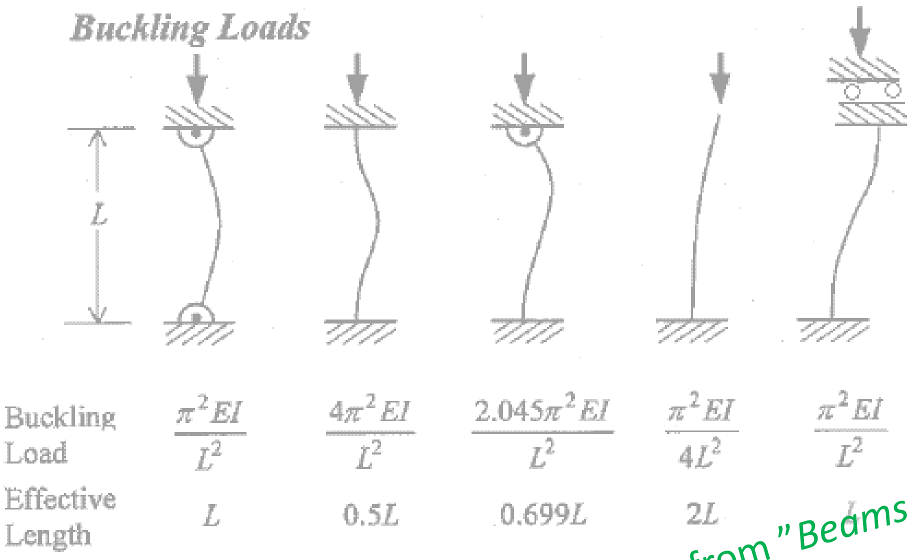
The well-known *Ayreton-Perry* design formula (Eurocode 3)



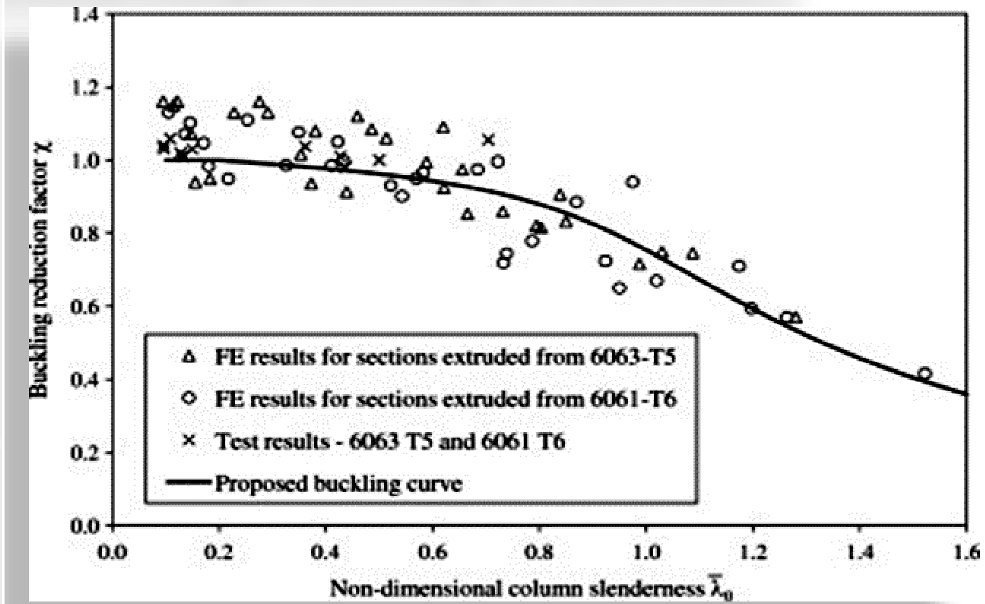




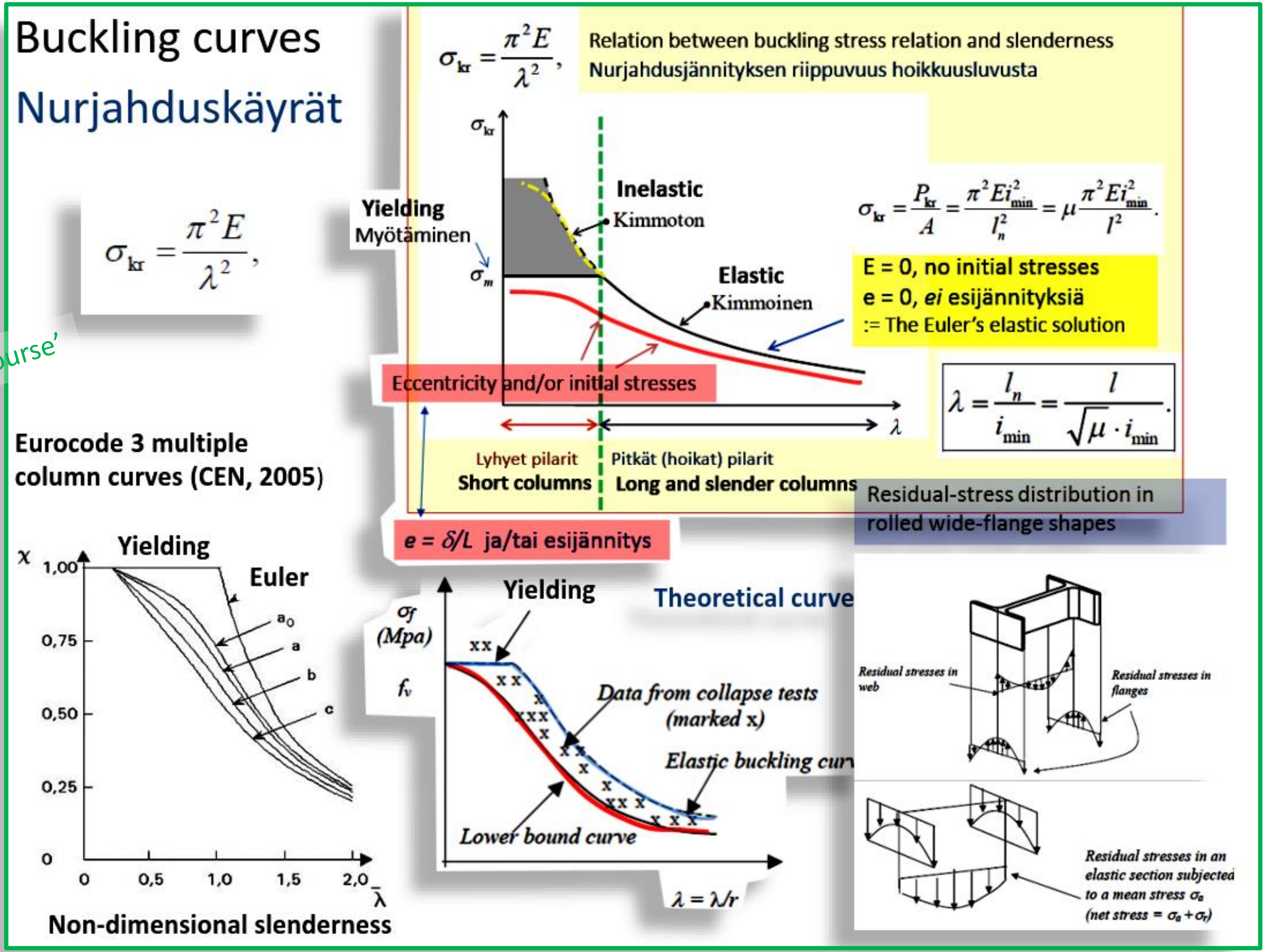
Mistä nurjahduskäyrät *tulevat*?



Slide from " Beams and Frames – course'



Slide from "Beams and Frames – course'



Effects of imperfections

Student's Readings

Ayreton-Perry design formula

(Eurocode 3)

6.3.1.2 Nurjahduskäyrät Buckling curves

(1) Aksiaalisesti puristetuille sauvoille muunnettua hoikkuutta $\bar{\lambda}$ vastaava pienennystekijä χ lask seuraavasta kaavasta käyttäen kyseeseentulevaa nurjahduskäyrää:

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{mutta } \chi \leq 1{,}0$$

missä $\Phi = 0{,}5\left[1 + \alpha\left(\bar{\lambda} - 0{,}2\right) + \bar{\lambda}^2\right]$

$$\bar{\lambda} = \sqrt{\frac{A f_y}{N_{cr}}} \quad \text{poikkileikkausluokille 1, 2 ja 3;}$$

$$\bar{\lambda} = \sqrt{\frac{A_{eff} f_y}{N_{cr}}} \quad \text{poikkileikkausluokalle 4;}$$

α on epätarkkuustekijä;

N_{cr} on kimmoteorian mukainen bruttopoikkileikkauksen mukaan laskettu kriittinen voima kyseeseen tulevassa nurjahdusmuodossa.

6.3.1.1 Nurjahduskestävyys

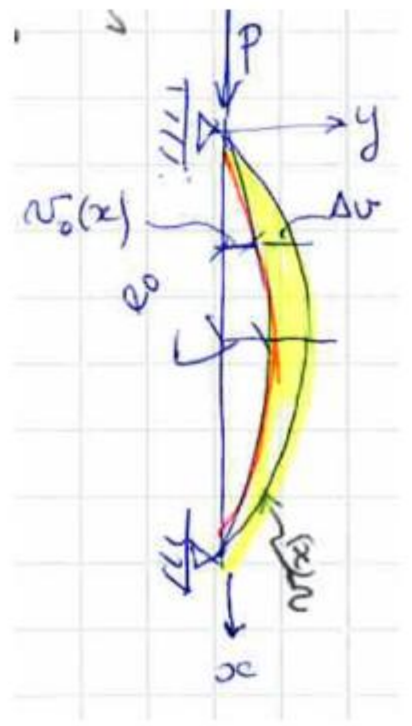
(1) Puristetut sauvat mitoiteta

$$\frac{N_{Ed}}{N_{b,Rd}} \leq 1{,}0$$

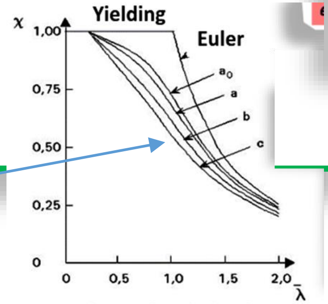
External axial load
Action

Resistance

$$N_{b,Rd} = \frac{\chi A f_y}{\gamma_{M1}}$$



Initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$.



Nurjahduskäyrä	a_0	a	b	c	d
Epätarkkuustekijä α	0,13	0,21	0,34	0,49	0,76

Effects of imperfections

Student's Readings

initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$

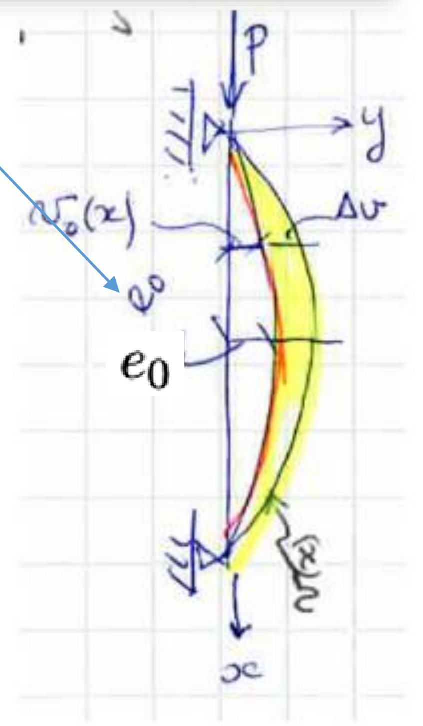
The well-known *Ayreton-Perry* design formula (Eurocode 3)

$$(EIv'')'' + Pv'' = 0$$
& four boundary conditions.

$\Longrightarrow$

Eccentricity

$$w(x) = \frac{e_0}{1-(\lambda/\pi)^2}\sin(\pi x/\ell), \qquad \lambda^2 = \frac{P\ell^2}{EI},$$



Initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$.

$$\sigma_x^{max} = \frac{N_{max}}{A} + \frac{M_{max}}{W} \le \sigma_y$$
$$\Downarrow \quad = \frac{P}{A} + \frac{M_{max}}{I}\frac{h}{2} \le \sigma_y.$$

Ayrton-Perry formula (of Eurocode 3)

$$\bar\lambda = \sqrt{A\sigma_y/N_E}.$$

$\Longrightarrow$

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar\lambda^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar\lambda + \bar\lambda^2\right]$$

Depends on *eccentricity*

$$M_{max} = M(\ell/2) = -EI(v''(\ell/2) - v_0''(\ell/2)),$$
$$\Downarrow \quad = P_{cr}e_0\frac{(\lambda/\pi)^2}{1-(\lambda/\pi)^2},$$
$$= P_{cr}e_0\frac{P/P_{cr}}{1-P/P_{cr}}.$$

$$a\bar\lambda = [e_0 h/2]/i^2$$
$$a = \pi\sqrt{E/\sigma_y}\,\frac{e_0}{\ell}\,\frac{h/2}{i}$$
$$\chi = P/P_y.$$

… and obtains:

Design formula:

$$N_s = P \le N_R = \chi\sigma_y A.$$

$$\sigma_y A = P + P_{cr}e_0\frac{h/2}{I/A}\frac{P/P_{cr}}{1-P/P_{cr}} \Longrightarrow$$
$$\frac{P_y}{P_{cr}} = \frac{P}{P_{cr}} + \frac{e_0 h/2}{i^2}\frac{P/P_{cr}}{1-P/P_{cr}}.$$

$$\bar\lambda^2 = a\bar\lambda\frac{\chi\bar\lambda^2}{1-\chi\bar\lambda^2} + \chi\bar\lambda^2 \Longrightarrow$$
$$\frac{1}{\chi} = a\bar\lambda\frac{1/\chi}{1/\chi - \bar\lambda^2} + 1$$
$$\Rightarrow \frac{1}{2}\frac{1}{\chi^2} - \underbrace{\frac{1}{2}\left[1 + a\bar\lambda + \bar\lambda^2\right]}_{\equiv\phi}\frac{1}{\chi} + \frac{1}{2}\bar\lambda^2 = 0,$$

Solve ϕ from this:

where $\bar\lambda$ is the column relative slenderness. How this formula is used in design of a steel column? Consider a steel section in compression with P. All the possible imperfections are now accounted through this buckling resistance reduction factor χ such that the inequality

Student's Readings

Ayreton-Perry design formula

(Eurocode 3)

6.3.1.2 Nurjahduskäyrät Buckling curves

(1) Aksiaalisesti puristetuille sauvoille muunnettua hoikkuutta $\bar{\lambda}$ vastaava pienennystekijä χ lask
seuraavasta kaavasta käyttäen kyseeseentulevaa nurjahduskäyrää:

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{mutta } \chi \leq 1{,}0$$

Eurocode 3

missä $\Phi = 0{,}5\left[1 + \alpha\left(\bar{\lambda} - 0{,}2\right) + \bar{\lambda}^2\right]$

$$\bar{\lambda} = \sqrt{\frac{Af_y}{N_{cr}}} \quad \text{poikkileikkausluokille 1, 2 ja 3;}$$

$$\bar{\lambda} = \sqrt{\frac{A_{eff} f_y}{N_{cr}}} \quad \text{poikkileikkausluokalle 4;}$$

α on epätarkkuustekijä;

N_{cr} on kimmoteorian mukainen bruttopoikkileikkauksen mukaan laskettu kriittinen voima
kyseeseen tulevassa nurj...

Theoretical, this lecture

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]$$

Nurjahduskäyrä					
Epätarkkuustekijä α	0,13	0,21	0,34	0,49	0,76

6.3.1.1 Nurjahduskestävyys

(1) Puristetut sauvat mitoitet...

$$\frac{N_{Ed}}{N_{b,Rd}} \leq 1{,}0$$

External axial load
Action

Resistance

$$N_{b,Rd} = \frac{\chi A f_y}{\gamma_{M1}}$$

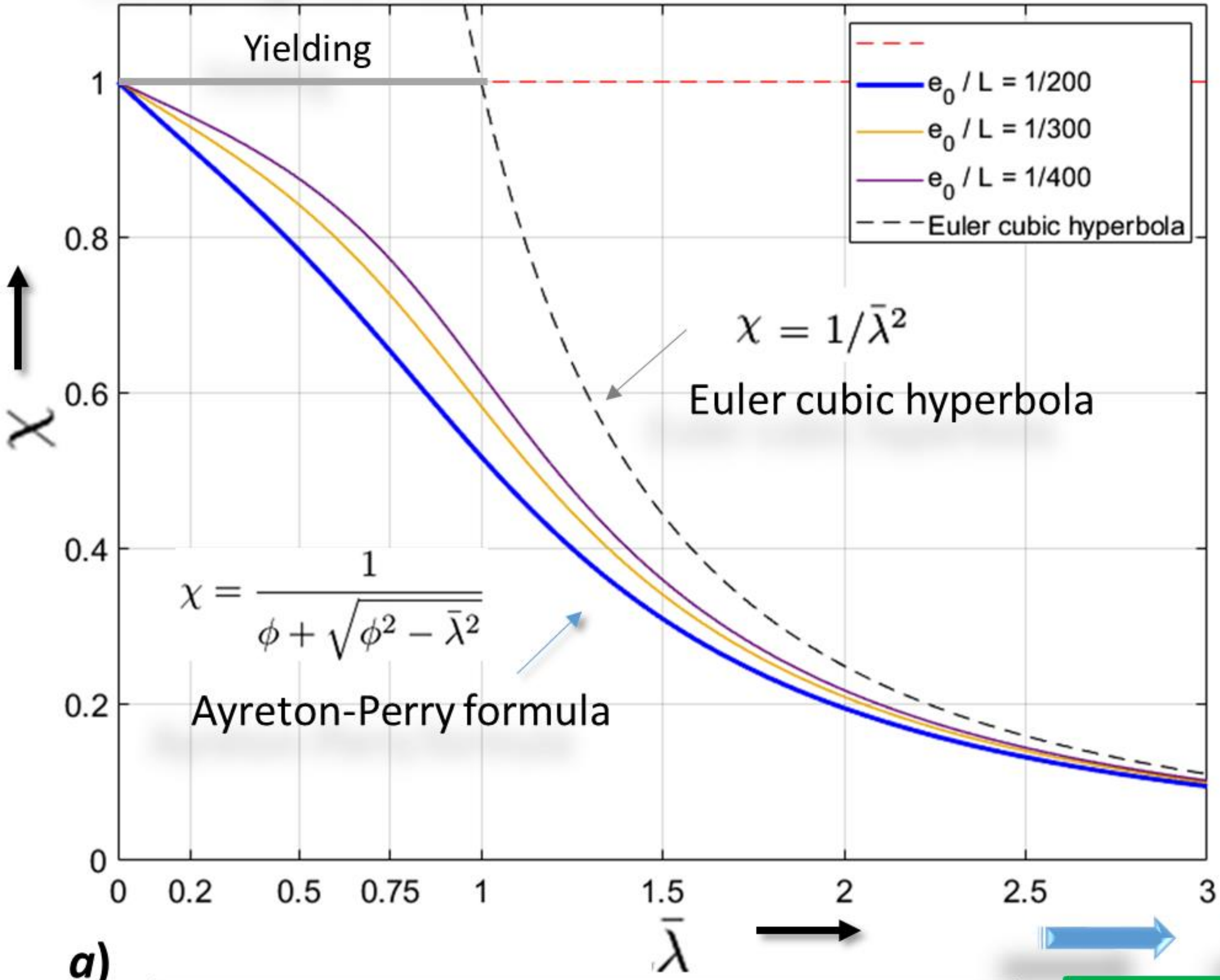
$$N_s = P \leq N_R = \chi \cdot \frac{\sigma_y A}{\gamma}$$

Ayreton-Perry design formula

Steel $\ell = 2\text{m}$,
$a = [0.446,\ 0.297,\ 0.223]$,
$i = 0.1714\ \text{m},\ h = 0.2\ \text{m}$,
$e_0/\ell = [1/400,\ 1/300,\ 1/400]$

Student's Readings

Buckling curves

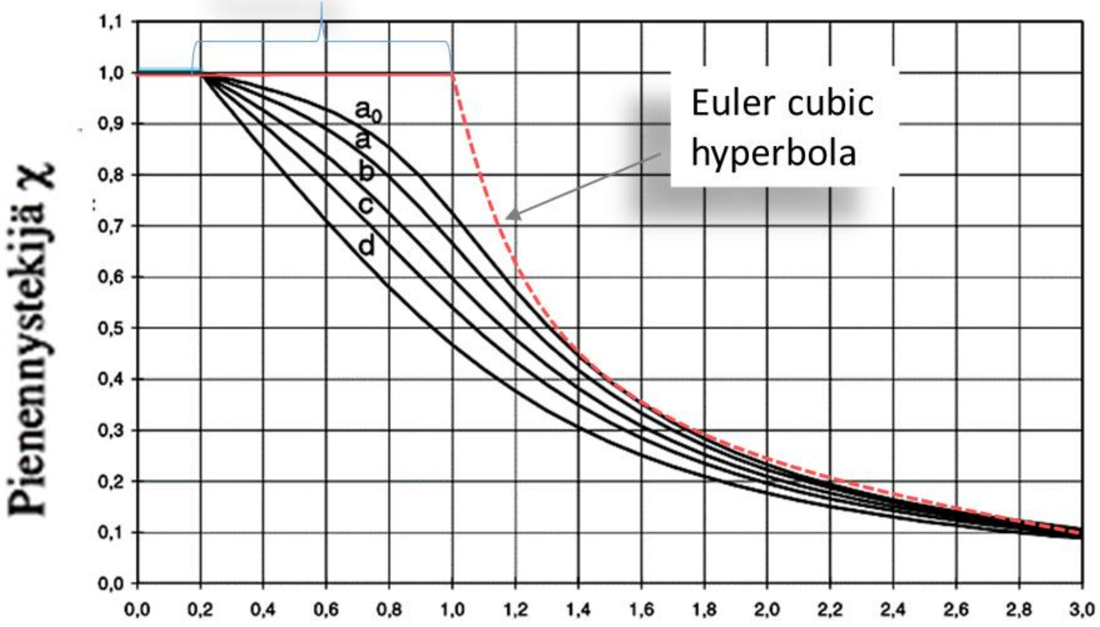


a)

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]$$

$$N_s = P \leq N_R = \chi \cdot \frac{\sigma_y A}{\gamma}$$

Eurocode buckling curves



b)

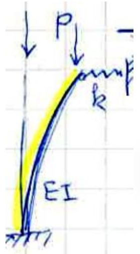
$$a\bar{\lambda} = [e_0 h/2]/i^2$$

$$a = \pi\sqrt{E/\sigma_y}\frac{e_0}{\ell}\frac{h/2}{i}$$

$$\chi = P/P_y.$$

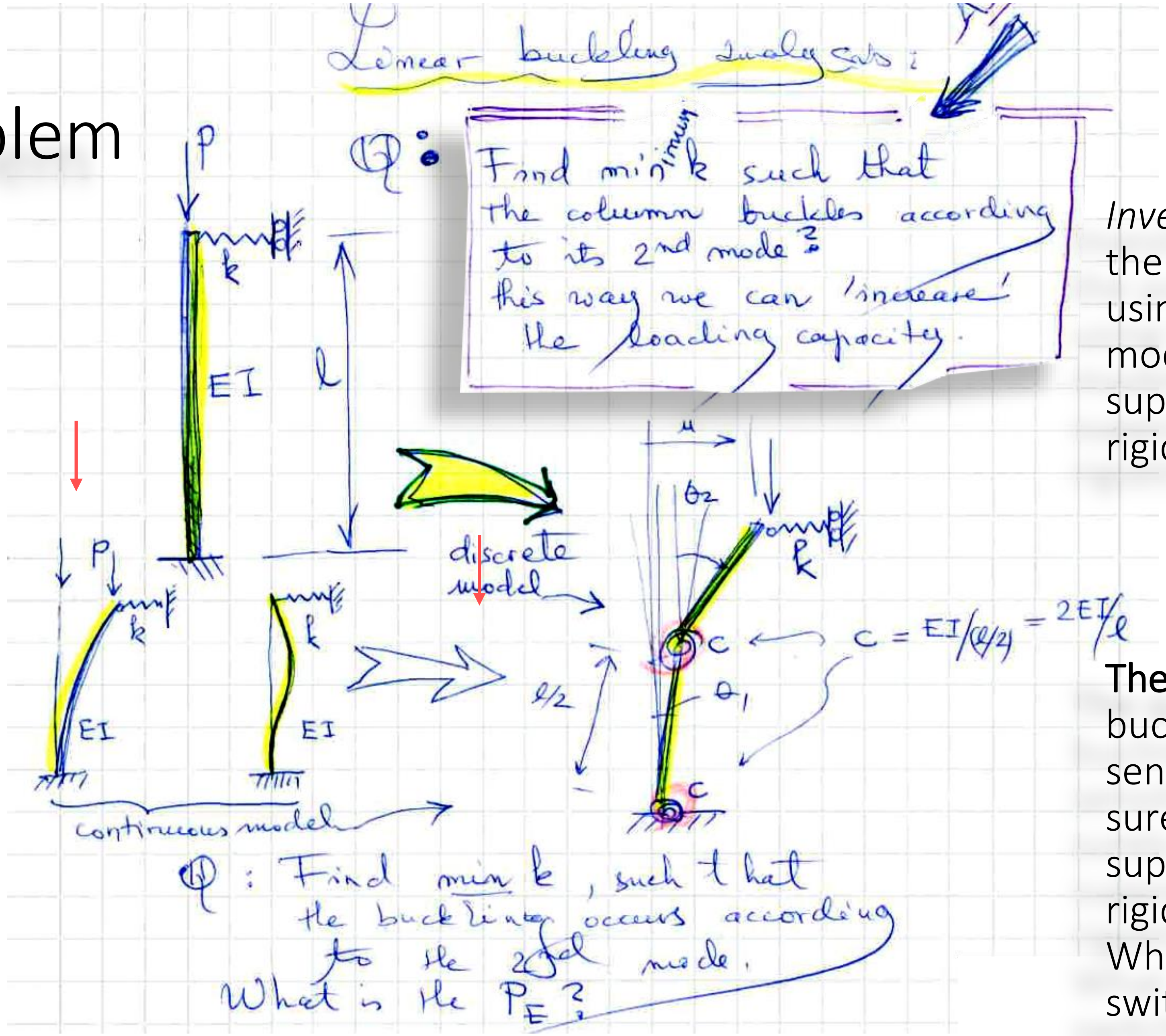
Adapted from Eurocode 3

Example of design problem



1st mode

2nd mode



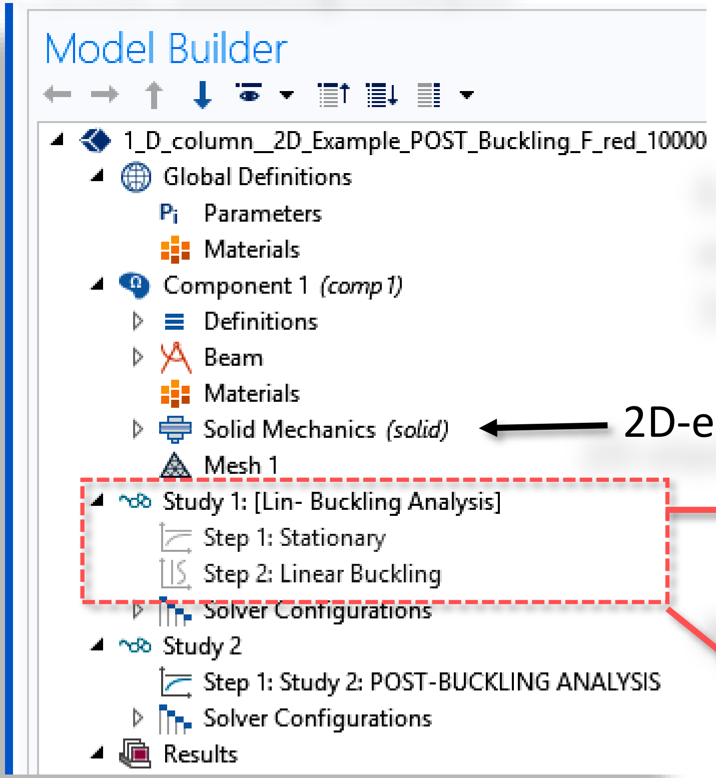
Investiagte 1) the buckling 2) the post-bucling behaviour using this discrete simple model for various ratios of support spring and rotational rigidities.

The question: are the post-bucling braches stable or not? sensitivity to imperfections? surely depends on the ratios of support spring and rotational rigidities.
What is the critical ratio for switching bto unstable mode,if any

Ref: example adapted from RK.

Linear buckling analysis of simply supported column

FE- Linear **Buckling Analysis**

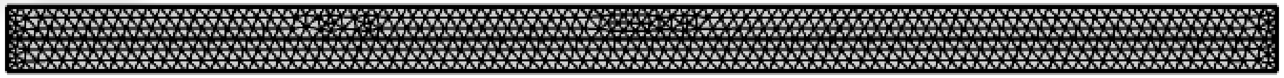


Euler analytical 1D

2D-elasticity

Buckling load

FE-Mesh



- In this FE-analysis, the column was treated as a two-dimensional elastic domain

rectangular cross-section

$$P_{E,1D} = \pi^2 EI/\ell^2.$$

height $h = 50$ mm,

width $b = \ell/10$,

$EI = 72.917$ kN.m^2

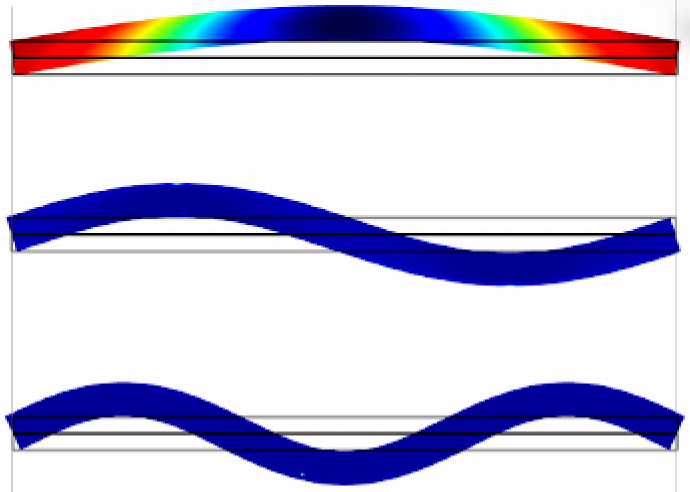
$E = 70$ GPa, $\nu = 0.33$.

$$P_{cr,2D} = 713 \text{ kN}$$

$$P_{E,1D}/P_{cr,2D} = 1.01,$$

FE- Linear **Buckling Analysis**

Buckling load

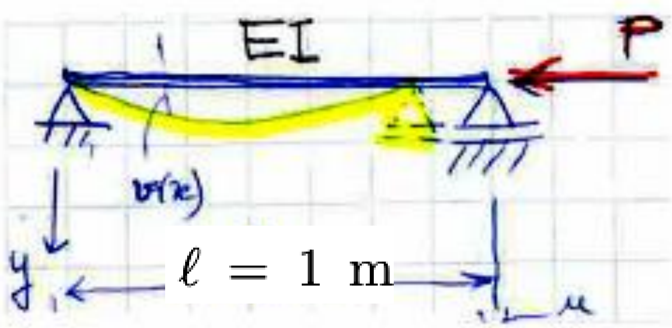


713.34 kN
λ = 1.0

2780 kN
λ = 3.9

5996 kN
λ = 8.4

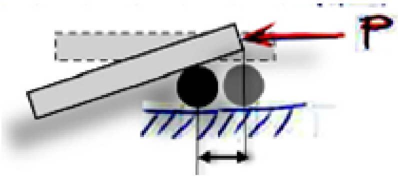
[FE-buckling analysis]
First three critical loads and respective buckling modes



P_E = 719.66 kN (analytical 1D)

$$P_E = 720 \text{ kN (1-D)},$$

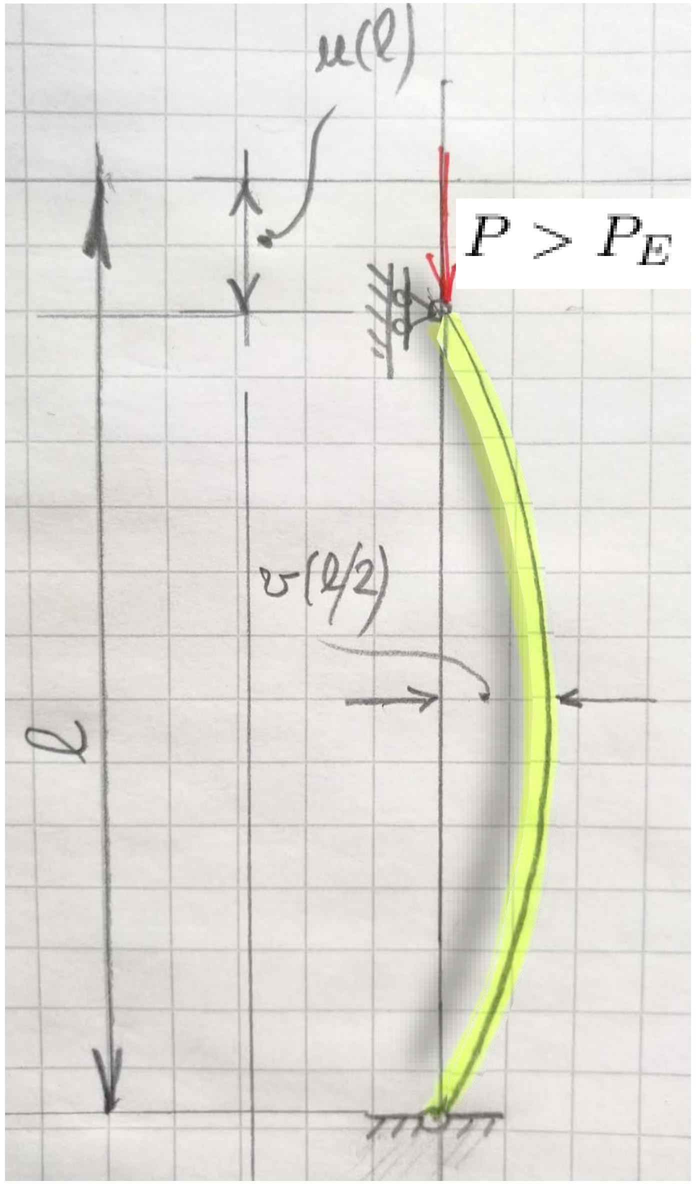
Asymptotic post-buckling analysis of simply supported column



Roller 'buckling' displacement.

$$\kappa = -\frac{v''}{\sqrt{1 - v'^2}}$$

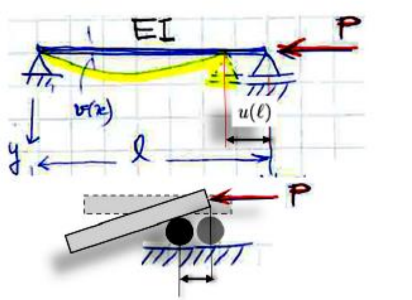
$$\mathrm{d}u = [1 - \sqrt{1 - v'^2}]\mathrm{d}x$$



Asymptotic post-buckling analysis of simply supported column

What to do: at buckling & for moderate increments

- ✓ estimate the **displacements**/rotation
- ✓ Study **stability** of post-buckling branch



$$u(\ell) \approx \frac{\pi^2 \ell}{4}\left(\frac{v_0}{\ell}\right)^2 + \frac{P_E \ell}{EA}$$

$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E} - 1\right)\cdot(P \geq P_E) + \frac{P_E \ell}{EA}$$

Roller 'buckling' displacement.

Derive the force-displacement relation

- analytical approach is used

load increase $P = P_E + \Delta P$

few percent

How to do it?

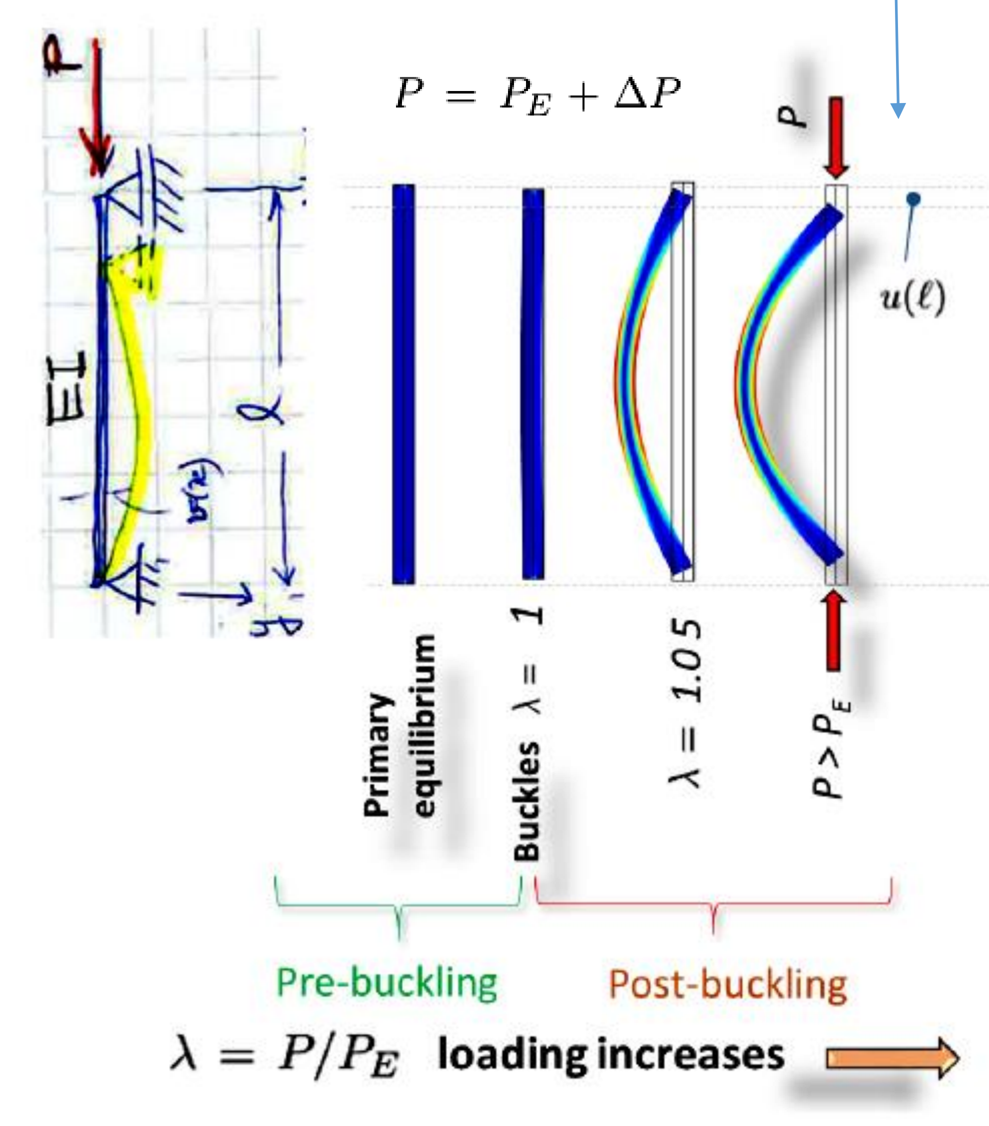
- we use the Lagrangian formulation
- assume a (bifurcational) flexural deflection mode $v(x) = v_0 \sin(\pi x/\ell).$

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI\kappa^2 dx - P\int_0^\ell \left[1 - \sqrt{1-(v')^2}\right] dx,$$

Lagrangian curvature $\kappa = -\frac{v''}{\sqrt{1-v'^2}}$

Shortening due to flexion $du/dx = 1 - \sqrt{1-(v')^2}$

FE- Post-Buckling Analysis



Post-buckling of simply supported column.

Asymptotic post-buckling analysis of simply supported column

- we use the Lagrangian formulation
- assume a (bifurcational) flexural deflection mode

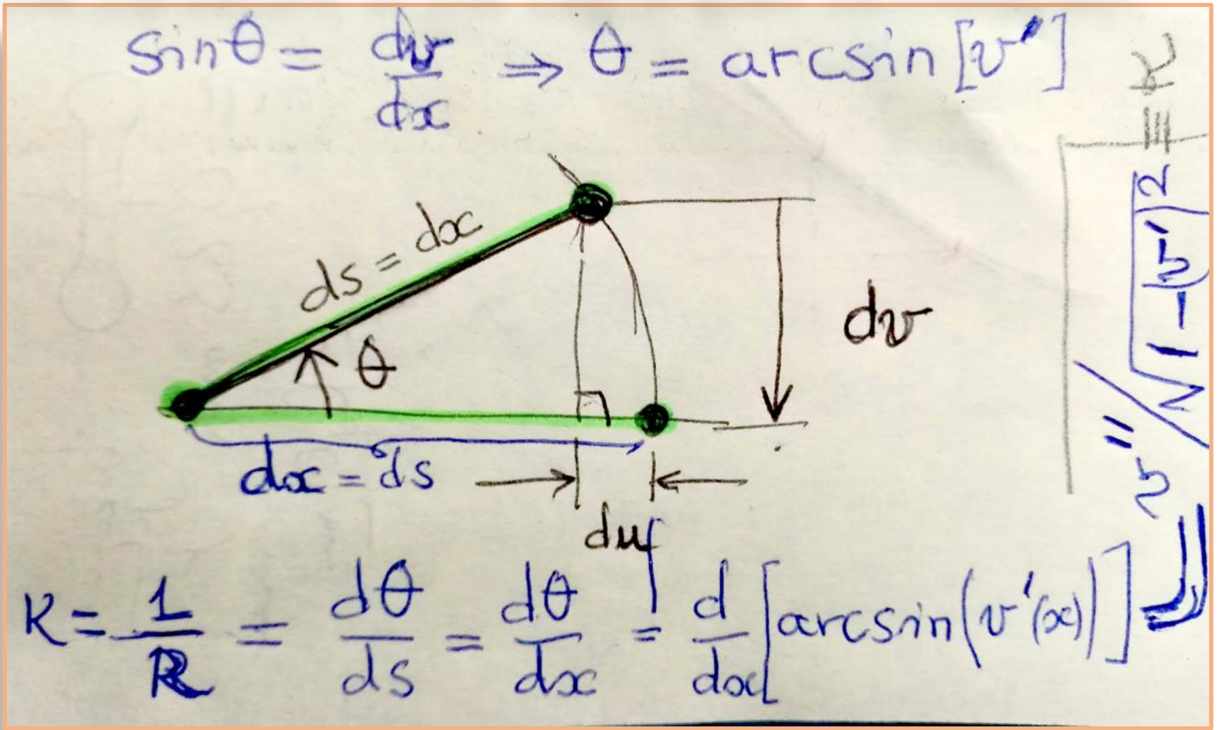
$$\Delta\Pi = \frac{1}{2}\int_0^{\ell} EI\kappa^2 \mathrm{d}x - P\int_0^{\ell}\left[1-\sqrt{1-(v')^2}\right]\mathrm{d}x,$$

Lagrangian curvature

Shortening due to bending

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}}$$

The curvature in the Lagrangian formulation:



The minus sign is because of sign convention for positive curvature

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}}$$

$$v' = \sin(\theta) \implies \theta = \arcsin(v')$$

$$\theta' = [\arcsin(v')]' = \frac{v''}{\sqrt{1-v'^2}} \equiv \kappa,$$

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}(y-Y)}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}x} = v' = \sin(\theta), \quad (\mathrm{d}s = \mathrm{d}x).$$

From the right-angle triangle (1.74) and using Pythagoras one obtains the shortening

$$\mathrm{d}u = [1-\sqrt{1-v'^2}]\mathrm{d}x$$

Asymptotic post-buckling analysis of simply supported column

What to do: at buckling & for moderate increments

- ✓ estimate the **displacements**/rotation
- ✓ Study **stability** of post-buckling branch ⟹ Derive the force-displacement relation

How to do it?

- we use the **Lagrangian** formulation
- assume a (bifurcational) flexural deflection mode $v(x) = v_0 \sin(\pi / x\ell)$

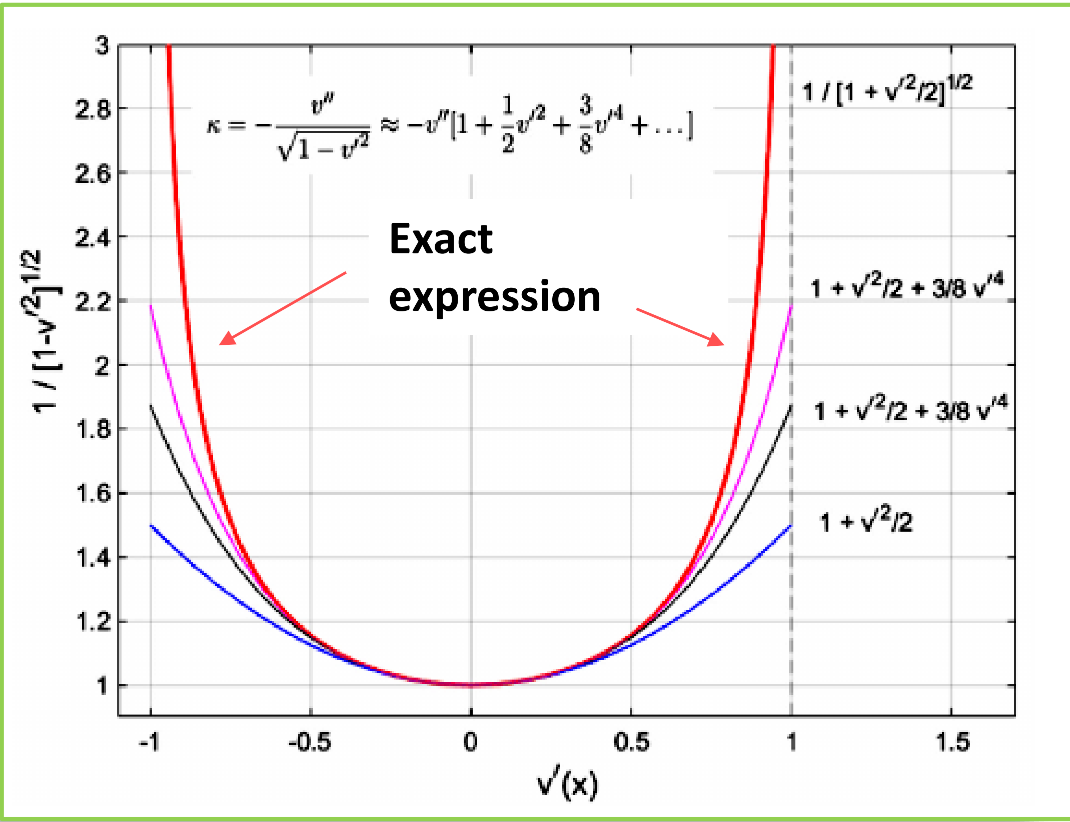
$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI\kappa^2 \mathrm{d}x - P\int_0^\ell \left[1-\sqrt{1-(v')^2}\right]\mathrm{d}x,$$

Lagrangian curvature

Shortening due to flexion

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}} \approx -v''[1+\frac{1}{2}v'^2+\frac{3}{8}v'^4+\ldots]$$

$$\mathrm{d}u/\mathrm{d}x = 1-\sqrt{1-(v')^2} \approx 1-[1-\frac{1}{2}v'^2] = \frac{1}{2}v'^2$$



Taylor expansions

Taylor expansions with only two terms

$$\Longrightarrow \quad \Delta\Pi \approx \frac{1}{2}\int_0^\ell EIv''^2[1+\frac{1}{2}v'^2]^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x,$$

Asymptotic post-buckling analysis of simply supported column

Assume a (bifurcational) flexural deflection mode $v(x) = v_0 \sin(\pi x/\ell)$

$$\Delta\Pi \approx \frac{1}{2}\int_0^\ell EI v''^2[1+\frac{1}{2}v'^2]^2 \mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2 \mathrm{d}x,$$

$$\Delta\Pi = -\frac{\pi^2}{4}P\ell\left(\frac{v_0}{\ell}\right)^2 + \frac{\pi^2 EI}{\ell^2}\cdot\frac{\pi^2}{128}\left(\frac{v_0}{\ell}\right)^2\cdot\ell\left[32+8\pi^2\left(\frac{v_0}{\ell}\right)^2+\pi^4\left(\frac{v_0}{\ell}\right)^4\right]$$

$$\Downarrow \quad = -\frac{\pi^2}{4}P\ell\delta^2 + P_E\cdot\frac{\pi^2\ell}{128}\delta^2\left[32+8\pi^2\delta^2+\pi^4\delta^4\right] \equiv \Delta\Pi(\delta,\lambda;\ell),$$

$$\delta(\Delta\Pi(v_0;P)) = 0 \implies \mathrm{d}\Delta\Pi(v_0;P)/\mathrm{d}v_0 = 0 \implies$$

$$\Downarrow \quad \implies P = \frac{\pi^2 EI}{\ell^2} + \frac{1}{2}\frac{\pi^2 EI}{\ell^2}\cdot\pi^2\left(\frac{v_0}{\ell}\right)^2 + \frac{3}{32}\frac{\pi^2 EI}{\ell^2}\cdot\pi^4\left(\frac{v_0}{\ell}\right)^4$$

$$P = P_E\left[1+\frac{1}{2}\cdot\pi^2\left(\frac{v_0}{\ell}\right)^2+\frac{3}{32}\cdot\pi^4\left(\frac{v_0}{\ell}\right)^4\right].$$

The asymptotic force-displacement relation

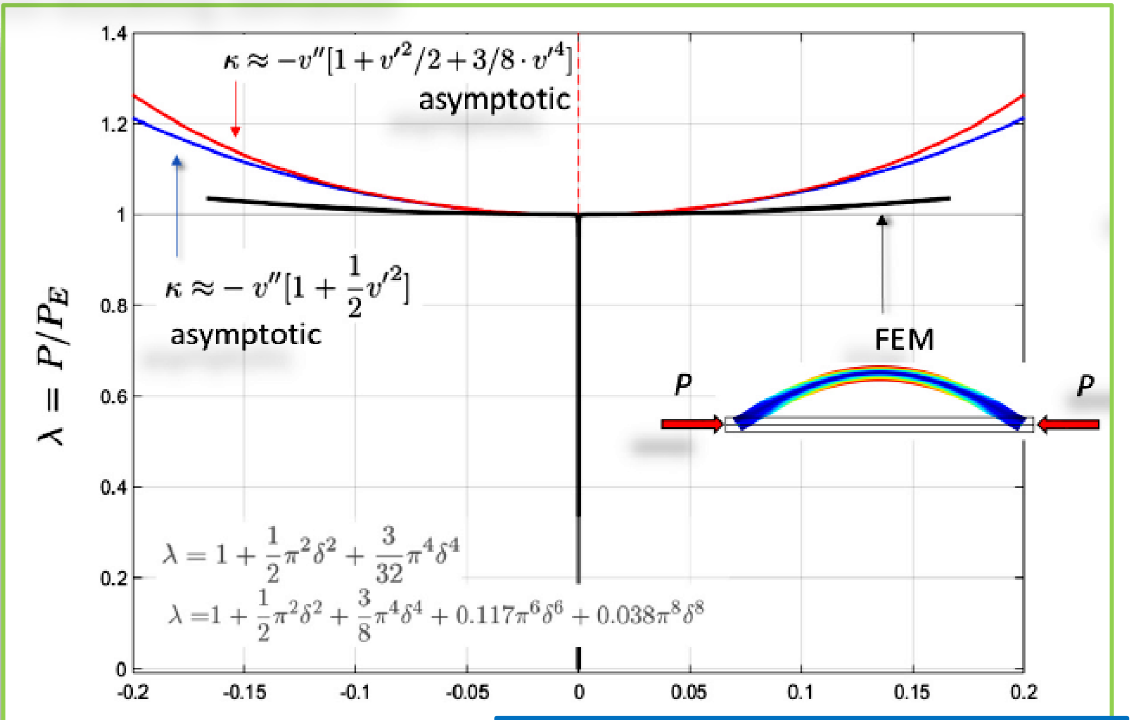
$$\lambda \approx 1+\frac{1}{2}\pi^2\delta^2+\frac{3}{32}\pi^4\delta^4 = 1+\frac{1}{2}\pi^2\delta^2\left[1+\frac{2\cdot 3}{32}\pi^2\delta^2\right]$$

$\delta = v_0/\ell$ $\qquad \lambda = P/P_E$

Taylor expansions with only two terms

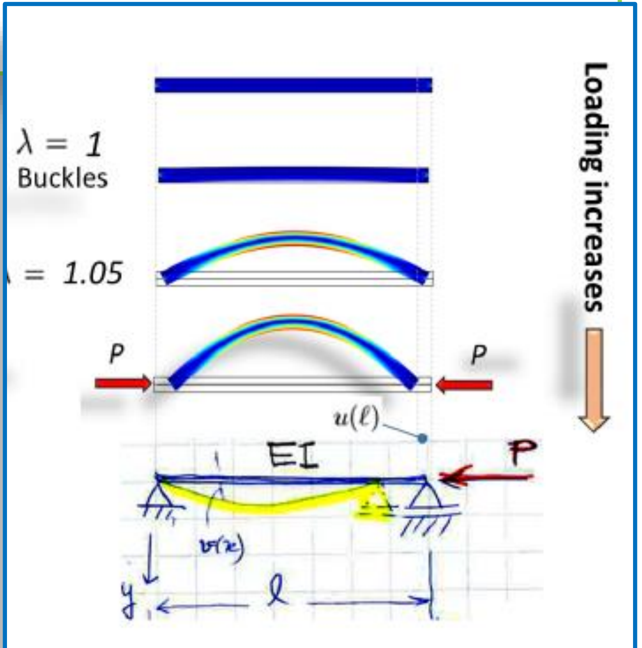
Matlab symbolic toolbox.

Post-buckling behavior



$\delta = v_0/\ell$

FE-Post-buckling behavior

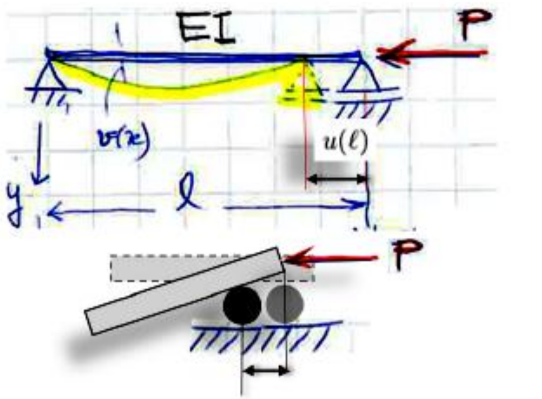


Asymptotic post-buckling analysis of simply supported column

The asymptotic post-buckling analysis provides also the value of column shortening and rotations at buckling

$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E} - 1\right) \cdot (P \geq P_E) + \frac{P_E \ell}{EA},$$

logical proposition $(P \geq P_E) = 1$ when **true**, otherwise, zero.



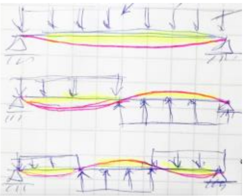
$$u(\ell) \approx \frac{\pi^2 \ell}{4}\left(\frac{v_0}{\ell}\right)^2 + \frac{P_E \ell}{EA}$$

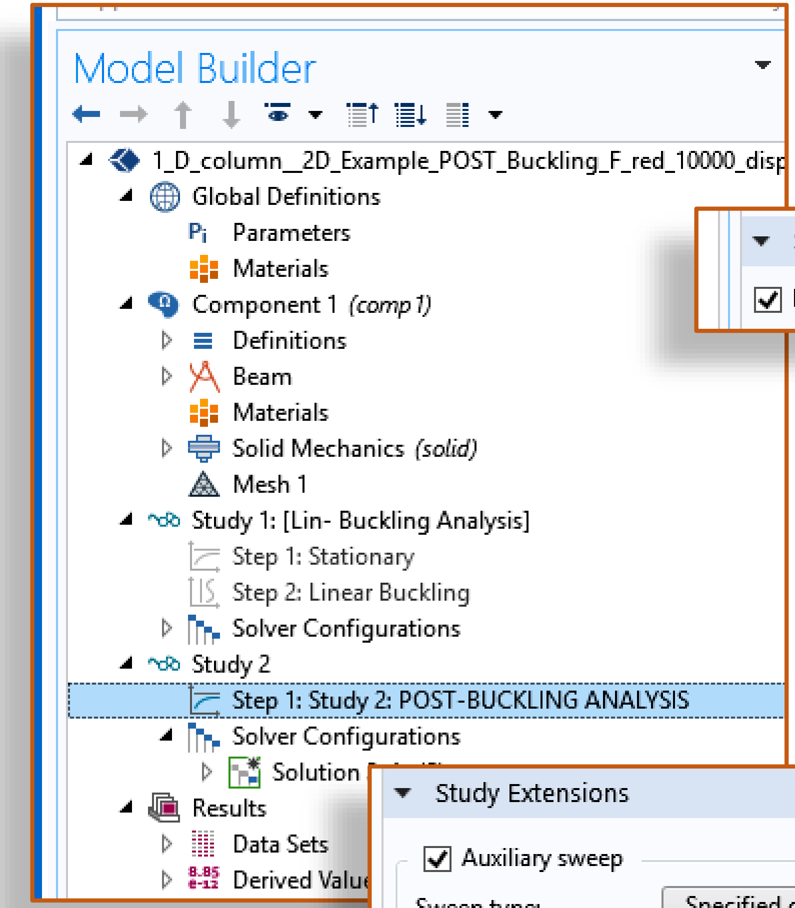
$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E} - 1\right) \cdot (P \geq P_E) + \frac{P_E \ell}{EA}$$

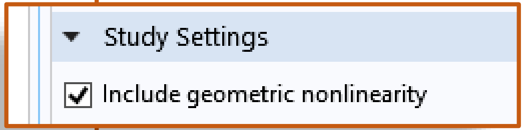
Roller 'buckling' displacement.

FE-based post-buckling analysis of axially compressed column

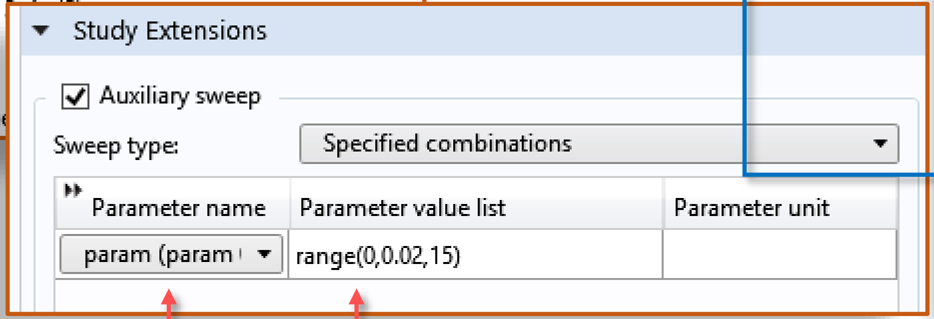
- Perturbed with tiny transversal distributed load
- Can also be given as initial shape imperfection



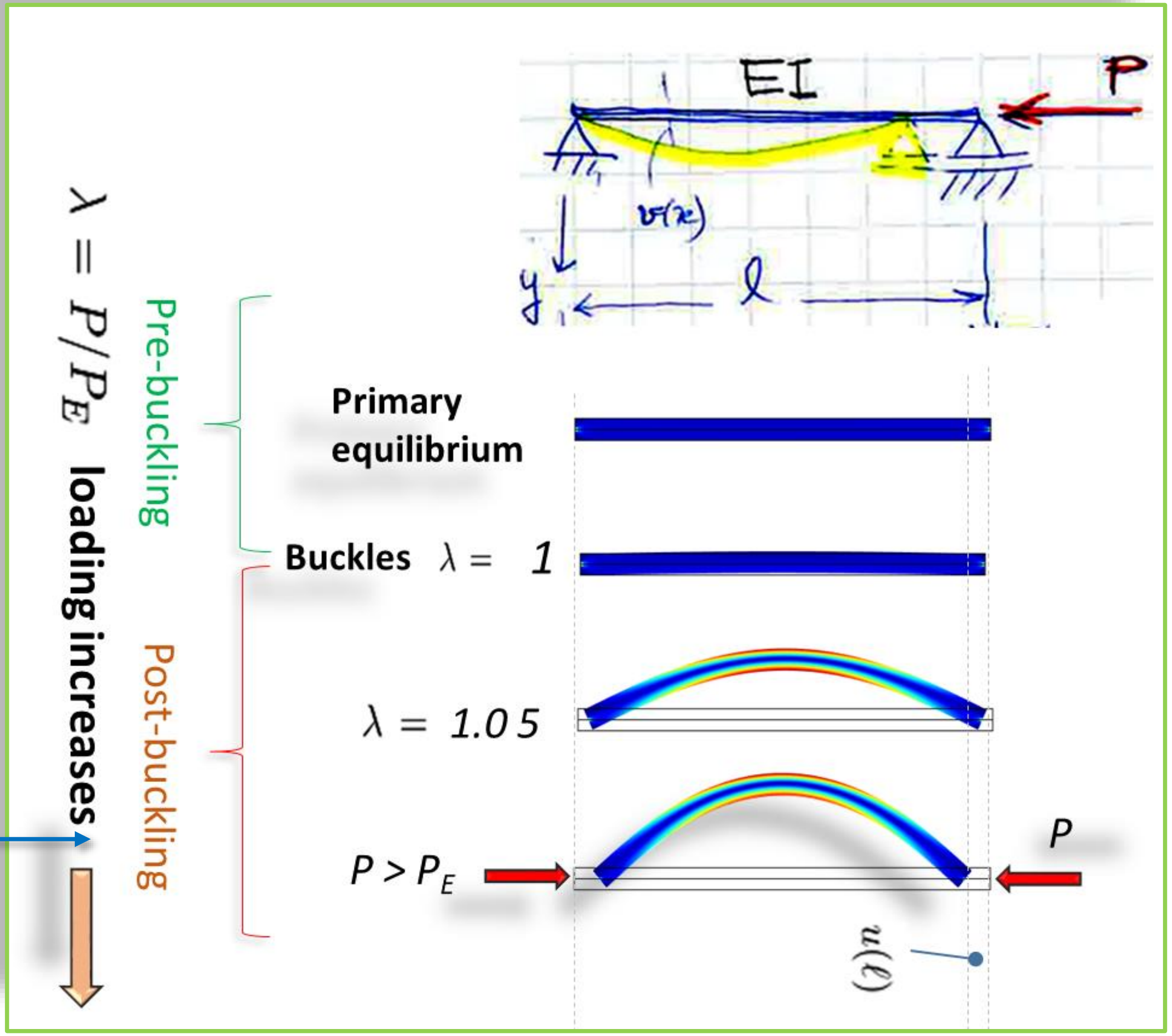




Uses: Finite strains and large displacements theory

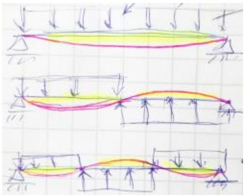


$$\lambda = P/P_E$$

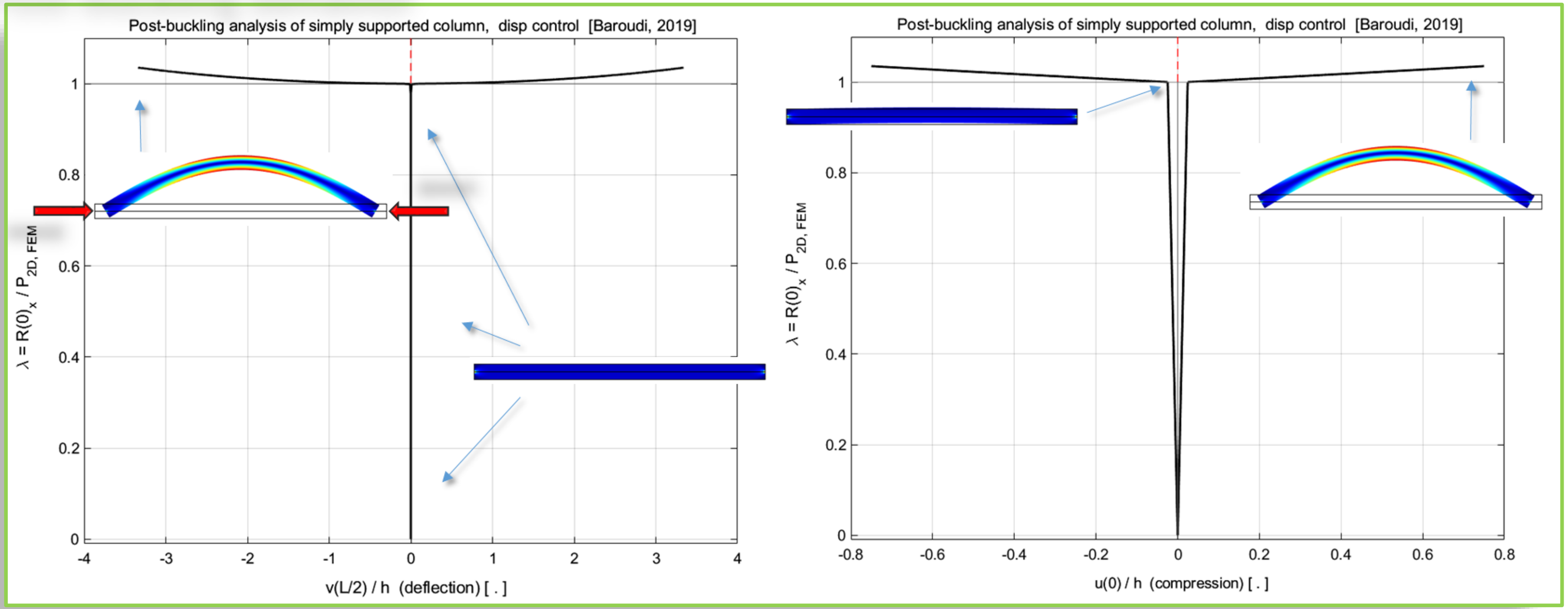


FE-based post-buckling analysis of axially compressed column

- Perturbed with tiny distributed load
- Can also be given as initial shape imperfection

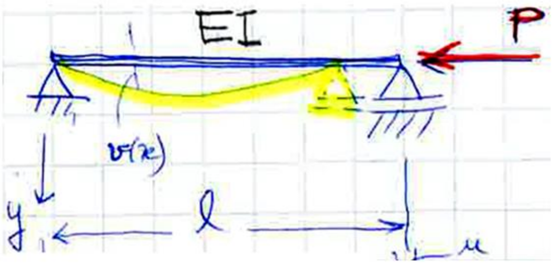


Post-buckling behavior



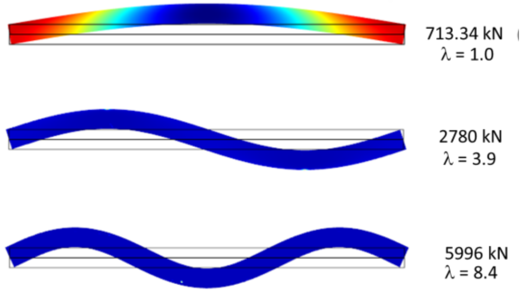
Flexural deflection v(L/2) / h

Axial shortening u / h

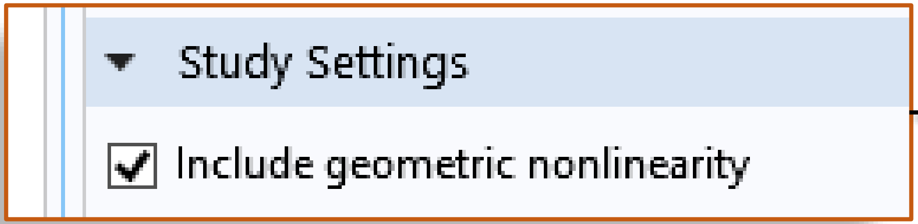


P_E = 719.66 kN (analytical 1D)

Linear buckling analysis



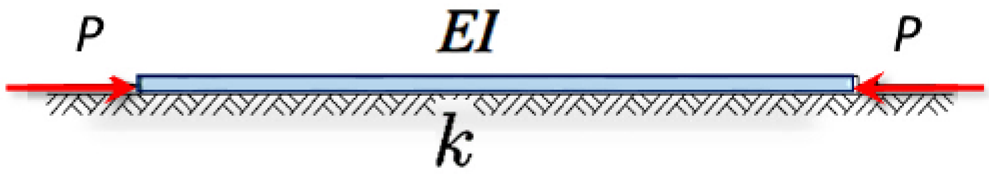
- at least, up-to the first mode is stable
- very shallow shape… no much increase in load bearing capacity



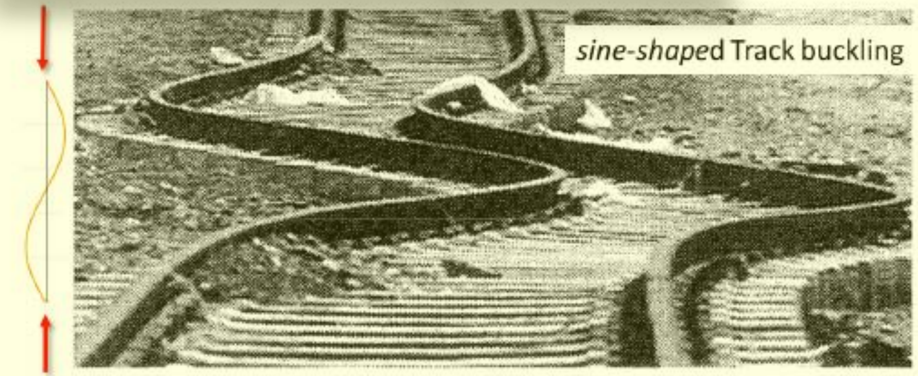
$$\epsilon_{ij}^{*} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

Uses: Finite strains and large displacements theory

Buckling of columns on elastic foundation

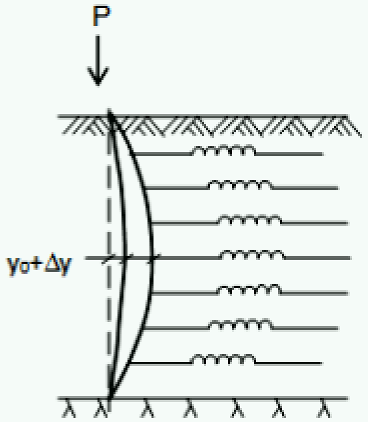


Application: 2) Buckling of rail track



Buckled rail track. Note the *sine-shaped* buckles

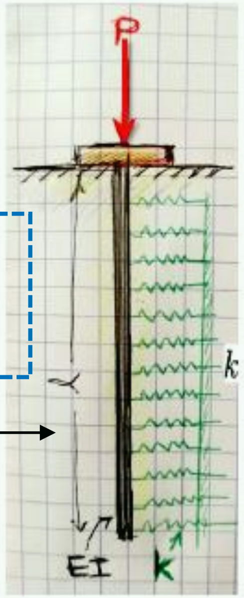
Application: 1) Buckling in pile design



Winkler model

$$r(x) = kv(x).$$

k, [N/m.m]



Shematic of foundation pile under axial thrust which is elastically restrained by the soil (geotechnical design; Eurocode 7).

Liikenneviraston ohjeita 13/2017
Eurokoodin soveltamisohje
Geotekninen suunnittelu – NCCI 7 (21.4.2017)

5.3.4 Nurjahduskestävyys STR/GEO

Nurjahdustarkastelu voidaan suorittaa rakennemallilla, jossa maan paalua tukeva vaikutus kuvataan jousilla.

Linear Buckling analysis

Sensitivity to imperfections
Post-buckling analysis
(Non-linear Buckling analysis)

Rakennemallissa (yleensä FEM) tulee paalun alkukaarevuus ja kuorman epäkeskisyys mallintaa. Jos paalu mallinnetaan suorana ja kuorma keskeisenä se ei laskennallisesti nurjahda.

***Cf.* Eurocode 7**

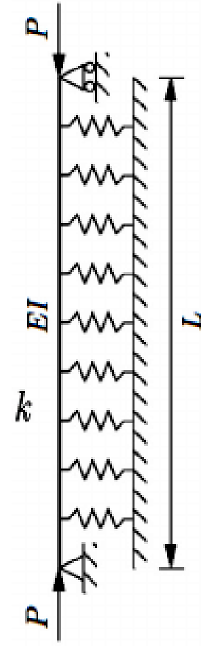
Buckling of columns on elastic foundation

$v(0) = v''(0) = 0,$
$v(\ell) = v''(\ell) = 0,$

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI[v''(x)]^2 + k[v(x)]^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}[v'(x)]^2 \mathrm{d}x.$$

Euler-Bernoulli beam

← Can be used to find approximate solutions **Rayleigh-Ritz**



Shematic of simply supported axially compressed column on elastic foundation.

energy criterion $\delta(\Delta\Pi) = 0$

$\Downarrow$

$$\int_0^\ell EIv''\delta v'' + kv\delta v \mathrm{d}x - P\int_0^\ell v'\delta v' \mathrm{d}x = 0, \ \forall \delta v$$

new term

$$\delta\left(\frac{1}{2}\int_0^\ell kv(x)^2 \mathrm{d}x\right) = \int_0^\ell \underbrace{kv}_{\text{new add to ODE}} \delta v \mathrm{d}x.$$

$\Downarrow$

which becomes after twice integration by parts

$$\int_0^\ell \underbrace{[EIv^{(4)} + kv + Pv'']}_{=0} \delta v \mathrm{d}x + [\underbrace{EIv''}_{-M} \delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0, \ \forall \delta v$$

Field equation — **Boundary conditions**

$\Downarrow$

The linearised buckling equation

$$EIv^{(4)} + Pv'' + kv = 0.$$

& Boundary conditions

← Can be used to find exact solutions

Solutions? →

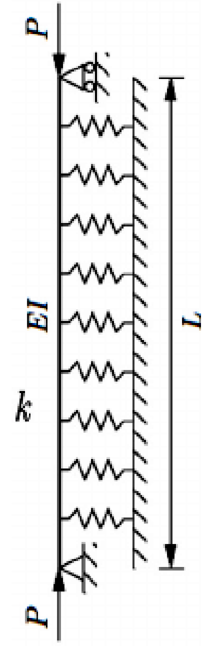
Buckling of columns on elastic foundation

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI[v''(x)]^2 + k[v(x)]^2\mathrm{d}x - P\int_0^\ell \frac{1}{2}[v'(x)]^2\mathrm{d}x.$$

Euler-Bernoulli beam

$v(0) = v''(0) = 0,$
$v(\ell) = v''(\ell) = 0,$

Shematic of simply supported axially compressed column on elastic foundation.



energy criterion $\delta(\Delta\Pi) = 0$

$\Downarrow$

$$\int_0^\ell EIv''\delta v'' + kv\delta v\mathrm{d}x - P\int_0^\ell v'\delta v'\mathrm{d}x = 0,\ \forall\delta v$$

new term

$$\delta\left(\frac{1}{2}\int_0^\ell kv(x)^2\mathrm{d}x\right) = \int_0^\ell \underbrace{kv}_{\text{new add to ODE}}\ \delta v\mathrm{d}x.$$

$\Downarrow$

which becomes after twice integration by parts

$$\int_0^\ell \underbrace{[EIv^{(4)} + kv + Pv'']}_{=0}\delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0,\ \forall\delta v$$

Field equation

Boundary conditions

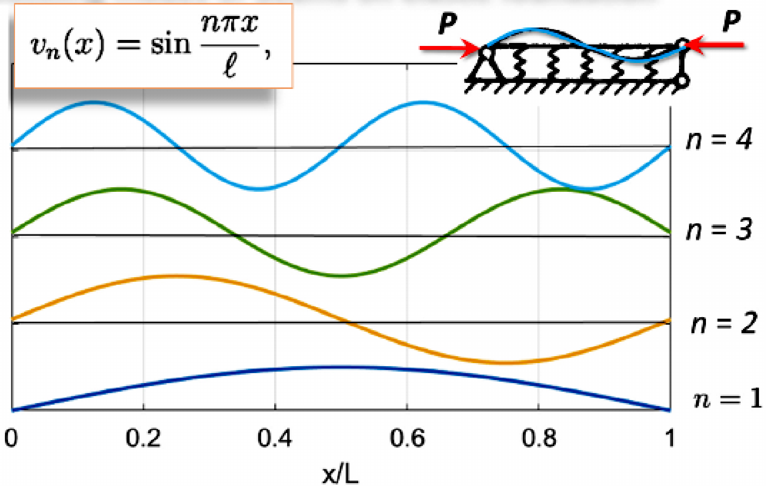
$\Downarrow$

The linearised buckling equation

&

$$EIv^{(4)} + Pv'' + kv = 0.$$

Buckling modes of beams on elastic foundation



Buckling of columns on elastic foundation

$v(0) = v''(0) = 0,$
$v(\ell) = v''(\ell) = 0,$

The linearised buckling equation

$$EIv^{(4)} + Pv'' + kv = 0.$$

& Boundary conditions

$\Downarrow$

The following trial satisfies the differential equations & the boundary conditions

$$v_n(x) = \sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$

$\Downarrow$

The critical load is now

$$P_n = \left(\frac{n\pi}{\ell}\right)^2 EI + \left(\frac{\ell}{n\pi}\right)^2 k$$

$$= n^2 \underbrace{\left[\frac{\pi^2 EI}{\ell^2}\right]}_{P_E} + \frac{1}{n^2}\frac{k\ell^2}{\pi^2} > n^2 P_E$$

$$= \left[\frac{\pi^2 EI}{\ell^2}\right]\left[n^2 + \frac{1}{n^2}\underbrace{\frac{k}{EI}\left(\frac{\ell}{\pi}\right)^4}_{\equiv\beta}\right], \; n = 1, 2, 3, \ldots$$

$$= P_E\left[n^2 + \frac{\beta}{n^2}\right].$$

which represents, graphically, a set of straight lines for $n = 1, 2, 3, \ldots$, etc. in function of the relative stiffness β. The graph $\bar{P}_n - \beta$ shows the lowest values for $\bar{P}_n$ which correspond to the critical loads as function of the parameter β.

$\Longrightarrow$

The buckling load is the smallest critical load:

The smallest critical load $P_{cr} = P_n$ depends on the half-wave number n.

$$\frac{dP_n}{dn} = 0 \implies n^2 = \sqrt{\beta},$$

$$P_{cr} = 2P_E\sqrt{\beta} = 2\sqrt{kEI}.$$

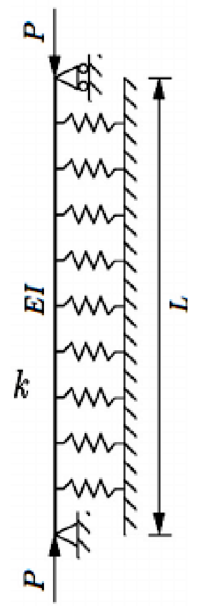
$$\beta = \frac{k\ell^4}{\pi^4 EI},$$

$$\bar{P}_n = \frac{P_n\ell^2}{\pi^2 EI} \equiv \frac{P_n}{P_E}$$

Constraint: half-wave number n should be an integer

Indeed, this is a *limit* for 'long' beams for which

$$\bar{\ell} \equiv \beta^{1/4} \geq 3$$



The buckling load:

'Long' beams:

$$P_{cr} \approx 2\sqrt{kEI}$$

$$\bar{\ell} \equiv \beta^{1/4} \geq 3$$

Beams of arbitrary length:

$$P_{cr} = k_{cr}\sqrt{kEI}$$

Buckling load

Buckling coefficient (see graph next slide)

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

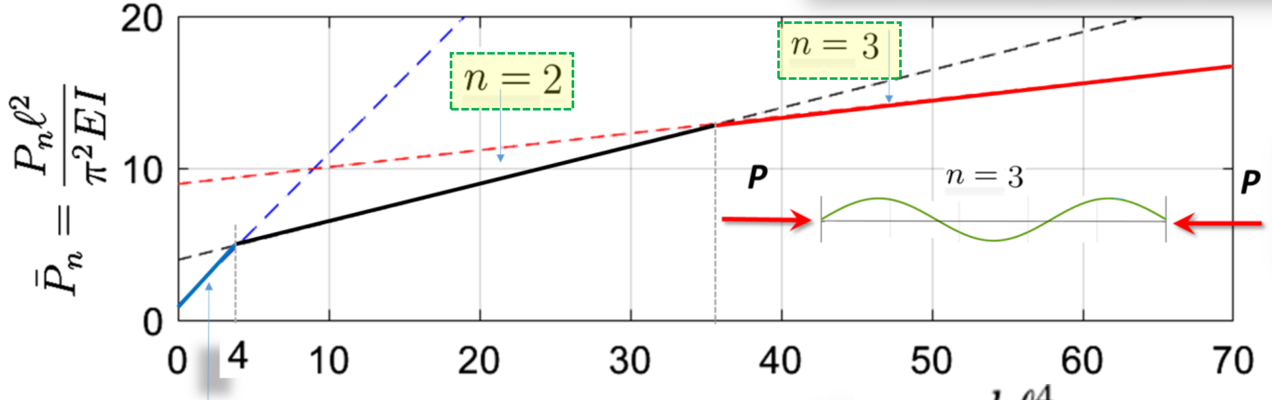
What is the corresponding buckling mode?

Buckling of columns on elastic foundation

What is the corresponding buckling mode?

Attention: The buckling mode corresponding to the critical buckling load is not always the first mode

$$\bar{P}_n = n^2 + \frac{\beta}{n^2}$$



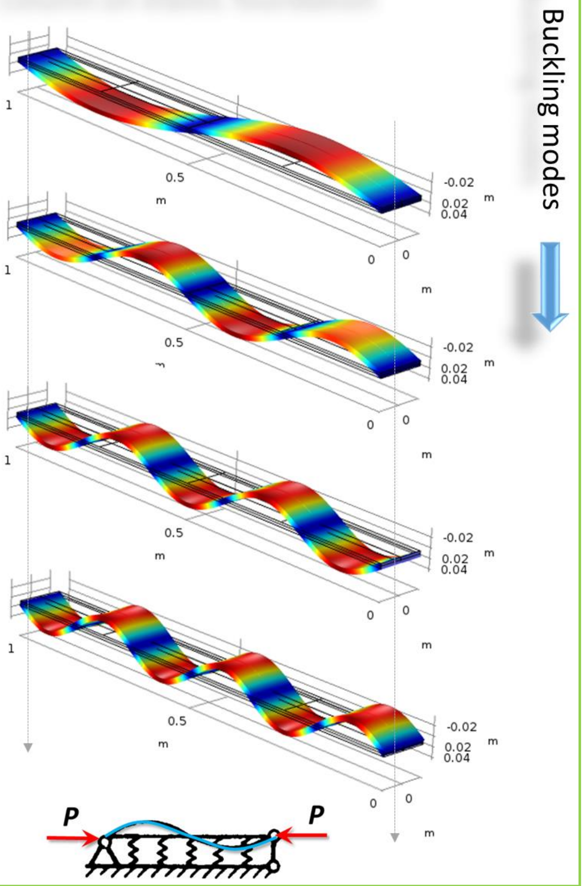
Relative buckling load

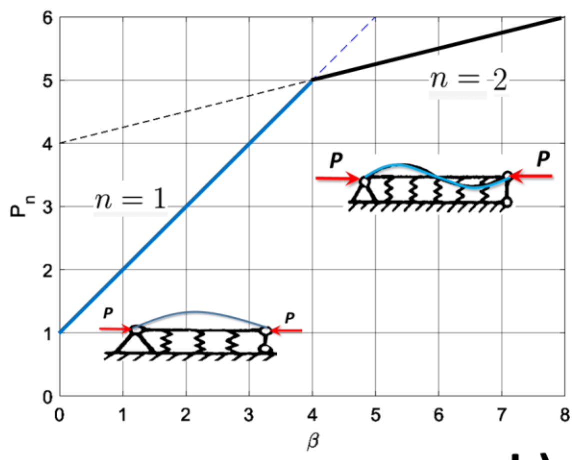
Relative stiffness $\beta = \frac{k\ell^4}{\pi^4 EI}$

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

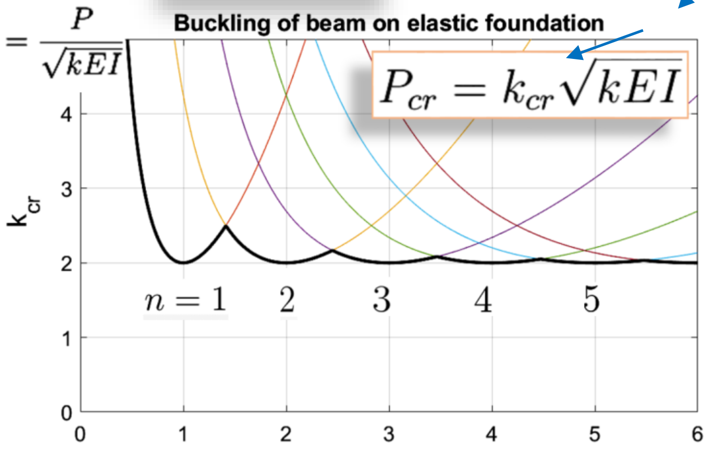
Buckling of axially compressed column on elastic foundation





b)

$$\bar{P} = \frac{P}{\sqrt{kEI}}$$



$$\bar{\ell} = \frac{\ell}{\pi}\left[\frac{k}{EI}\right]^{1/4}$$

c)

$$P_{cr} = k_{cr}\sqrt{kEI}$$

Buckling load

Buckling coefficient

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

Buckling of a column on elastic foundation - a summary

Other types of boundary conditions

- **For general types of BCs** one should obtain a complete solution of the **ODE**

$$v^{(4)} + \frac{P}{EI}v'' + \frac{k}{EI}v = 0$$

$$v^{(4)} + \lambda_P^2 v'' + \frac{\beta_k^4}{4}v = 0$$

$$\lambda_P^2 \equiv P/EI\,(= p^2)$$

$$\beta_k^4 \equiv 4k/EI\,(= 4b^4)$$

The general solution $v(x) = Ae^{rx}$

- $\lambda_P > \beta_k,$

$$v(x) = C_1 \cos px + C_2 \sin px + C_3 \cos qx + C_4 \sin qx$$

$$p = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} + \tfrac{1}{2}\sqrt{\lambda_P^2 - \beta_k^2} \;\;\&\;\; q = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} - \tfrac{1}{2}\sqrt{\lambda_P^2 - \beta_k^2}$$

- $\lambda_P < \beta_k,$

$$v(x) = C_1 \cosh px + C_2 \sinh px + C_3 \cosh qx + C_4 \sinh qx$$

$$p = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} + \tfrac{1}{2}\sqrt{\beta_k^2 - \lambda_P^2} \;\;\&\;\; q = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} - \tfrac{1}{2}\sqrt{\beta_k^2 - \lambda_P^2}$$

- $\lambda_P = \beta_k,$

$$v(x) = (C_1 + C_2 x)\cos\left(\lambda_k/\sqrt{2}\right) + (C_3 + C_4 x)\sin\left(\lambda_k/\sqrt{2}\right)$$

- **For general types of BCs** one should obtain a complete solution of the **ODE**

$$\beta_k^4 \equiv 4k/EI \; (= 4b^4) \qquad \lambda_P^2 \equiv P/EI \; (= p^2)$$

$$v^{(4)} + \frac{P}{EI} v'' + \frac{k}{EI} v = 0$$

$$v^{(4)} + \lambda_P^2 v'' + \frac{\beta_k^4}{4} v = 0$$

$$v(x) = C_1 \cos k_1 x + C_2 \sin k_1 x + C_3 \cos k_2 x + C_4 \sin k_2 x$$

$$(k_1, k_2) = \sqrt{a^2 \pm \sqrt{\Delta}}$$

$$v(-L) = v(L) = 0, \quad v'(-L) = v'(L) = 0.$$

$$\begin{bmatrix} \cos k_1 L & \sin k_1 L & \cos k_2 L & \sin k_2 L \\ \cos k_1 L & -\sin k_1 L & \cos k_2 L & -\sin k_2 L \\ -k_1 \sin k_1 L & k_1 \cos k_1 L & -k_2 \sin k_2 L & k_2 \cos k_2 L \\ k_1 \sin k_1 L & k_1 \cos k_1 L & k_2 \sin k_2 L & k_2 \cos k_2 L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_{cr} = \mu \cdot 2\sqrt{kEI},$$

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

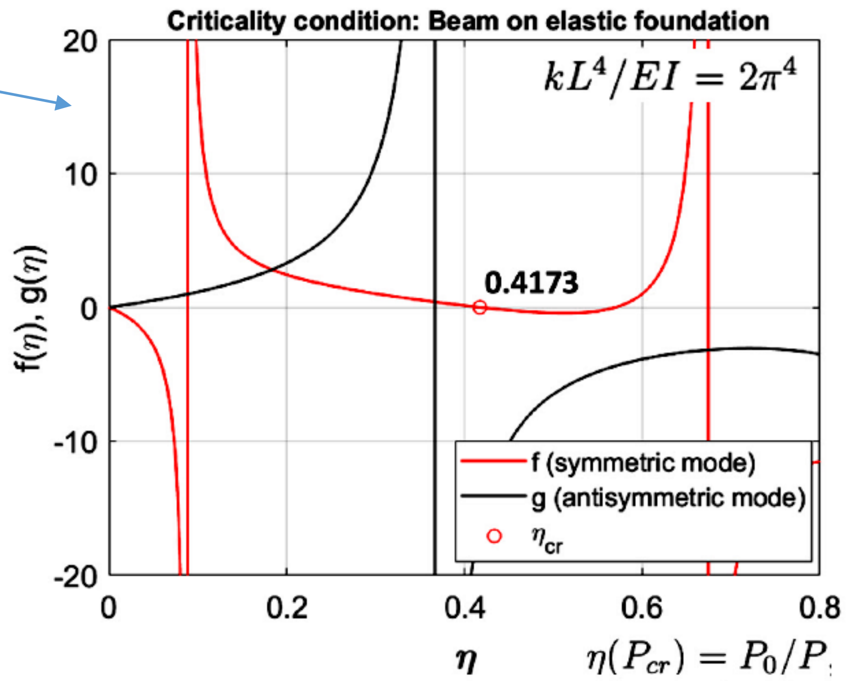
To obtain from the smallest zero of the determinant

Buckling load (symmetric mode)

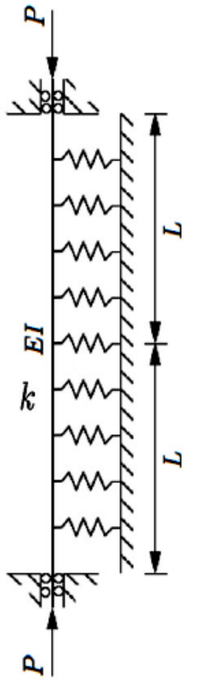
$$P_{cr} = P_0/\eta_{cr} \approx \underbrace{2.4}_{\mu} \cdot \underbrace{2\sqrt{kEI}}_{P_0},$$

Let's fix the value In this example: $kL^4/EI = 2\pi^4$.

- One should consider, separately, symmetric and asymmetric buckling
- The smallest critical load → buckling load



$\eta(P_{cr}) = P_0/P$,
$P_0 = 2\sqrt{kEI}$



The zeros of the determinant for the buckling of a column on elastic foundation.

Read the details in the pdf-notes I provided

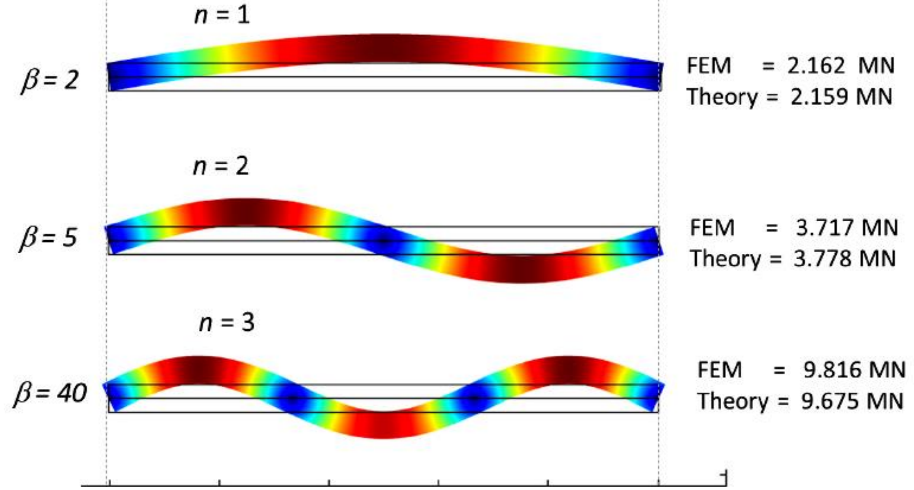
Buckling of columns on elastic foundation

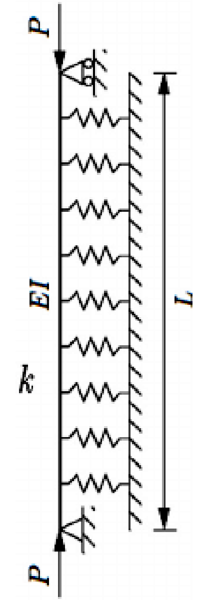
$$v(0) = v''(0) = 0,$$
$$v(\ell) = v''(\ell) = 0,$$

The linearised buckling equation

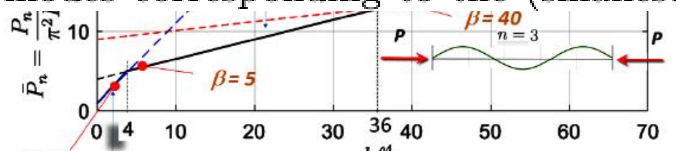
$$EIv^{(4)} + Pv'' + kv = 0.$$

& Boundary conditions





In the following, for illustrative pedagogical purposes, we analyse a simply supported column on elastic foundation with centric axial compressive load P. Simulation data: $\ell = 1$ m, $b = \ell/10$, $h = 50$ mm. $E = 70$ GPa ($\nu = 0.33$). We investigate, how the relative 'stiffness number' $\beta \equiv k\ell^2/(\pi^4 EI)$ determine the number n of half-waves of the buckling modes corresponding to the (smallest) buckling load P_{cr},



The buckling load:

$$P_{cr} = k_{cr}\sqrt{kEI}$$

Buckling load

Buckling coefficient

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

(see graph next slide)

Linear FE-buckling analysis. Buckling of axially compressed

Table 1.1: FE-linear buckling analysis. The loads are given in [MN] units.

β	$\bar{\ell}$	n	$P_{cr}^{\text{lim.}}$	k_{cr}	$P_{cr}^{(\text{theor.})}$	P_{cr}^{FEM}	$P_{cr}^{(\text{theor.})}/P_E$	k [N/m²]
2	1.189	1	2.04	2.121	2.159	2.162	3	14.2
5	1.495	2	3.22	2.348	3.778	3.717	5.3	35.5
40	2.515	3	9.10	2.126	9.675	9.816	13.4	284.1

What is the corresponding buckling mode?

Post-buckling analysis of columns on elastic foundation

FE-based post-buckling analysis

Buckles here
(2D elasticity solution with tiny initial imperfection)

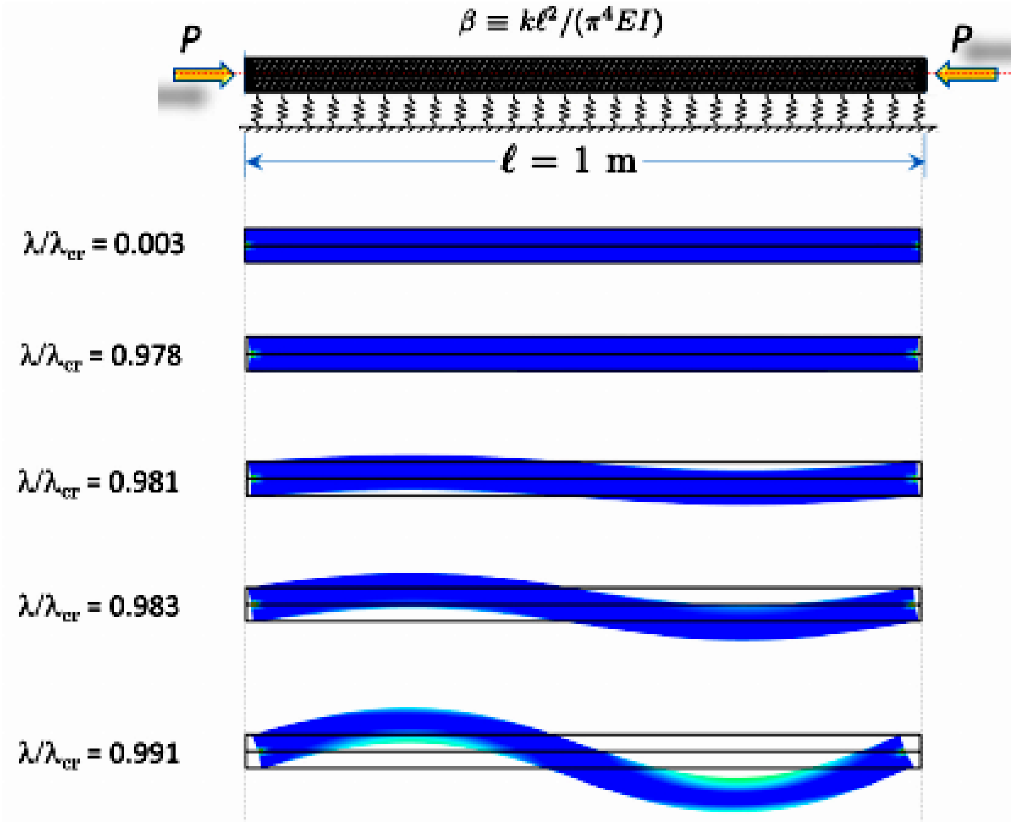
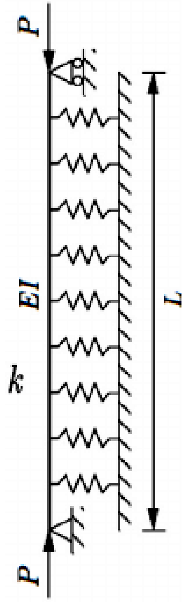


Figure 1.89: Post-buckling displacements in 1:1 scale (FE simulation). The perturbation scale $\epsilon = 1/1000$. After $\lambda/\lambda_{cr,FE} > 0.991$, the behaviour seems (in this simulation) to become unstable and could not be captured because of force control approach used (I will do a displacement control soon). ($EI = 72917$ N.m^2, $\beta = 5$ ($n = 2$)), theoretical 1-D value for $P_{cr} = 3.778$ MN (2D-elasticity FE based linear buckling analysis gave $P_{cr,FE} = 3.720$ MN). .

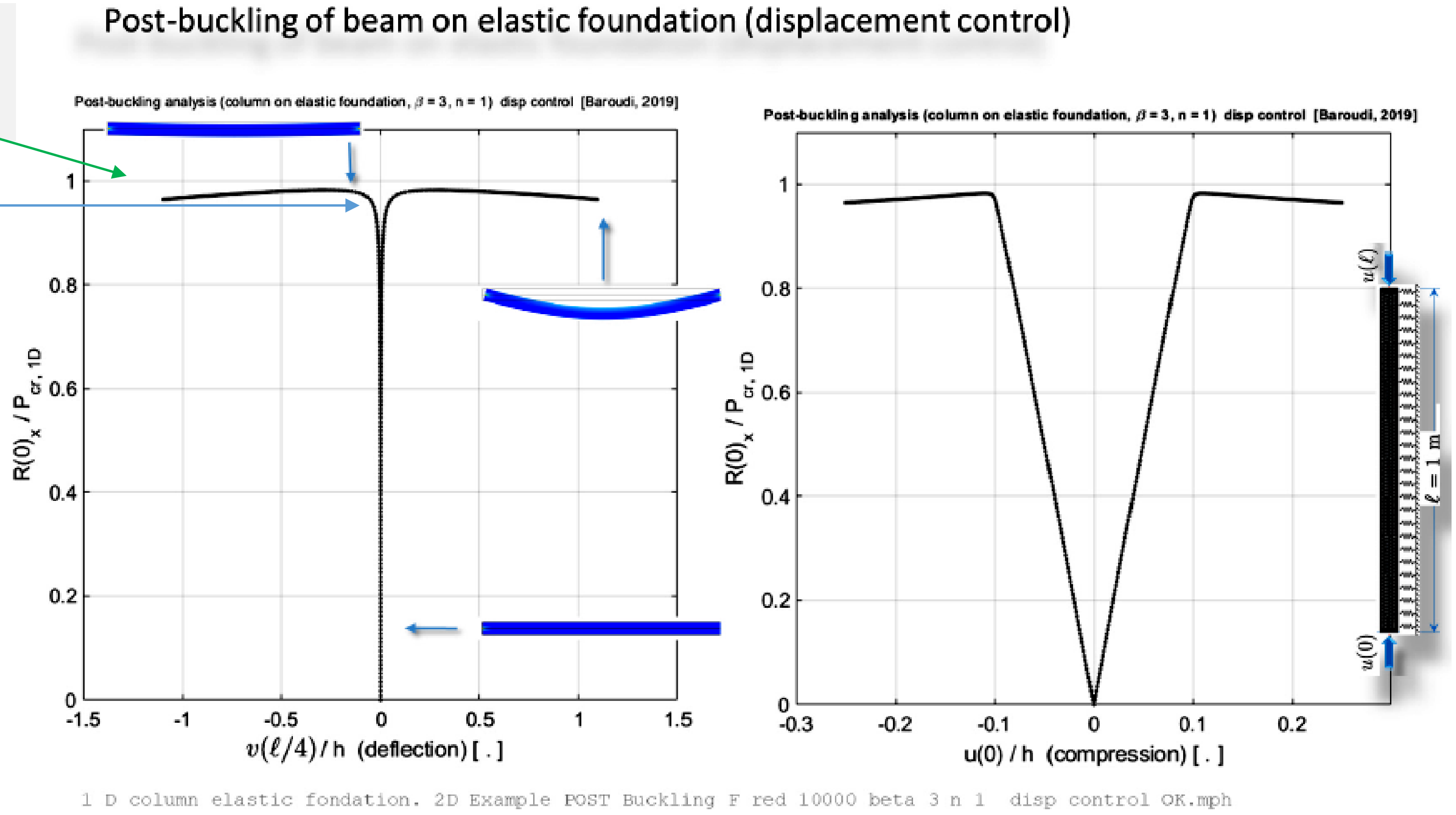
The column-beam is simply supported (kuvasta puuttuu nivelet)



Effect of foundation stiffness on post-buckling behaviour

Buckles here (1D theoretical ideal solution)

Buckles here (2D elasticity solution with imperfection)



The column-beam is simply supported

(kuvasta puuttuu nivelet)

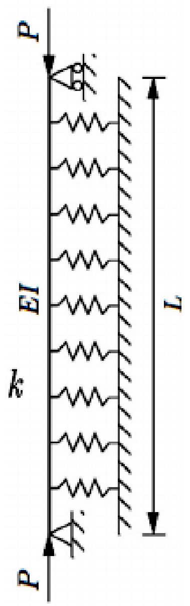
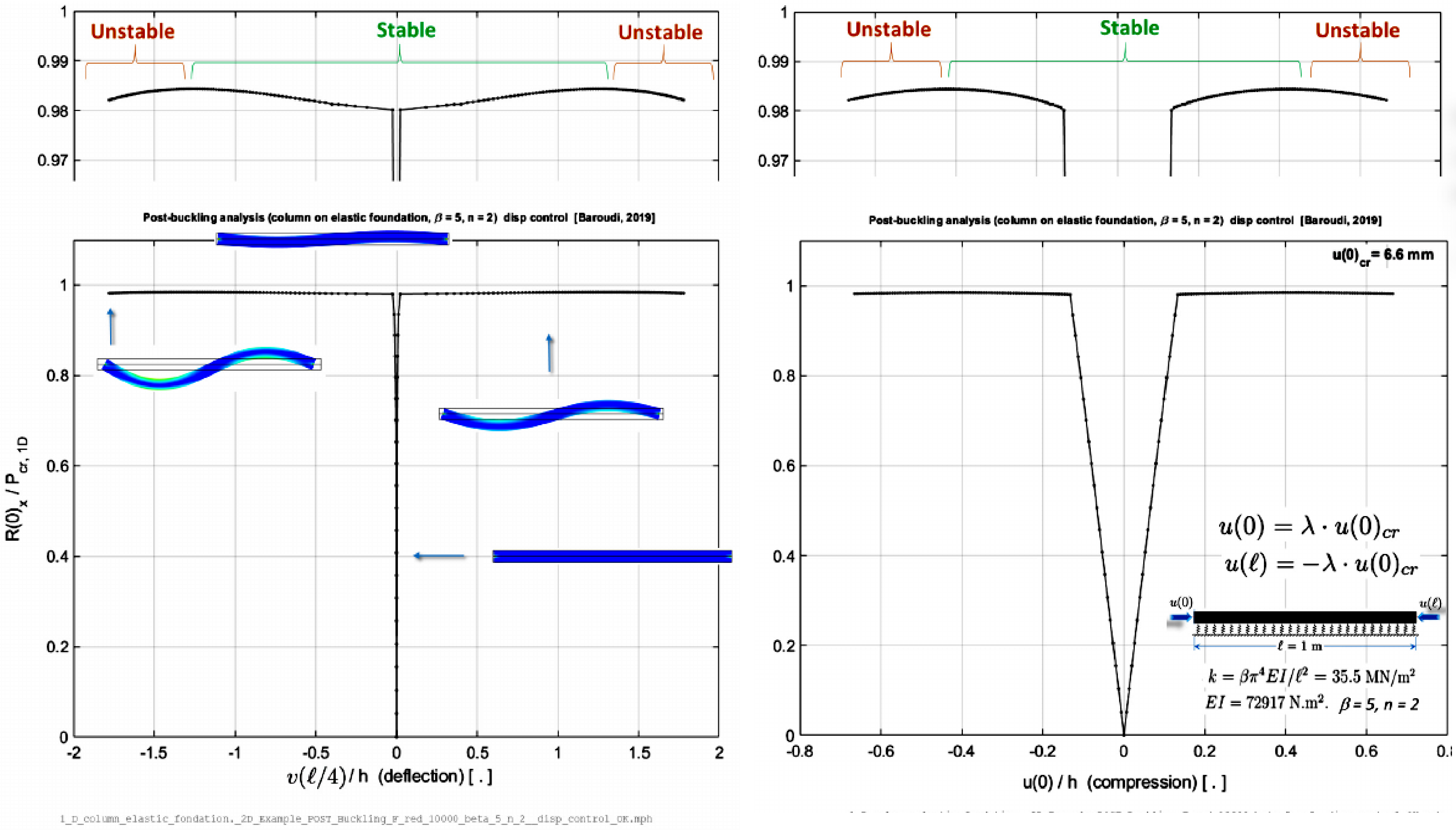


Figure 1.93: Post-buckling equilibrium paths (FE-simulation, displacement-control) of a uniformly compressed column on elastic foundation. The ends-load is centric. The parameters ℓ, k and EI are such that $\beta = 3$ and the initial post-buckling mode corresponds to one-half waves ($n = 1$). The perturbation scale for the transverse loads was $\epsilon = 1/1000$.

Effect of foundation stiffness on post-buckling behaviour



The column-beam is simply supported

(kuvasta puuttuu nivelet)

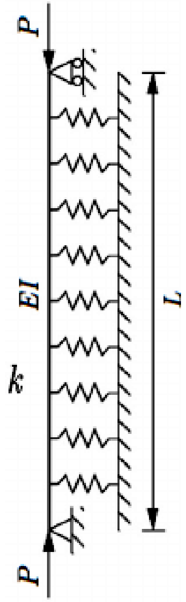


Figure 1.91: Post-buckling equilibrium paths; $v(\ell/4)$ *versus* P/P_{cr}, (FE-simulation, displacement-control). The parameters ℓ, k and EI are such that $\beta = 5$ and the initial post-buckling mode corresponds to two-half waves ($n = 2$). The perturbation scale for the transverse loads was $\epsilon = 1/10000$. (the post-buckled displacements are in scale 1:1 in the deformed column).

Discrete energy method - FEM

The starting point for deriving the elementary matrices above is the total potential energy functional (1.233) or more directly, its variation which is known as *Virtual Work Principle*. The idea is the write the variation of the total functional as a sum over the elements

$$v^{(e)}(x) = \sum_{i=1}^{M} \phi_i(x) a_i^{(e)} \equiv \mathbf{N}(x)\mathbf{a}^{(e)},$$

$$\mathbf{a}^{(e)} = \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2. \end{bmatrix}^{\mathrm{T}}$$

$$\delta v(x) = \mathbf{N}(x)\delta\mathbf{a}^{(e)},$$

$N_j(x)$ are the shape functions

$$\delta(\Delta\Pi) = \sum_{e=1}^{N} \left[\int_0^{\ell^{(e)}} EIv''(x)\delta v'' + kv(x)\delta v(x)\mathrm{d}x - P^{(e)} \int_0^{\ell^{(e)}} v'(x)\delta v'(x)\mathrm{d}x \right] = 0$$

$$= \sum_{e=1}^{N} (\delta\mathbf{a}^{(e)})^{\mathrm{T}} \left[\underbrace{\int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x) \cdot EI \cdot \mathbf{N}''(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(B)}} + \underbrace{\int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x) \cdot k \cdot \mathbf{N}(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(F)}} + \right.$$

$$\left. \underbrace{- \int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x) \cdot P^{(e)} \cdot \mathbf{N}'(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{G}}} \right] \mathbf{a}^{(e)} = 0, \ \forall \delta\mathbf{a}^{(e)}$$

where $P^{(e)} = -N^0(x)$ and $N^0(x)$ being the membrane stress-resultant

Discrete energy method - FEM

The starting point for deriving the elementary matrices above is the total potential energy functional (1.233) or more directly, its variation which is known as *Virtual Work Principle*. The idea is the write the variation of the total functional as a sum over the elements

$$\delta(\Delta\Pi) = \sum_{e=1}^{N}\left[\int_0^{\ell^{(e)}} EIv''(x)\delta v'' + kv(x)\delta v(x)\mathrm{d}x - P^{(e)}\int_0^{\ell^{(e)}} v'(x)\delta v'(x)\mathrm{d}x\right] = 0$$

$$= \sum_{e=1}^{N}(\delta \mathbf{a}^{(e)})^{\mathrm{T}}\left[\underbrace{\int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x)\cdot EI\cdot \mathbf{N}''(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(B)}} + \underbrace{\int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x)\cdot k\cdot \mathbf{N}(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(F)}} + \right.$$

$$\left. \underbrace{-\int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x)\cdot P^{(e)}\cdot \mathbf{N}'(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{G}}}\right]\mathbf{a}^{(e)} = 0,\ \forall \delta\mathbf{a}^{(e)}$$

where $P^{(e)} = -N^0(x)$ and $N^0(x)$ being the membrane stress-resultant

Discrete energy method - FEM

linearised stiffness matrix for bending

$$\mathbf{K}_L^{(B)} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

consistent stiffness matrix from the elastic foundation

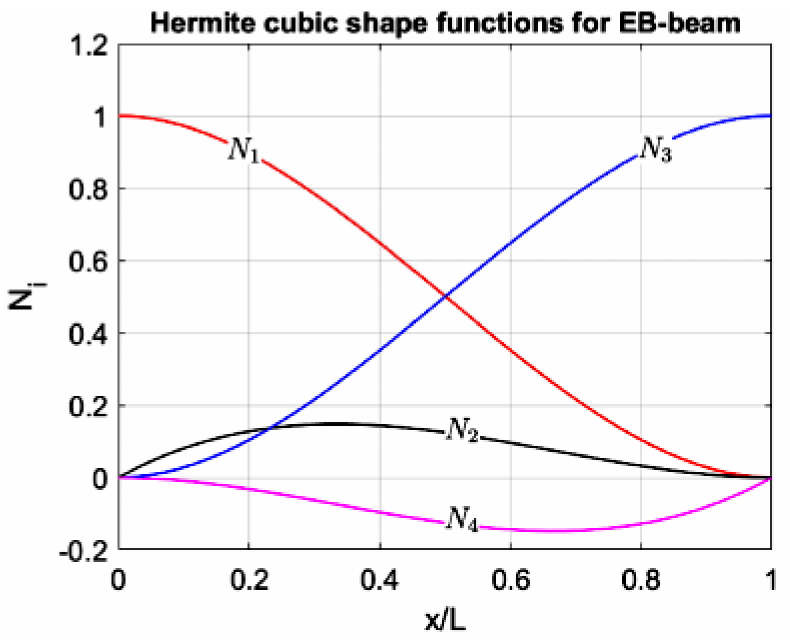
$$\mathbf{K}_L^{(F)} = \frac{k\ell}{70}\begin{bmatrix} 26 & 11\ell/3 & 9 & -13\ell/6 \\ 11\ell/3 & 2\ell^2/3 & 13\ell/6 & -\ell^2/2 \\ 9 & 13\ell/6 & 26 & -11\ell/3 \\ -13\ell/6 & -\ell^2/2 & -11\ell/3 & 2\ell^2/3 \end{bmatrix}$$

Diagonalized foundation stiffness matrix:

$$\mathbf{K}_L^{(F)} = \frac{k\ell}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Euler-Bernoulli beam element

$$N_1(x) = 1 - 3(x/\ell)^2 + 2(x/\ell)^3,$$
$$N_2(x) = x(1 - x/\ell)^2,$$
$$N_3(x) = 3(x/\ell)^2 - 2(x/\ell)^3,$$
$$N_4(x) = x((x/\ell)^2 - x/\ell)$$



geometric elementary matrix is

$$\mathbf{K}_G = -\frac{P}{30\ell}\begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

compression load $P = -N^0(x) > 0.$

$$\mathbf{K}_L^{(\mathbf{B})} = \int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x) \cdot EI \cdot \mathbf{N}''(x)\mathrm{d}x,$$
$$\mathbf{K}_L^{(F)} = \int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x) \cdot k \cdot \mathbf{N}(x)\mathrm{d}x,$$
$$\mathbf{K}_G = -\int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x) \cdot P^{(e)} \cdot \mathbf{N}'(x)\mathrm{d}x.$$

[A result from FEA] The convergence rate *k* for Euler-Bernoulli beam element for the Eigen-values is *k* = 4

Application example

$\mathbf{a} = [\phi_1, \phi_2]^{\mathrm{T}}$

DO: Determine the critical load and the corresponding mode by the "handy-FE" method (stiffness method)

Assembly:

$$K_{11} = K_{44}^{(1)} = \frac{EI}{\ell^3}4\ell^2 - \frac{P}{30\ell}4\ell^2 \qquad P^{(1)} = P$$

$$K_{12} = K_{21} = K_{42}^{(1)} = \frac{EI}{\ell^3}2\ell^2 + \frac{P}{30\ell}\ell^2 \qquad P^{(2)} = 3P$$

$$K_{22} = K_{22}^{(1)} + K_{44}^{(2)} = \frac{EI}{\ell^3}4\ell^2 - \frac{P}{30\ell}4\ell^2 + \frac{2EI}{\ell^3}4\ell^2 - \frac{3P}{30\ell}4\ell^2$$

The global linearised stiffness and geometric matrices

$$\Downarrow \quad \mathbf{K}_{\mathrm{L}} = \frac{2EI}{\ell}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}, \qquad \mathbf{K}_{\mathrm{G}} = -\frac{P\ell}{30}\begin{bmatrix} 4 & -1 \\ -1 & 16 \end{bmatrix}$$

$$\left(\frac{2EI}{\ell}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} - \frac{P\ell}{30}\begin{bmatrix} 4 & -1 \\ -1 & 16 \end{bmatrix}\right)\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Downarrow$

$$P_{1,cr} = 1.62\pi^2\frac{EI}{\ell^2}$$

$$P_{2,cr} = 3.97\pi^2\frac{EI}{\ell^2}$$

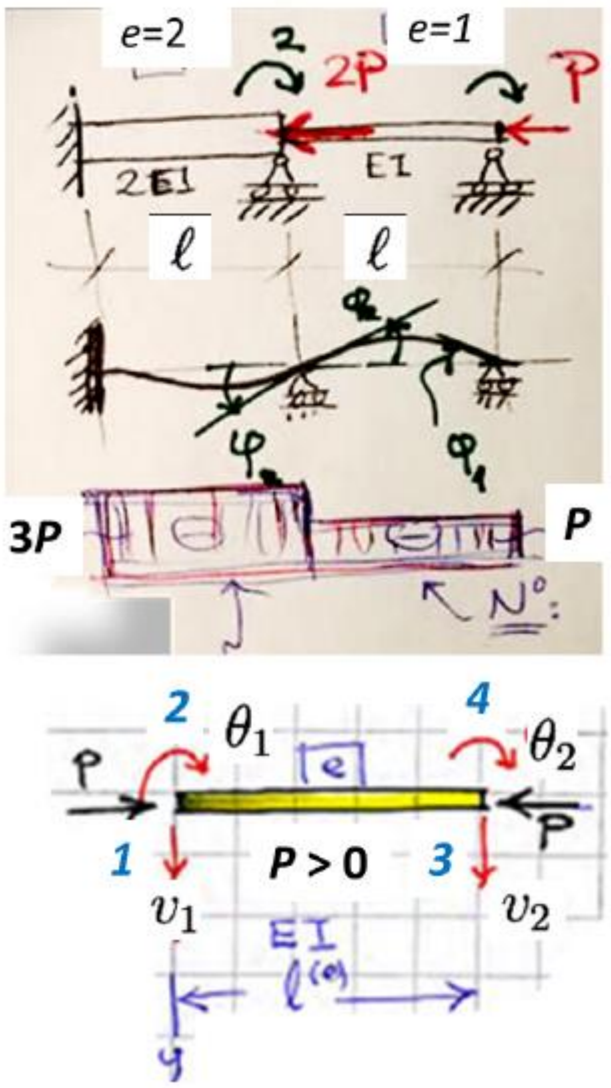
Buckling Load & mode

$$\phi_1 = \begin{bmatrix} 0.266 \\ -0.196 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -0.428 \\ -0.159 \end{bmatrix}$$

Buckled state

Initial membrane stress (pre-buckling)



NB. On should refine the FE-mesh until convergence …

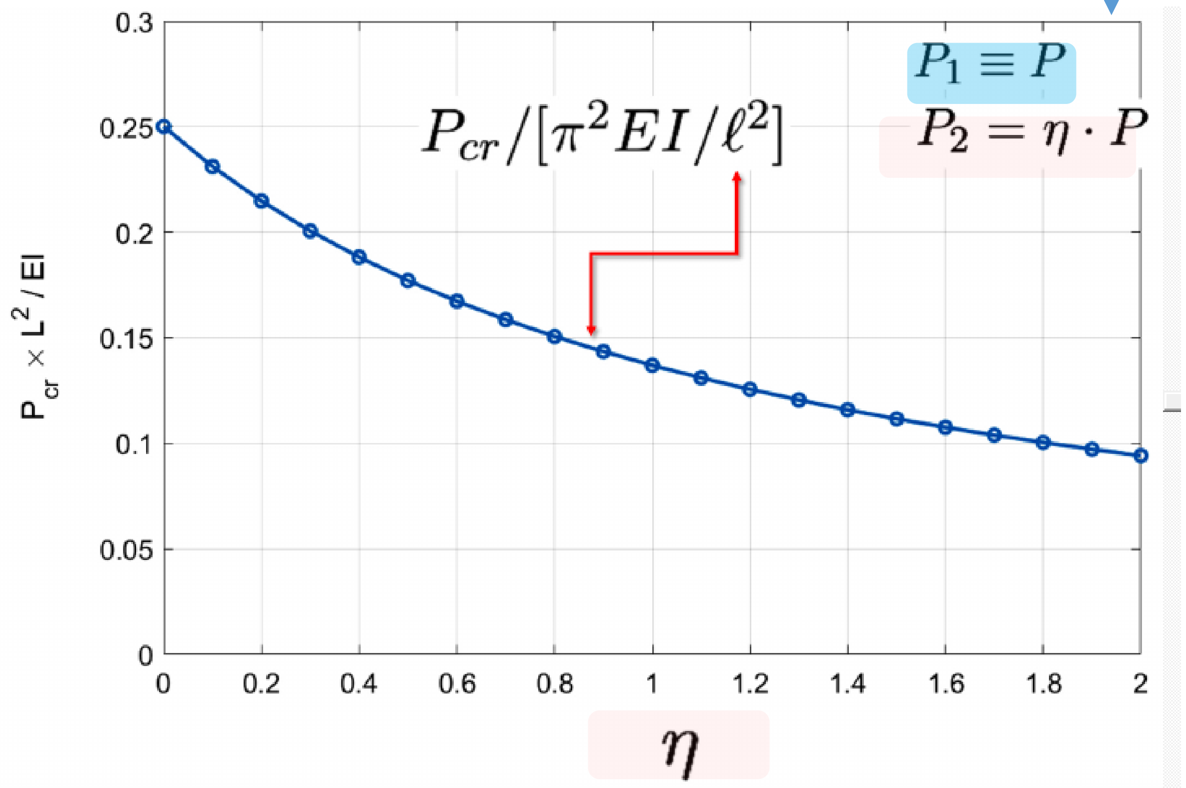
Application example — buckling load of column supporting two levels

Eigenvalue problem = lobal equilibrium equations in the tiny buckled state:

Non-proportional loading

$$\left(\begin{bmatrix}-12 & -6 & -12 & -6\\ -6 & 4 & 6 & 2\\ -12 & 6 & 24 & 0\\ -6 & 2 & 0 & 8\end{bmatrix} - \frac{\bar{\ell}^2}{30EI}\begin{bmatrix}36P_1 & -3P_1 & -36P_1 & -3P_1\\ -3P_1 & 4P_1 & 3P_1 & -P_1\\ -36P_1 & 3P_1 & 36(2P_1+P_2) & -3P_2\\ -3P_1 & -P_1 & -3P_2 & 4(2P_1+P_2)\end{bmatrix}\right)\begin{bmatrix}v_1\\ \phi_1\ell\\ v_2\\ \phi_2\ell\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

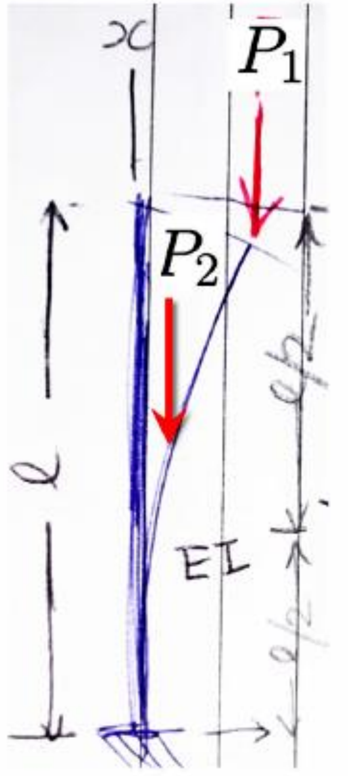
$$(\mathbf{K}_{4\times4} - \lambda\mathbf{K}_{G,4\times4})\,\mathbf{u} = \mathbf{0}, \qquad \bar{\ell} \equiv \ell/2$$



P_{cr} - refernce critical load P when both loads P_1 and P_2 are acting

$$P_E = \pi/4 \cdot EI/\ell^2$$

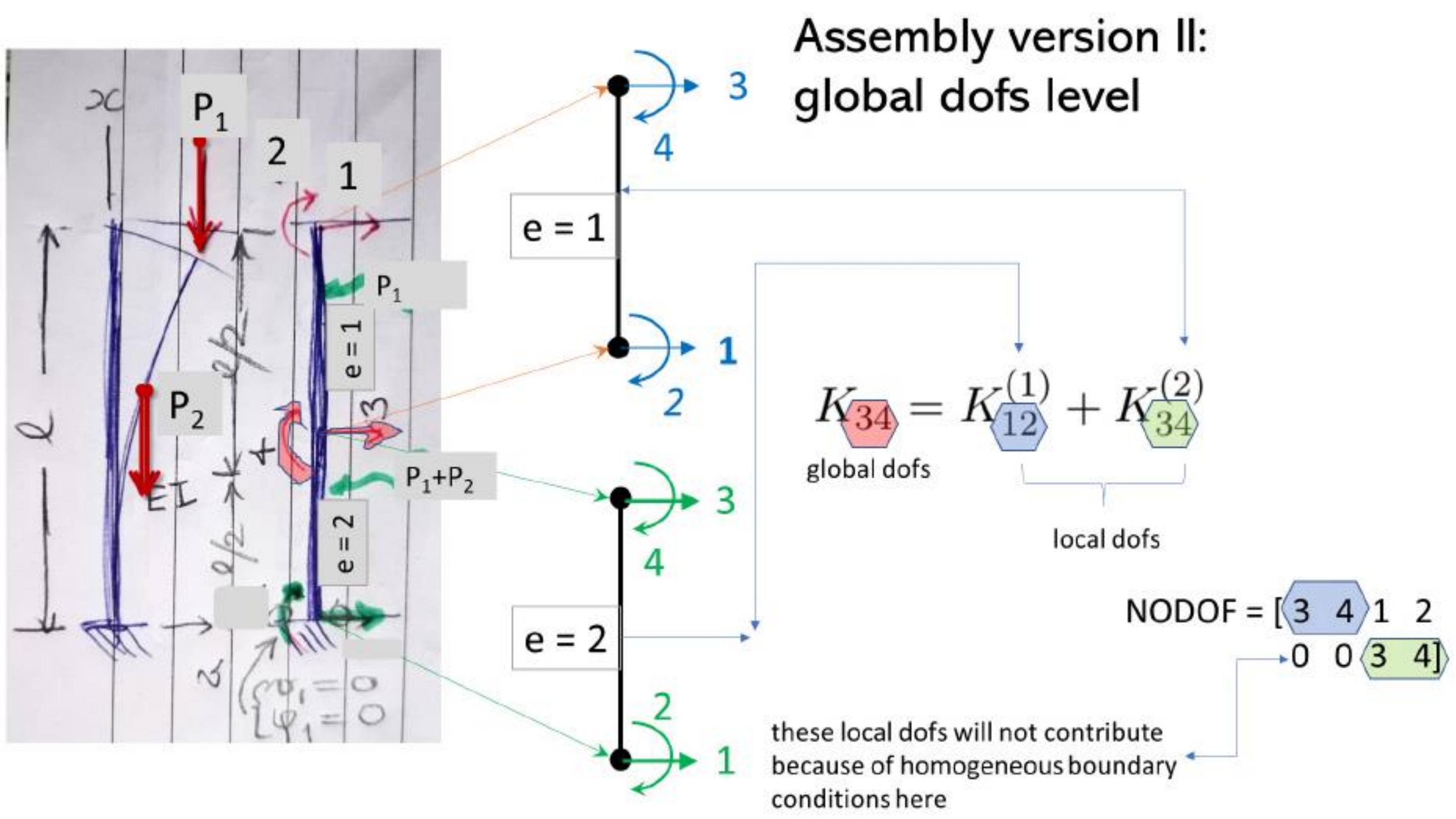
when $P_2 = 0$



For details on the assembly, please refer to my additional pdf-notes in MyCourses

Assembly of global matrices. $\quad (\mathbf{K}_{4\times4} - \lambda \mathbf{K}_{G,4\times4})\ \mathbf{u} = \mathbf{0},$

$$\left(\begin{bmatrix} -12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 24 & 0 \\ -6 & 2 & 0 & 8 \end{bmatrix} - \frac{\bar{\ell}^2}{30EI}\begin{bmatrix} 36P_1 & -3P_1 & -36P_1 & -3P_1 \\ -3P_1 & 4P_1 & 3P_1 & -P_1 \\ -36P_1 & 3P_1 & 36(2P_1+P_2) & -3P_2 \\ -3P_1 & -P_1 & -3P_2 & 4(2P_1+P_2) \end{bmatrix}\right)\begin{bmatrix} v_1 \\ \phi_1\ell \\ v_2 \\ \phi_2\ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$K_{34} = K_{12}^{(1)} + K_{34}^{(2)}$$

the global geometric stiffness matrix elements

$$K_{11} = K_{33}^{(1)} = 12,\ K_{12} = K_{34}^{(1)} = -6\bar{\ell},$$
$$K_{13} = K_{31}^{(1)} = -12,\ K_{14} = K_{32}^{(1)} = -6\bar{\ell}$$
$$K_{22} = K_{44}^{(1)} = 4\bar{\ell}^2,\ K_{23} = K_{41}^{(1)} = 6\bar{\ell},\ K_{24} = K_{42}^{(1)} = 2\bar{\ell}^2$$
$$K_{33} = K_{11}^{(1)} + K_{33}^{(2)} = 12 + 12 = 24\bar{\ell},$$
$$K_{34} = K_{12}^{(1)} + K_{34}^{(2)} = 6\bar{\ell} - 6\bar{\ell} = 0,$$
$$K_{44} = K_{22}^{(1)} + K_{44}^{(2)} = 4\bar{\ell}^2 + 4\bar{\ell}^2 = 8\bar{\ell}^2.$$

$$\bar{\ell} \equiv \ell/2$$

global linearised stiffness matrix elements

$$K_{G,11} = K_{G,33}^{(1)} = -36P_1,\ K_{G,12} = K_{G,34}^{(1)} = 3P_1\bar{\ell},$$
$$K_{G,13} = K_{G,31}^{(1)} = 36P_1,\ K_{G,14} = K_{G,32}^{(1)} = 3P_1\bar{\ell}$$
$$K_{G,22} = K_{G,44}^{(1)} = -4P_1\bar{\ell}^2,$$
$$K_{G,23} = K_{G,41}^{(1)} = -3P_1\bar{\ell},\ K_{G,24} = K_{G,42}^{(1)} = P_1\bar{\ell}^2$$
$$K_{G,33} = K_{G,11}^{(1)} + K_{G,33}^{(2)} = 36(2P_1 + P_2)\bar{\ell},$$
$$K_{G,34} = K_{G,12}^{(1)} + K_{G,34}^{(2)} = -3P_2,$$
$$K_{G,44} = K_{G,22}^{(1)} + K_{G,44}^{(2)} = 4(2P_1 + P_2)\bar{\ell}.$$

Accelerating convergence

refine the FE-mesh until convergence …

About convergence … and Richardson extrapolation toward the limit

- Assume we have an priori knowledge on the **convergence rate** of some **quantity** λ (can be always estimated)

e.g.. buckling load

The numerical solution is proportional to Ch^k

Convergence rate (k)

Positive constant (C)

h step-size or characteristic mesh-size (length of the largest element)

- The above **extrapolated solution** is **much closer to the exact** one than the solutions 1 and 2

Two solution with two different mesh-size: $h_2 < h_1$

$$\begin{cases} \lambda_1 = \lambda_{ex} + Ch_1^k = \lambda(h_1) \\ \lambda_2 = \lambda_{ex} + Ch_2^k = \lambda(h_2) \end{cases}$$

Richardson extrapolation: is a **sequence convergence acceleration** method

$$C = (\lambda_1 - \lambda_{ex})h_1^{-k}$$

$$\lambda_2 = \lambda_{ex} + (\lambda_1 - \lambda_{ex})\left(\frac{h_2}{h_1}\right)^k$$

$$\lambda_{ex} = \frac{\lambda_2 - \lambda_1\left(\frac{h_2}{h_1}\right)^k}{1 - \left(\frac{h_2}{h_1}\right)^k} \Longleftarrow$$

Extrapolated value

[A result from FEA] The convergence rate k for Euler-Bernoulli beam element for the Eigen-values is k = 4. (We can also estimate k from log-log plot of convergence rate (graph of changes in lambda *versus* changes in h)

NB. This extrapolated value is much more accurate than if would refine *substantially* the mesh further

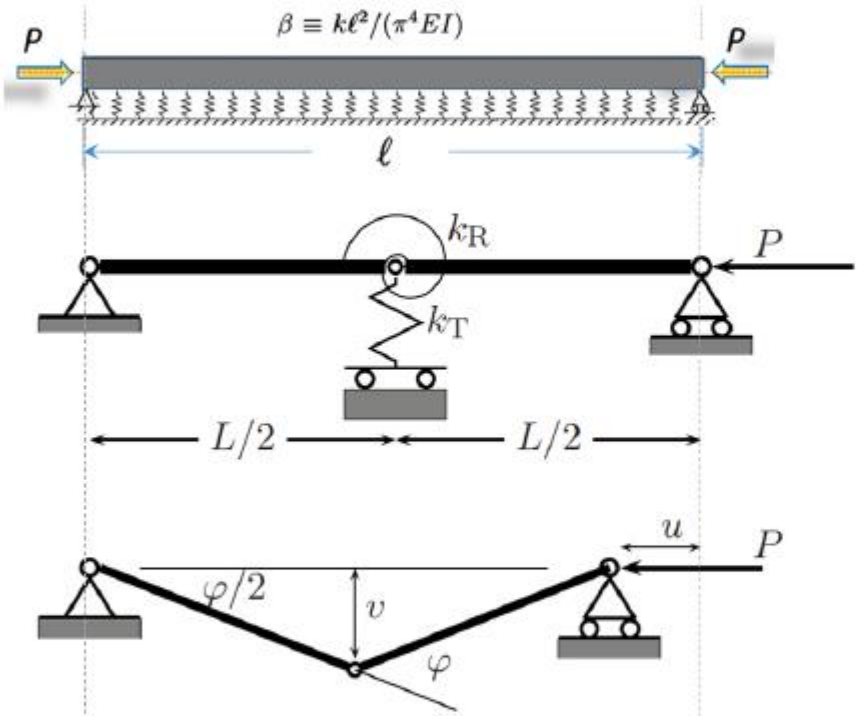
Physical discrete model based post-buckling analysis

Simplified model of elastically restrained column

translational spring
$k_T = k\ell/2$

rotational spring
$k_R = 1/4\pi^2 EI/\ell$

$\beta = k\ell^4/[\pi^2 EI]$



$$u = 2 \cdot \frac{L}{2}(1 - \cos(\varphi/2)), \qquad v = \frac{L}{2}\sin(\varphi/2)$$

Questions:

- Study the equilibrim paths
- how buckling load and stability depends on the relative rigidity
  $\beta = k\ell^4/[\pi^2 EI]$

Solution

$$u = 2 \cdot \frac{L}{2}(1-\cos(\varphi/2)), \qquad v = \frac{L}{2}\sin(\varphi/2)$$

$$\Pi = \frac{1}{2}k_R\varphi^2 + \frac{1}{2}k_T v^2 - Pu$$

$$= \frac{1}{8}\pi^2\frac{EI}{L}\varphi^2 + \frac{1}{16}\beta\pi^2\frac{EI}{L}\sin^2(\varphi/2) - PL(1-\cos(\varphi/2))$$

$\varphi = 0$

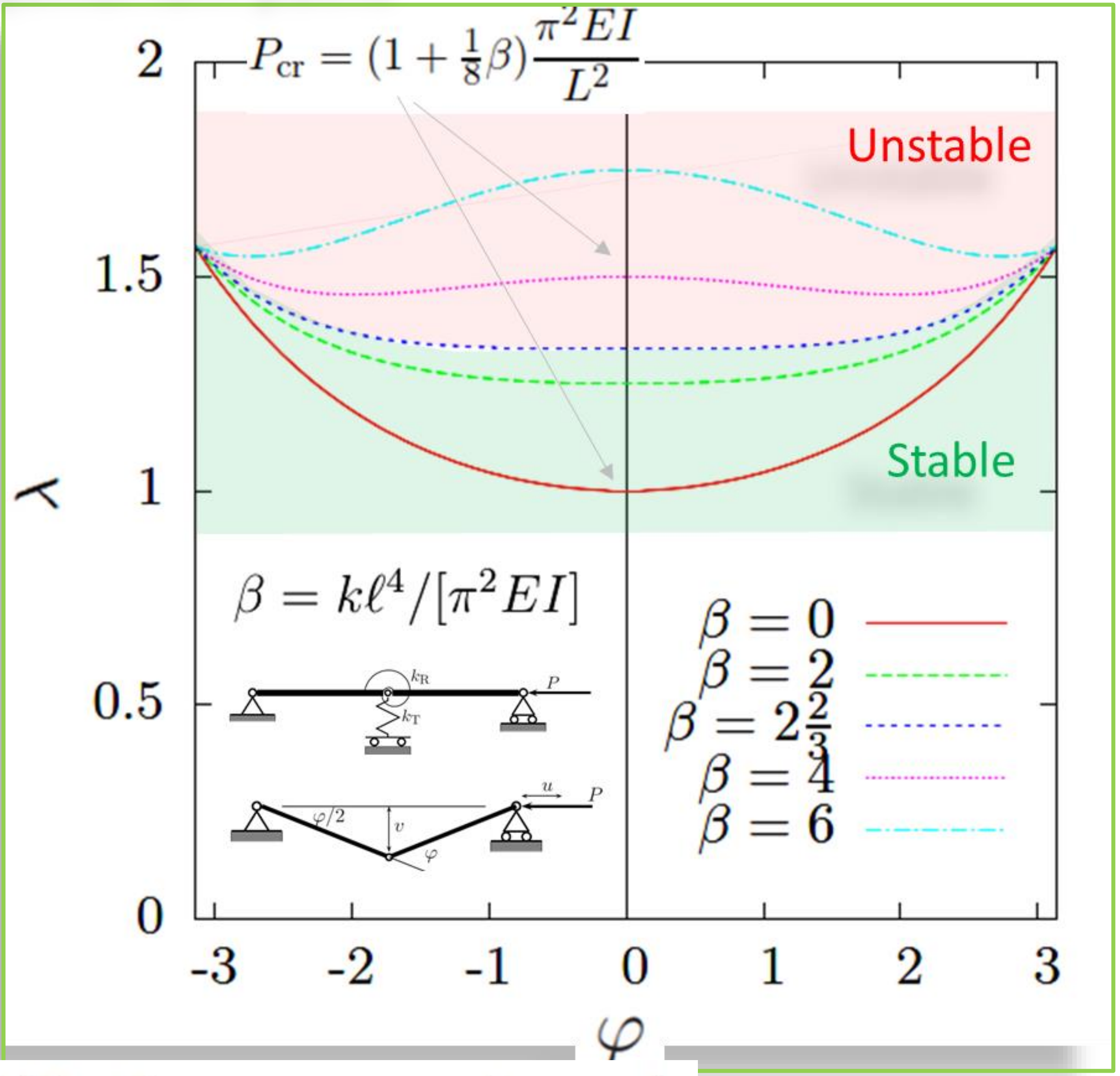
$$\lambda = \frac{\varphi/2}{\sin(\varphi/2)} + \frac{1}{8}\beta\cos(\varphi/2) \qquad \varphi = 0,\ \lambda = 1 + \tfrac{1}{8}\beta,$$

$$P_{cr}(\beta) = (1+\frac{\beta}{8})\frac{\pi^2 EI}{\ell^2}.$$

λ

$$P = \lambda\frac{\pi^2 EI}{L^2}$$

Equilibrium paths



$$\frac{d^2\tilde{\Pi}}{d\varphi^2} = \frac{d}{d\varphi}\left(\frac{1}{4}\varphi + \frac{1}{16}\beta\sin(\varphi/2)\cos(\varphi/2) - \frac{1}{2}\lambda\sin(\varphi/2)\right)$$

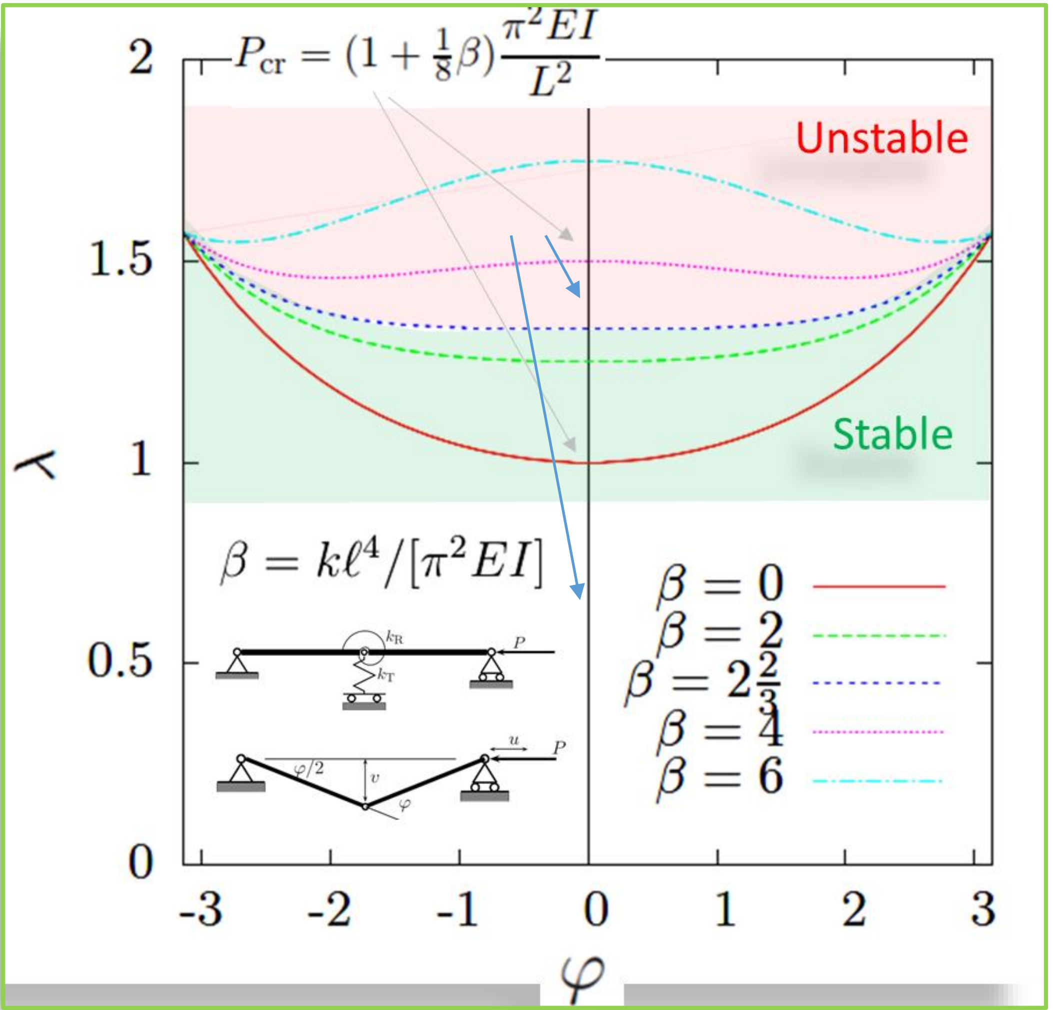
$$= \frac{1}{4} + \frac{1}{32}\beta\cos\varphi - \frac{1}{4}\lambda\cos(\varphi/2)$$

(Ref: This problem is provided by R. Kouhia)

Solution

$$P_{cr}(\beta) = \underbrace{(1 + \frac{\beta}{8})}_{\lambda}\frac{\pi^2 EI}{\ell^2}.$$

Equilibrium paths



From this study we conclude that

- the buckling load increases with the increase of the stiffness of the foundation.
- **However**, at the same time, the **bifurcation** switches from stable to becomes of unstable after a critical value $\beta > 8/3$

$$\beta = k\ell^4/[\pi^2 EI]$$

What to take with you? From the above study we can conclude that: *the buckling load increases with the increase of the stiffness of the foundation. However, at the same time, the bifurcation switches from stable to becomes of unstable-type after a critical value for $\beta > 8/3$.*

Appendix

No need to distribute the followin slides(or read)

Stability theorem of Lagrange-Dirichlet

Self-reading

RECALL

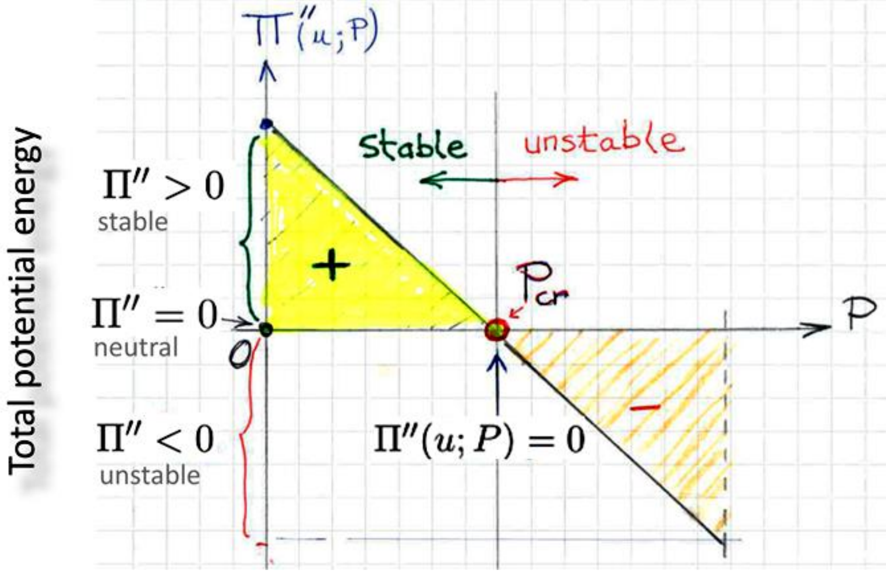
Trefftz condition
for stability of an equilibrium:

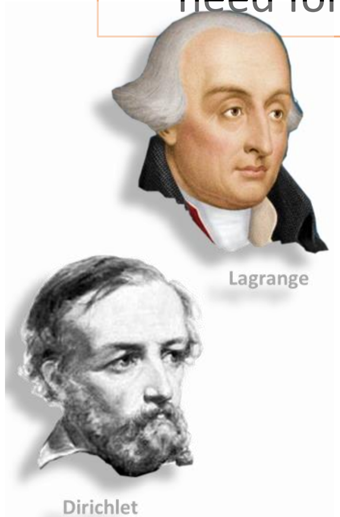
$$\begin{cases} \delta^2\Pi(u) > 0, & \text{stable,} \\ \delta^2\Pi(u) = 0, & \text{neutral,} \\ \delta^2\Pi(u) < 0, & \text{unstable.} \end{cases}$$

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

- Is a *global energy criterion for stability*
- will be used systematically to derive the all the equations of stability (loss) we need for all elastic structure s

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$




Lagrange
Dirichlet

Lagrange-Dirichlet theorem and investigate the sign of the increment

$$\Delta\Pi = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \delta^4\Pi + \dots$$

(More general than Trefftz)

Trefftz is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

Energy criteria for determination **loss of stability** of elastic structures

Self-reading

The general[64] **Trefftz** (1930, 1933) criterion says that the loss or change in stability of an elastic structure occurs when the variation of the second variation[65] of the total potential energy Π of the structure vanishes, *i.e.*,

$$\boxed{\delta(\delta^2\Pi) = 0.}$$

Later, while discussing about bifurcational loss of stability, it will be shown that Trefftz stability condition (Eq. 1.85) is essentially an energetic criterion saying that during loss of stability and for the critical load, the equilibrium holds also in the perturbed state $u^* = u^0 + \delta u$, *i.e*, then $\delta(\Delta\Pi) = 0$. It will be discussed later that, indeed all these energy criteria for loss of stability: $(\Delta\Pi = 0; P_{min} = P_{cr})$, $\delta(\Delta\Pi) = 0$ and $\delta(\delta^2\Pi) = 0$ - which look at first glad different, are indeed equivalent[66]

Self-reading

$$\Pi^* = \Pi[u^0 + \delta u, P^0] = \Pi[u^0, P^0] + \underbrace{\delta\Pi|_{u^0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u^0} + \frac{1}{3!}\delta^3\Pi|_{u^0} + \dots \quad (1.125)$$

The idea is now to develop the increment of total potential energy up-to second or higher when the second, third and so on, variation vanishes.

Then the energy criterion for the stability loss is unchanged and is (physically, an equilibrium condition for the perturbed state $u^* = u^0 + \delta u \equiv u^0 + \hat{u}$):

$$\delta(\Delta\Pi^*) = 0, \forall\delta u \quad \text{kin. admissible} \quad (1.126)$$

$$\delta(\Pi[u^0 + \delta u, P^0] = \delta[\Pi[u^0, P^0] + \underbrace{\delta\Pi|_{u^0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u^0} + \frac{1}{3!}\delta^3\Pi|_{u^0} + \dots)] = 0, \forall\delta u \quad (1.127)$$

$$\delta(\Pi[u^0 + \delta u, P^0]) = \underbrace{\delta[\Pi[u^0, P^0]]}_{=0} + \delta[\frac{1}{2}\delta^2\Pi|_{u^0}] + \delta[\frac{1}{3!}\delta^3\Pi|_{u^0}] + \delta[\dots] = 0, \forall\delta u \quad (1.128)$$

$$\underbrace{\delta(\Pi[u^0 + \delta u, P^0]) - \Pi[u^0, P^0])}_{\delta(\Delta\Pi)=0} = \underbrace{\delta[\frac{1}{2}\delta^2\Pi|_{u^0}] + [\frac{1}{3!}\delta^3\Pi|_{u^0}] + \delta[\dots]}_{=0} = 0, \forall\delta u. \quad (1.129)$$

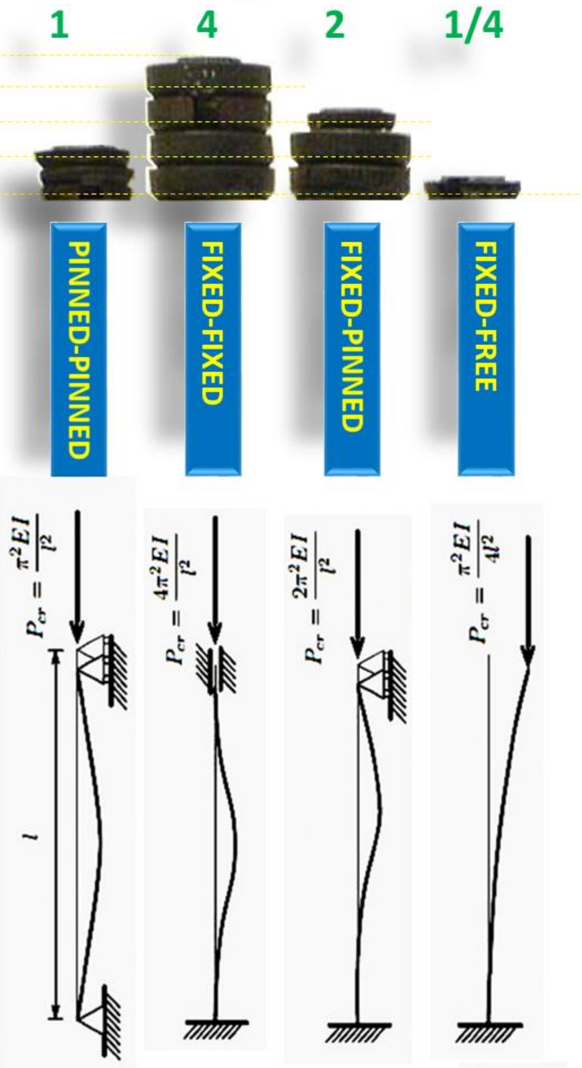
When we keep terms only up-to the second order we obtain the energy criterion for stability loss in the familiar Trefftz form too as:

We will use systematically this more general energy criterion:

$$\boxed{\delta(\Delta\Pi) = \delta[\delta^2\Pi|_{u^0}] = 0, \forall\delta u, \quad \text{kin. admissible},} \quad (1.130)$$

Trefftz stability loss criteria in its canonical form

Effects of boundary conditions – experimental evidence for Euler's buckling formulas

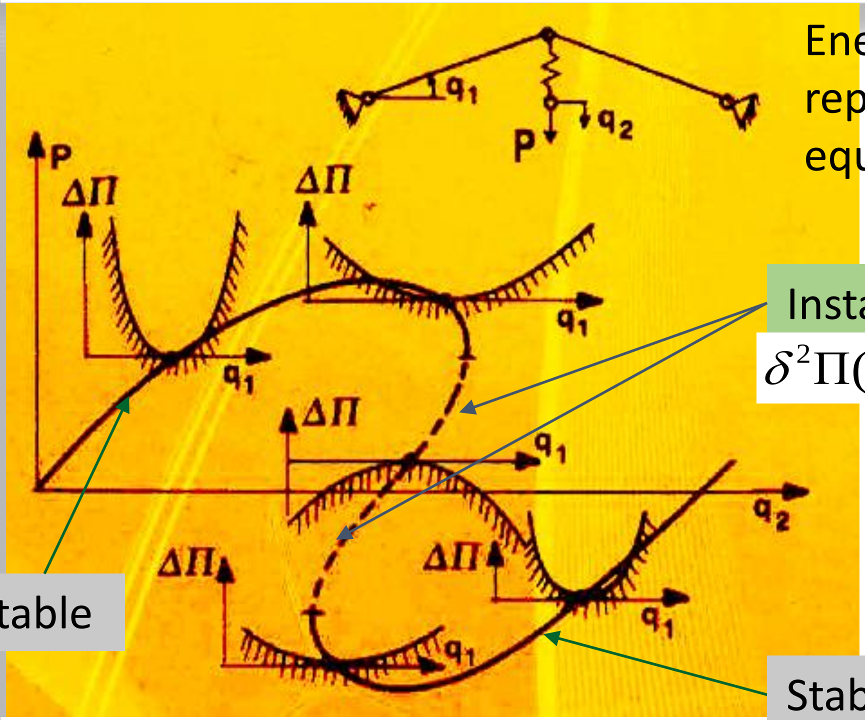


$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad \frac{4\pi^2 EI}{l^2} \quad \frac{2\pi^2 EI}{l^2} \quad \frac{\pi^2 EI}{4l^2}$$

1 4 2 1/4

Examples – snap-through

Note that loss of stability may happen also without bifurcation through limit points as here



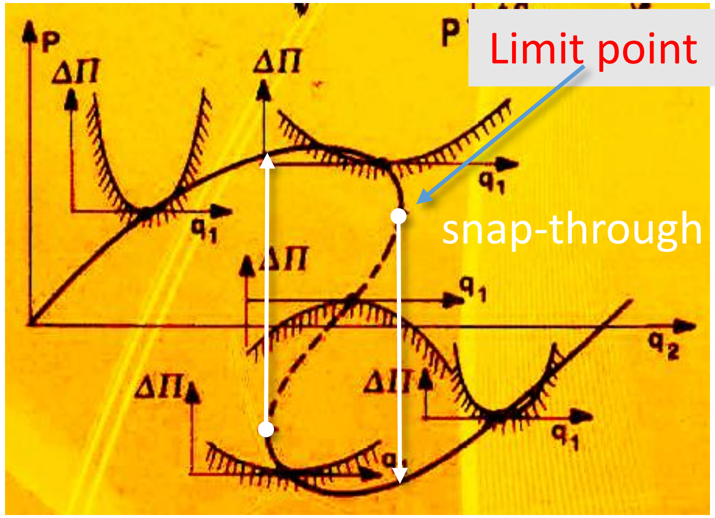
Energy space representation and equilibrium paths

$$\delta^2\Pi(q_1, q_2) < 0$$

$$\delta^2\Pi(q_1, q_2) > 0$$

Ref: Bazant's classical textbook on stability

$$\Delta\Pi = \delta\Pi + \delta^2\Pi + \dots$$



The Rayleigh-quotient

Problème eulerien : la condition de Legendre s'écrit toujours dans la forme d'un problème de type :

$$EIv'''' - P_{cr}v'' = 0\,.$$

D'après le lemme fondamental de la mécanique on peut aussi écrire

$$\int_0^l EIv''''\psi\,dx - P_{cr}\int_0^l v''\psi\,dx = 0\ \forall\psi\,,$$

et donc en particulier

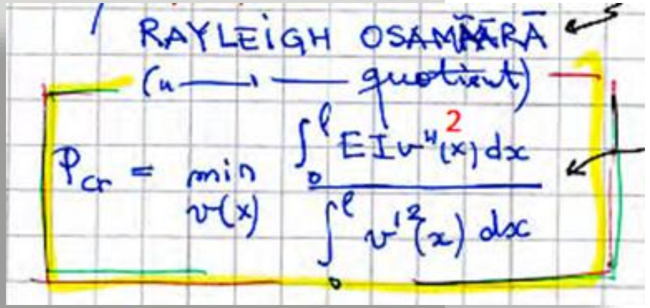
$$\int_0^l EIv''''v\,dx - P_{cr}\int_0^l v''v\,dx = 0\,.$$

En intégrant par parties et pour n'importe quelles conditions au bord de liaison parfaite,

$$\int_0^l EI(v'')^2\,dx - P_{cr}\int_0^l (v')^2\,dx = 0\,,$$

$$\Downarrow$$

$$P_{cr} = \frac{\int_0^l EI(v'')^2\,dx}{\int_0^l (v')^2\,dx}\,.$$



Condition nécessaire pour que P_{cr} soit la charge critique de la structure, avec v déformée en équilibre avec P_{cr}.

Slide from "Beams and Frames – course'

https://eductv.enpc.fr/videos/mecanique-des-structures-seance-8/ (5.10.2017)