Stability of thin plates

$$\sigma_{cr} = K \cdot \underbrace{\frac{\pi^2 E}{12(1-\nu^2)} \left[\frac{h}{\ell}\right]^2}_{\text{reference buckling stress}}$$

buckling coefficient

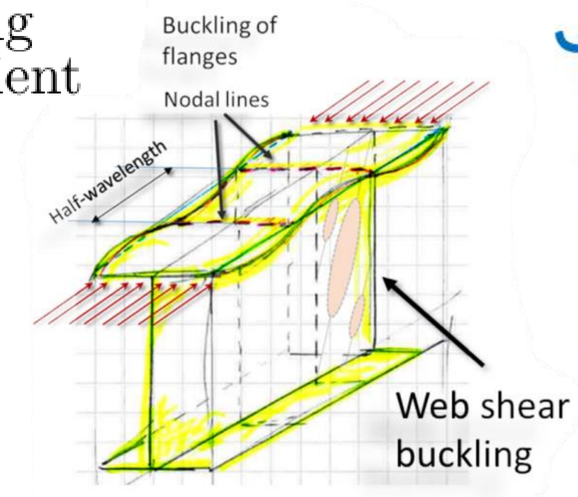
Content

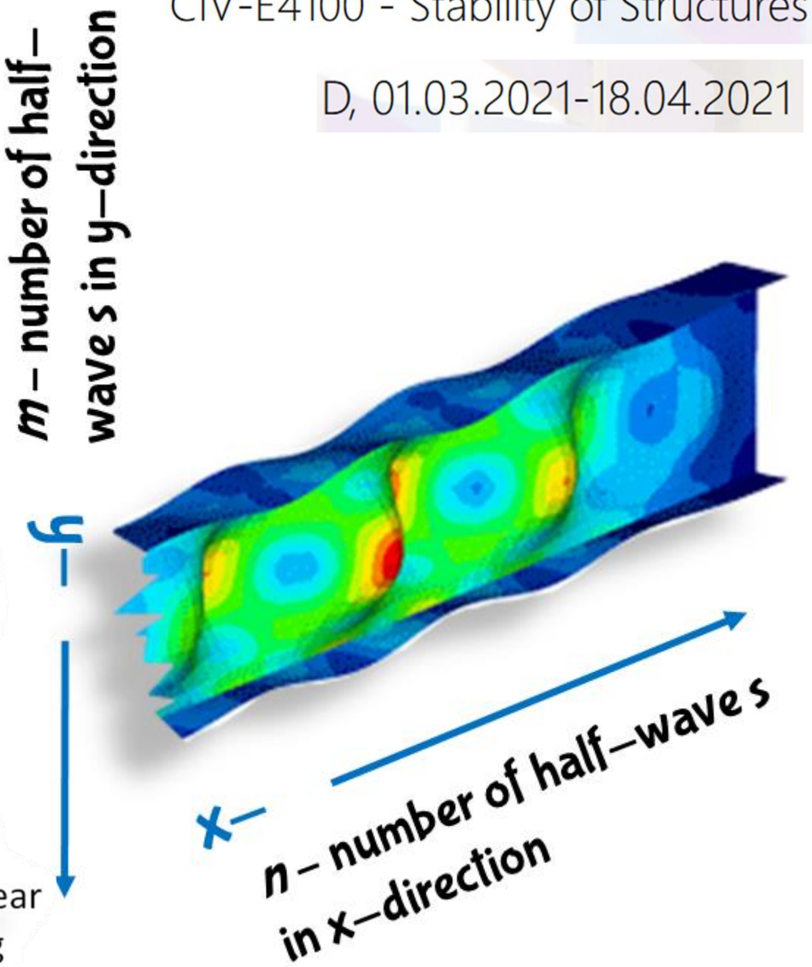
0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability

Weeks #3-4 – Lectures series

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. **Buckling of thin plates**
5. Buckling of shells (lommahdus)

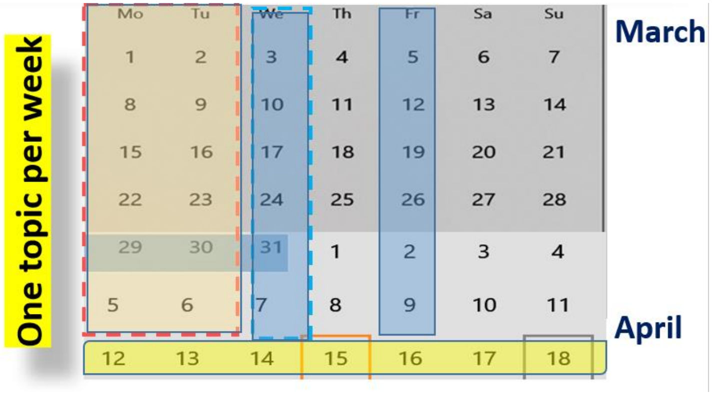
Using
1) **energy** and
2) **differential** approaches





Plane starin assumption:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \underbrace{\frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}}_{\mathbf{E}} \begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$



Lecturer
Djebar Baroudi, Dr.
Civil Engineering Department
Aalto University

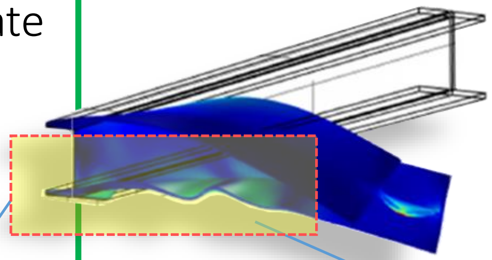
version109.4.2021

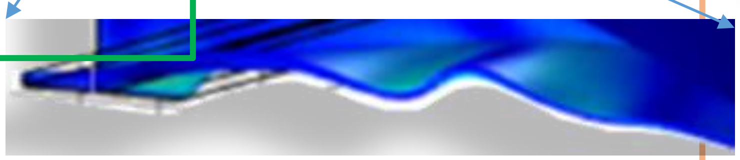
Elastic Stability of Structures

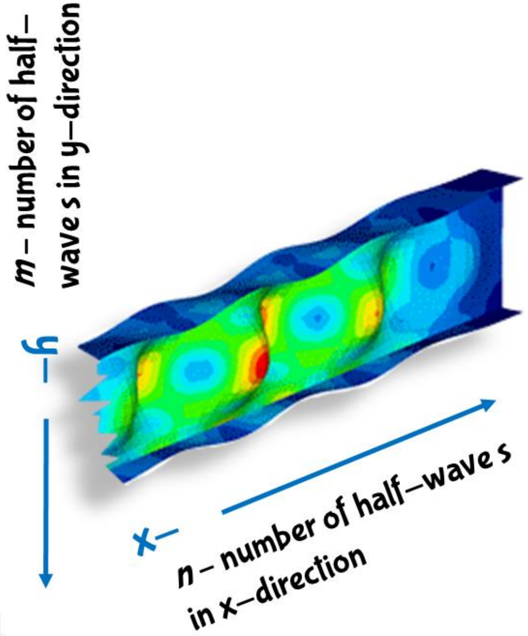
Week #4 – Lecture series

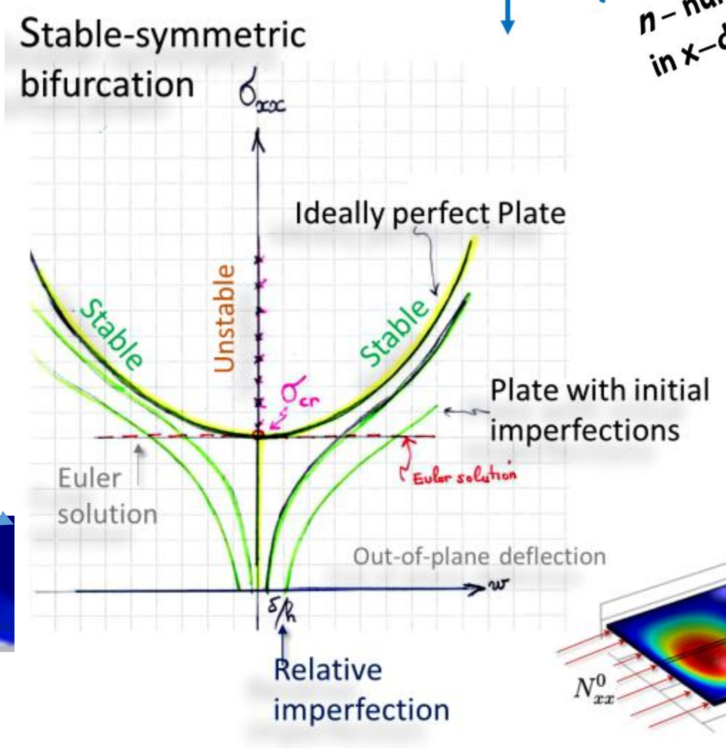
Stability of plates

- **Introductory example**
  - ➢ Cylindrical plate buckling
- **Deriving the Equation of loss of stability**
- Some classical cases
  - ➢ simply supported rectangular plate under one-side compression
  - ➢ simply supported rectangular plate under in-plane bending and compression
  - ➢ Shear buckling of a rectangular plate
- **FEA linear buckling example**
- **FEA post-buckling example**









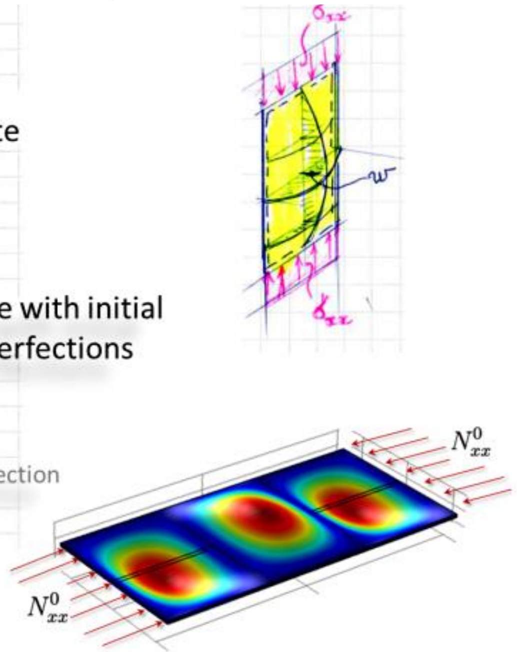


Plate buckling

Lecture slides for internal use only
D. Baroudi, PhD

Post-buckling analysis

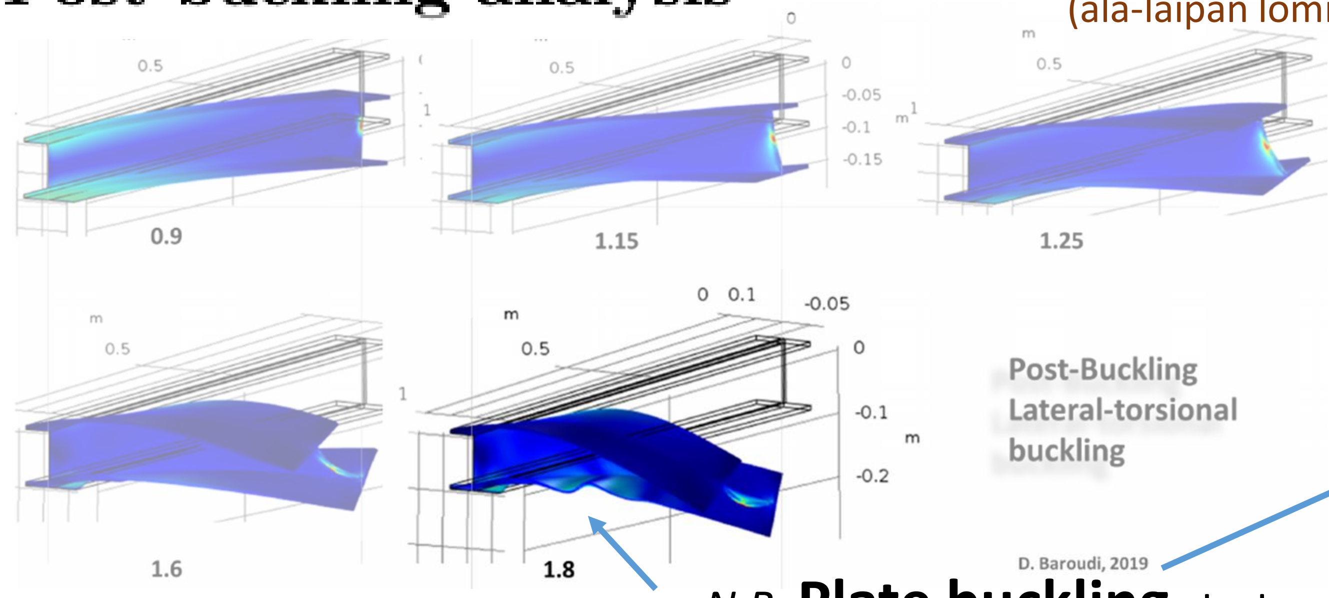
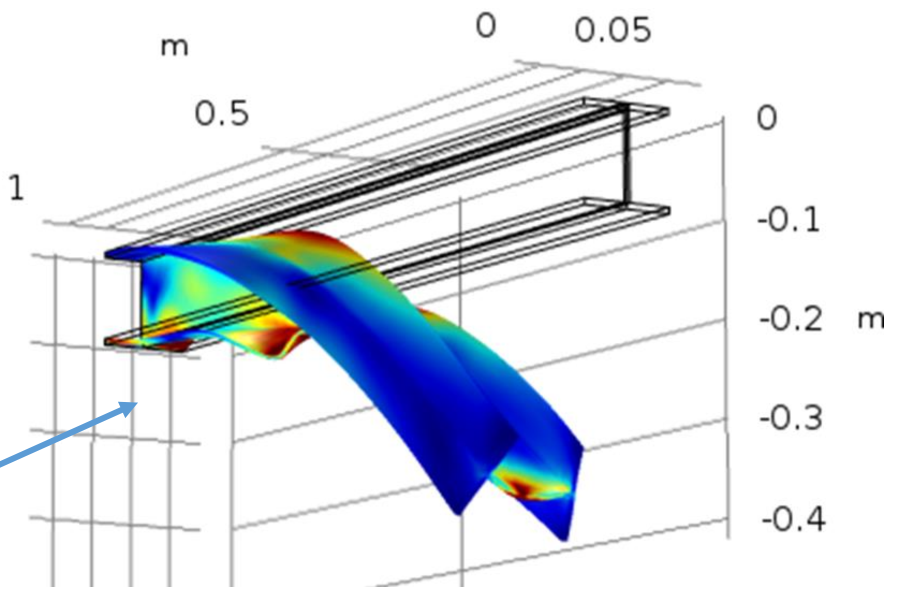


Plate buckling

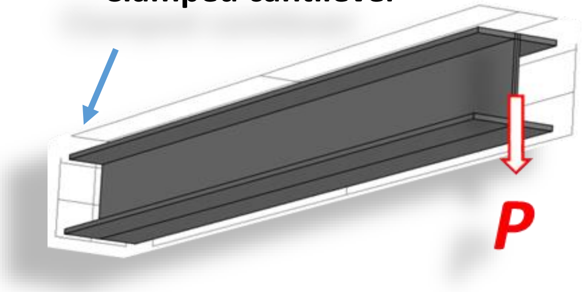
(ala-laipan lommahdus)



N.B. **Plate buckling** starts

(ala-laipan lommahdus)

Clamped cantilever



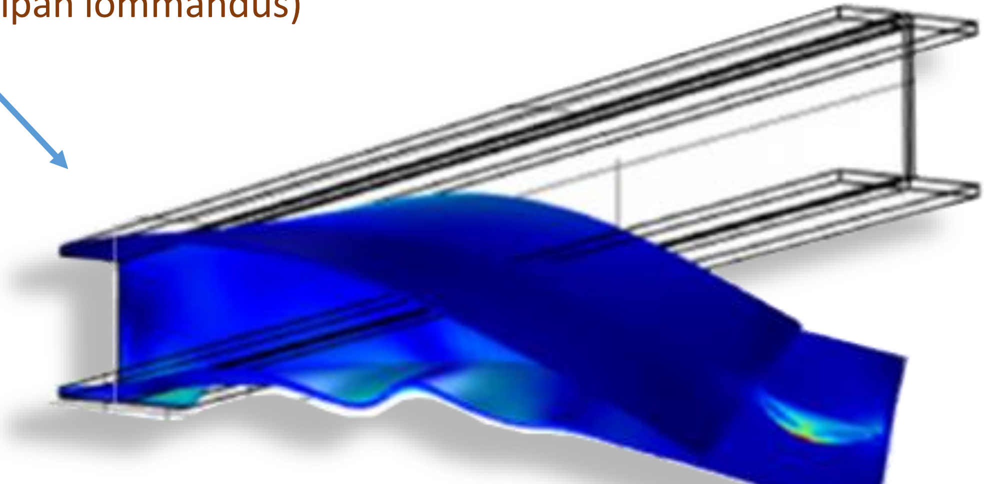


Plate buckling

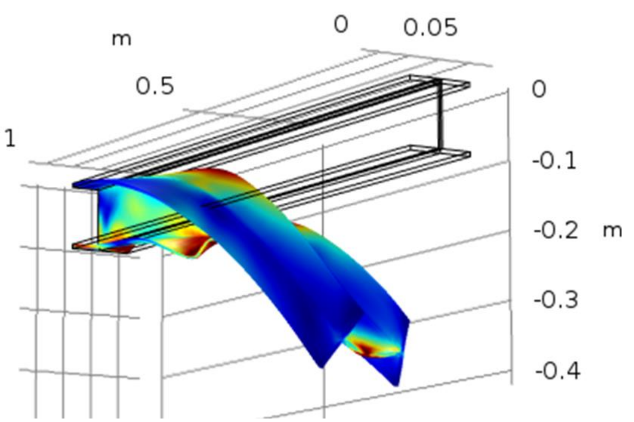
Waking–up ... for me too

N.B. Here, we did elastic post-buckling analysis as it is the topic of this course.

The next analysis step is to add possibility for yield of the material (GMNA) postuckling analys.

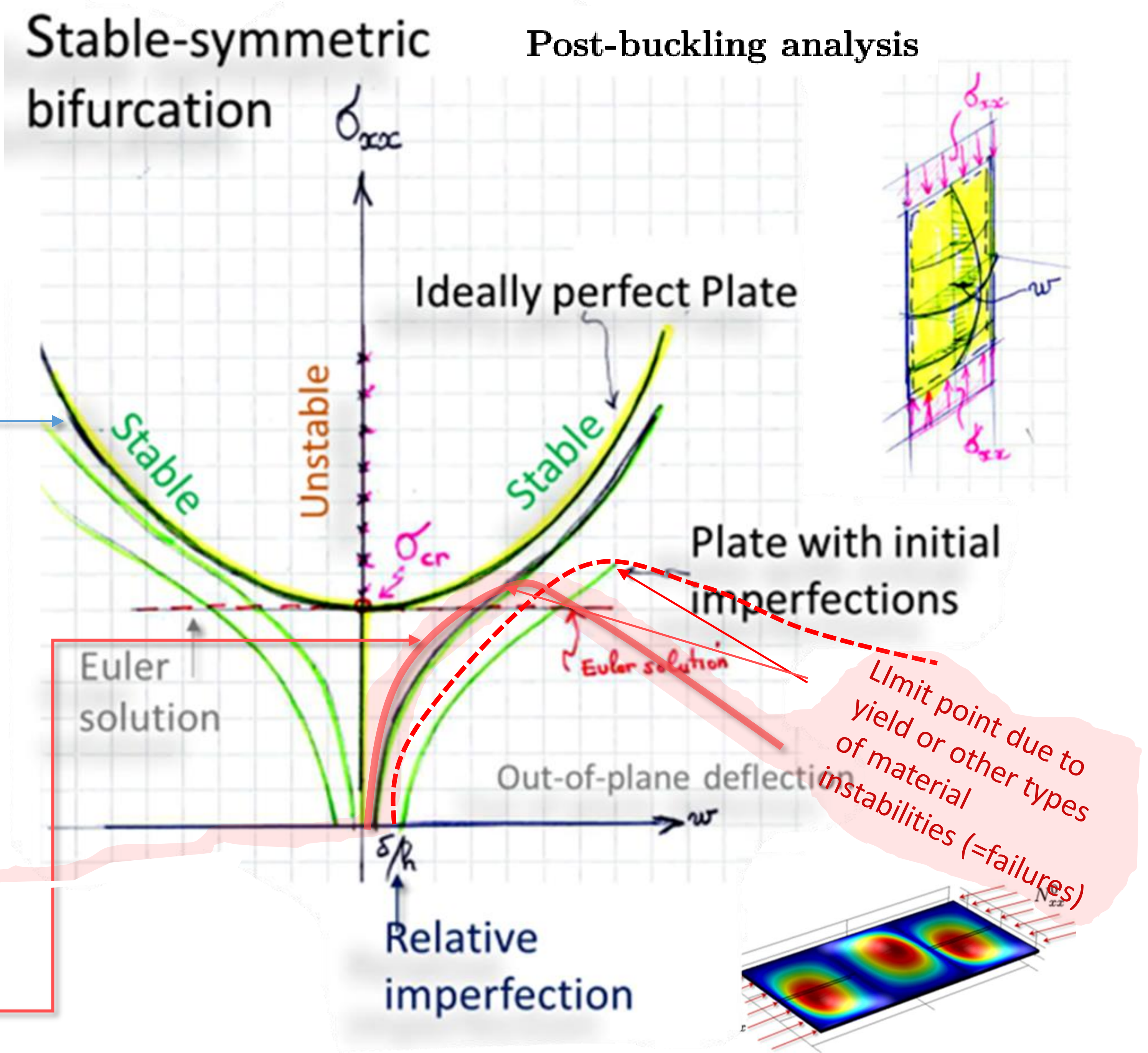
Recall about GNA and G**M**NA

aterial



- Here, we do **G**eometrically **N**on-linear Elastic post-buckling **A**nalysis (GNA) as it is the topic of this course

- The next analysis step to do in is to add possibility for yield or other types of failure of the **M**aterial. Such postuckling analys is known as G**M**NA .



Waking–up   Example from my research work

Buckling load parameter \lambda versus foundation stiffness \gamma parameter for the free edge. (NB. 1/over \gamma ~ k/D = E(cold)/E(hot), is a measure of the relative softening of the hot layer)

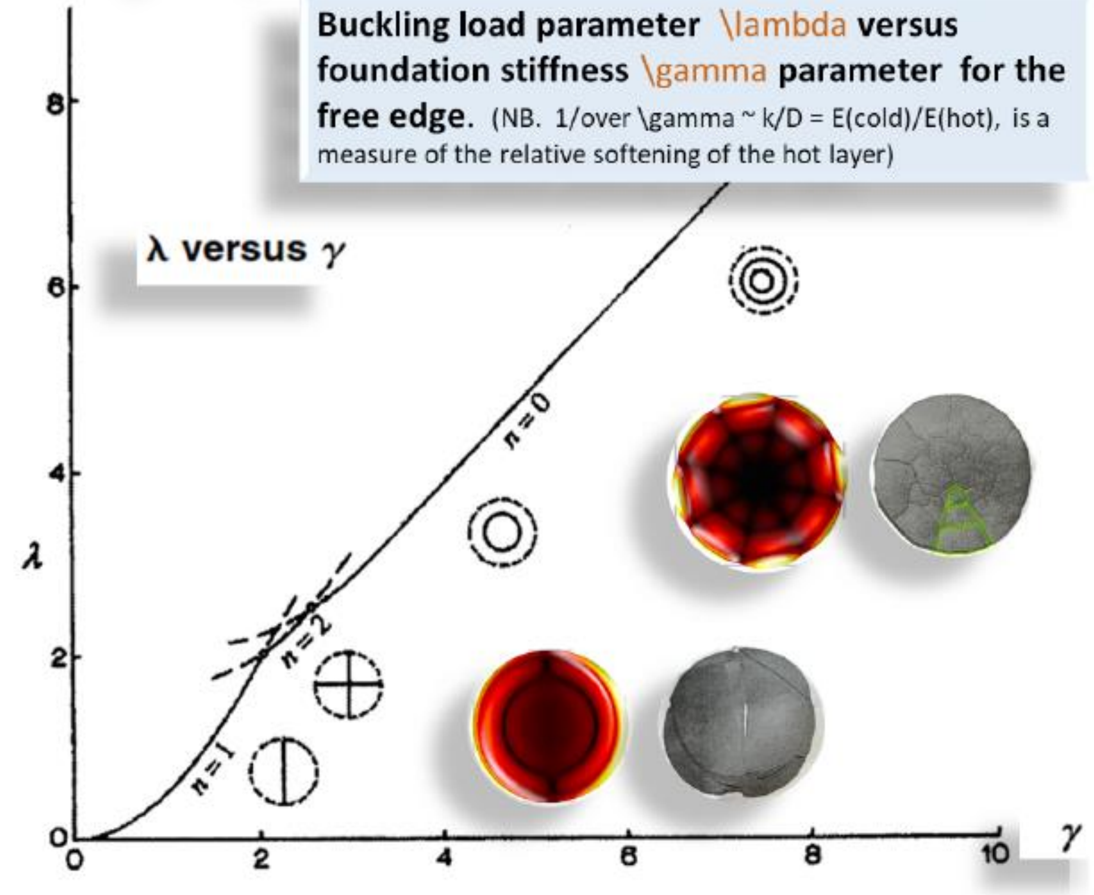
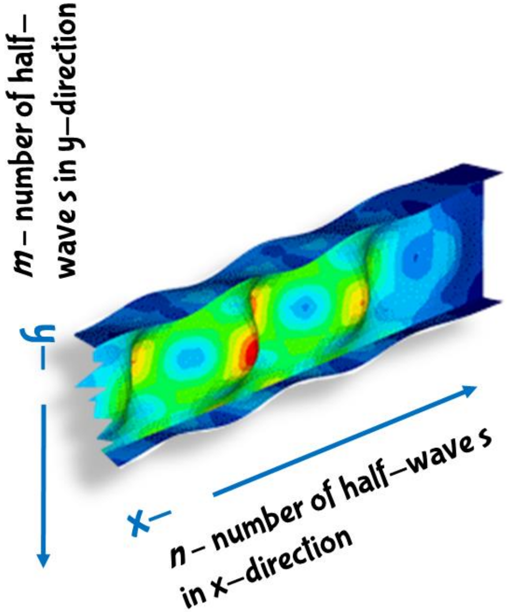
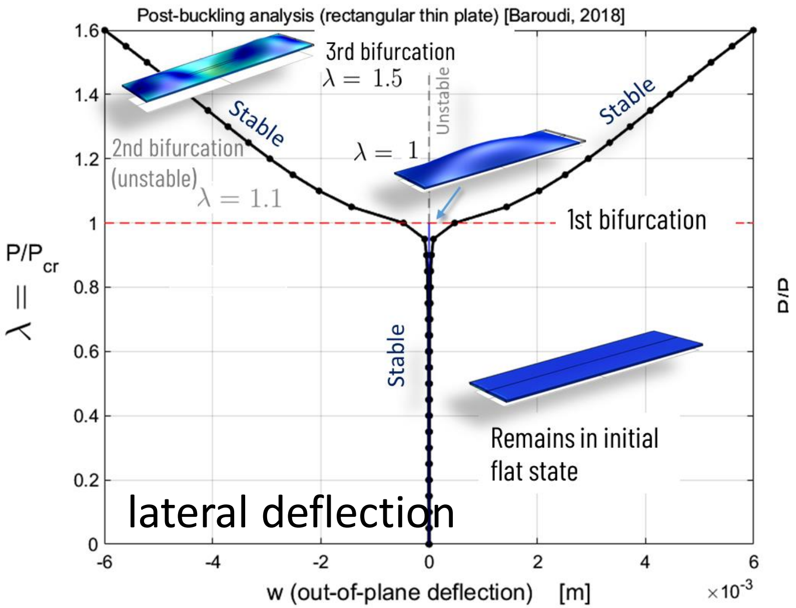


Fig. 3 λ versus γ for the free edge. Legend is similar to that of Fig. 1.

Buckling load parameter \lambda versus foundation stiffness \gamma parameter for the free edge.
Only the solid line represents the buckling load. Small circles denote a change in buckling modes. The dotted line is from the n=2 mode. The buckling modes are shown schematically as disks with nodal lines.

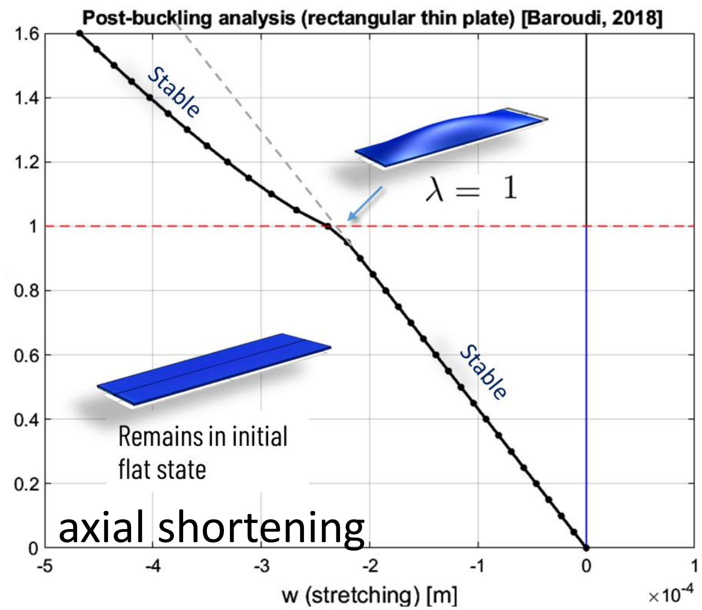


Post-Buckling Analysis



2nd bifurcation (unstable) was not observed in simulation

Voima-ohjattuna



Homework #5

1 Exercise: Plate buckling and stiffeners

As a future or already an active structural designer you want to design stiffeners for a simply supported square thin metallic plate under the compressive edge load (Figure 1). The purpose is to increase the buckling strength (buckling stress) of the plate at least to $10 \times \pi^2 D/a^2$.

The stiffeners[1] are directed along the loading direction and are equal spaced beams of length a and rectangular cross-section $h \times 4h$, where h bei the height of the plate. Read the margin comment.

The material of the stiffeners and the plate is the same isotropic wi Poisson's ratio about 0.3.

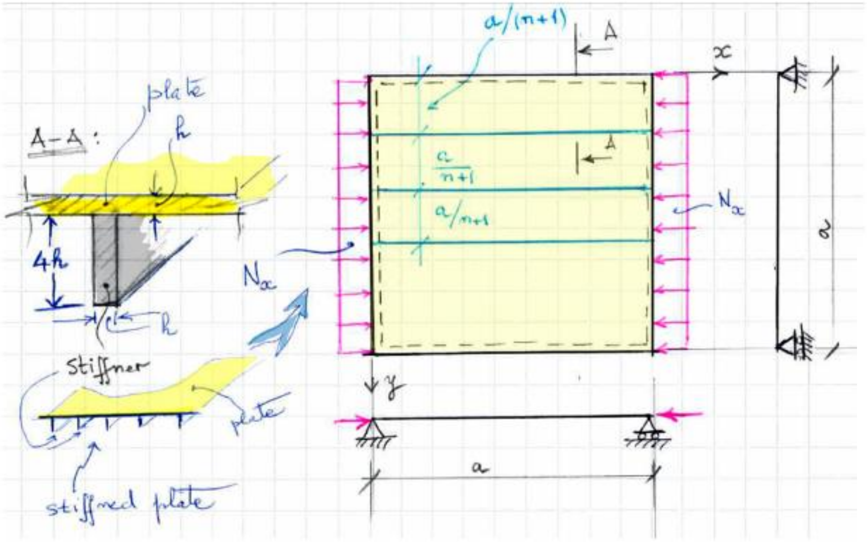
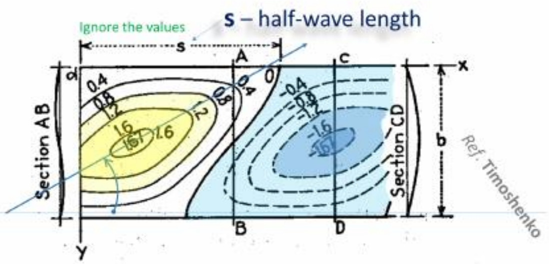


Figure 1: Simply supported square plate and stiffeners. Assume for simpl that the the stiffener-beam is divided in two parts as shown in the ma

Shear buckling of thin plates of finite length a

Approximate for finite slab: $k = 5.35 + 4(b/a)^2$

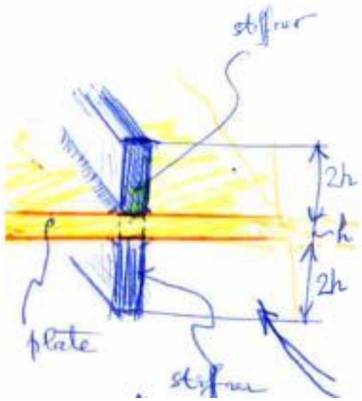
Shear buckling of thin plates



a)

The exact solution of the problem for an infinitely long strip with simply supported edges gives

Exact for infinitely long slab: $\tau_{cr} = 5.35 \dfrac{\pi^2 D}{b^2 h}$



2 Exercise: Column local and global buckling

A metallic column of length L (Figure 2) is being loaded centrically by a thrust P. The metal being isotropic ($\nu = 0.3$). The stress at the loaded cross-section is constant and transmitted to the column through a stiffening end-plate. The cross-section is a square thin tube of thickness t and width b such that $L/b = 20$.

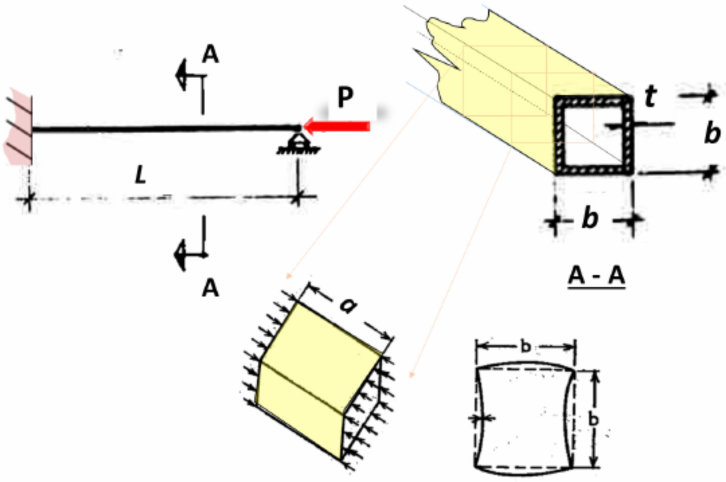


Figure 2: A thin-walled tubular column.

3 Exercise: Shear buckling

Consider Shear buckling of thin metallic isotropic plate of a very long[4] strip ($L \gg b$) (sub-figure b) in Figure 3) is loaded at horizontal edges by a constant shear stress τ. The bending rigidity of the plate is D. Determine

[4]You can approximate it as an infinite sheared plate strip.

$$w(x, y) \approx A \sin(\pi y/b) \sin[\pi(x - \alpha y)/s]$$

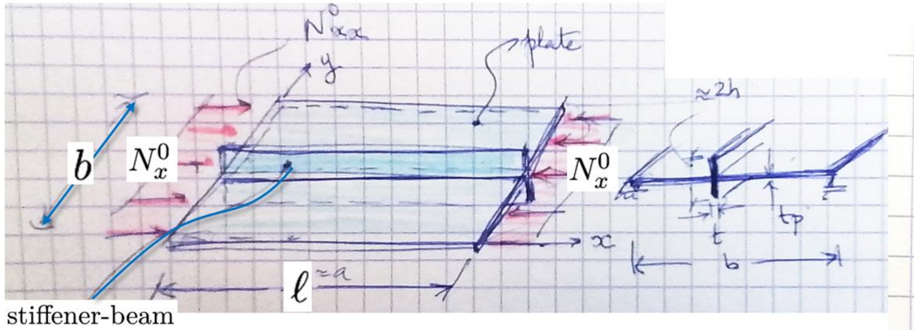
Energy principles to estimate buckling load ... example coming soon....

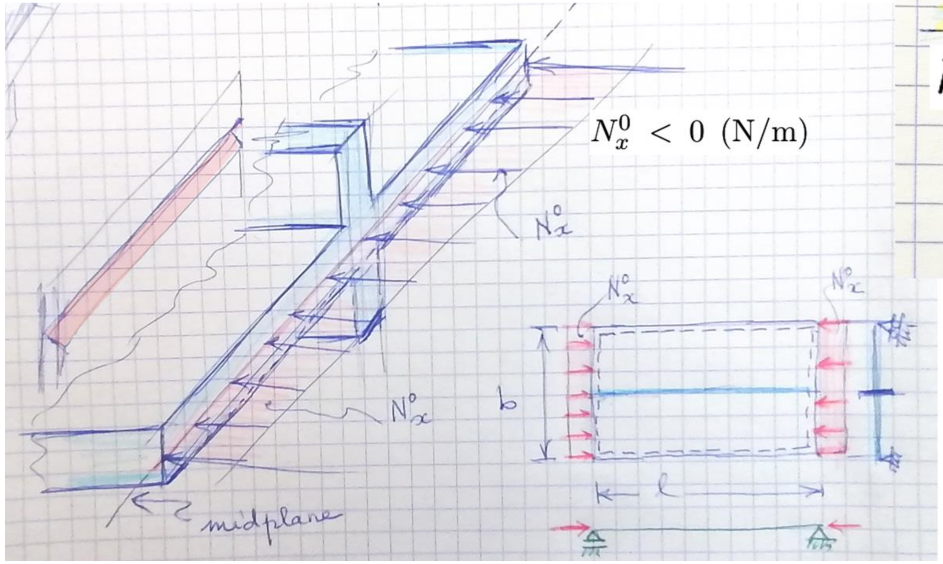
Buckling of a stiffened plate

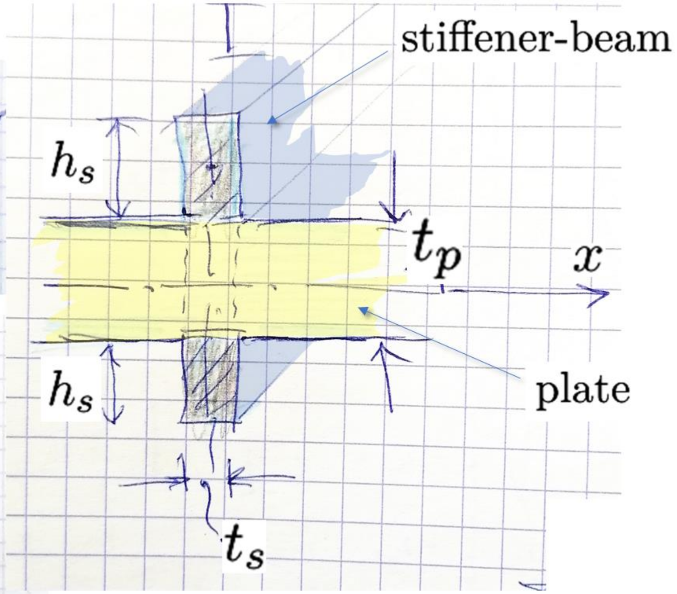
Consider a thin elastic plate (Fig. XXXX) of thickness t_p having length ℓ and width b. The effective bending rigidity of the plate being D. The external compressive load is applied along the lines $x = 0$ and $x = \ell$ with intensity $N_x^0 < 0$ (N/m). A stiffener[381] is designed along the loading direction. The bending rigidity of the stiffener-beam being EI_s. In this example, we assume that ratio b/ℓ is such that the first buckling mode of the plate is in form of a bubble. The stiffener has half-height h_s and thickness t_s. Notice that the inertia moment

$$\delta(\Delta\Pi) = 0$$

Buckling of a stiffened plate





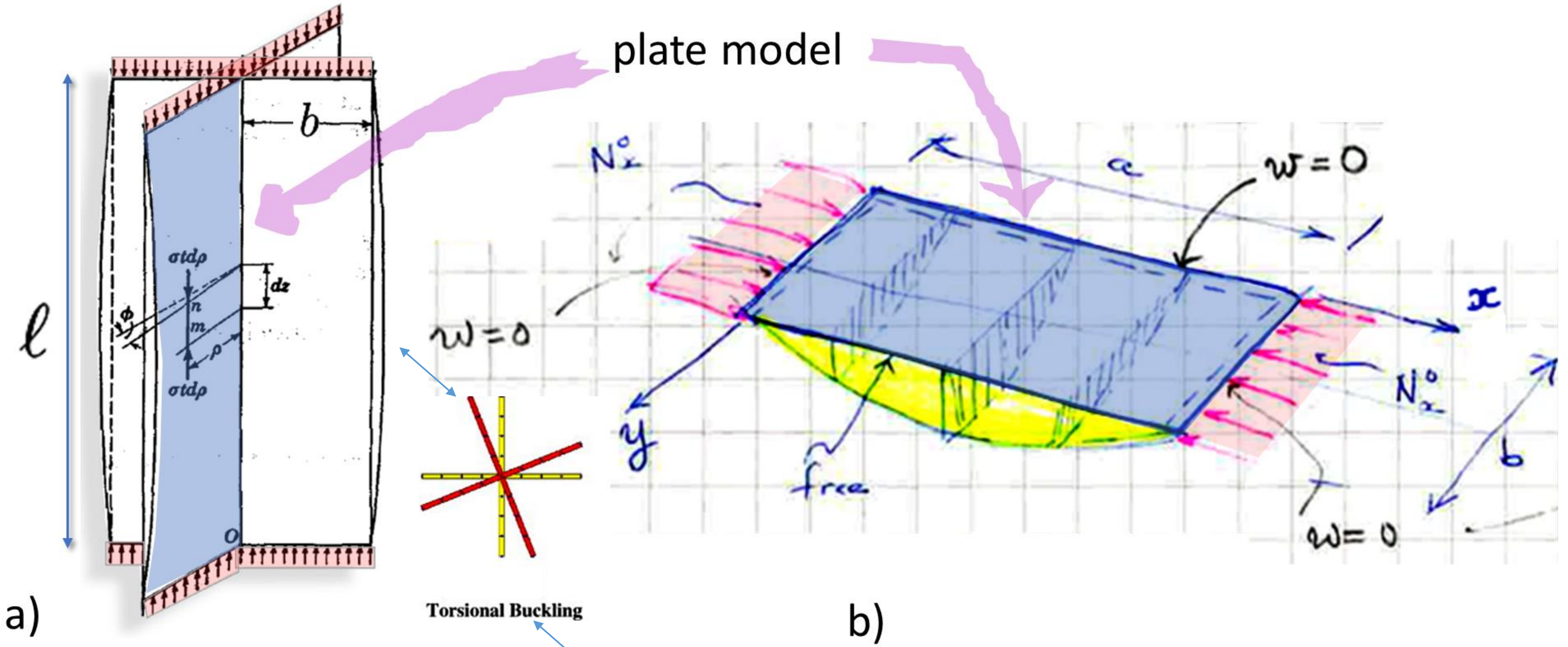


$$I_s = \frac{t_s \cdot (2h_s + t_p)^3}{12} - \frac{t_s t_p^3}{12}$$

```
% ---------------------------------------------
delta_U(L, b, D, w0) =
(D*L*w0^2*pi^4)/(8*b^3) + (D*b*w0^2*pi^4)/(8*L^3) +
(D*nu*w0^2*pi^4)/(4*L*b) - (D*w0^2*pi^4*(nu - 1))/(4*L*b)

d2x_w_beam =
-(w0*pi^2*sin((pi*x)/L))/L^2

delta_U_beam(L, b, D, w0) =
(EI*w0^2*pi^4)/(4*L^3)

delta_W(L, b, w0, N0x) =
(N0x*b*w0^2*pi^2)/(8*L)

delta_Pi =
(EI*w0^2*pi^4)/(4*L^3) + (D*L*w0^2*pi^4)/(8*b^3) +
(D*b*w0^2*pi^4)/(8*L^3) + (N0x*b*w0^2*pi^2)/(8*L) +
(D*nu*w0^2*pi^4)/(4*L*b) - (D*w0^2*pi^4*(nu - 1))/(4*L*b)

delta_Pi_w =
(w0*pi^2*(D*pi^2*L^4 + N0x*L^2*b^4 +
2*D*pi^2*L^2*b^2 + D*pi^2*b^4 + 2*EI*pi^2*b^3))/(4*L^3*b^3)

% ------------------------------------------------------------
texti =
'Remember Nox is now negative : = - Nox_ref'
% -------------------------------------------------------
```

before solving that, let's start from the begining

Pure **torsional buckling and plate buckling**



The primary mechanism the **rotational mode** of the thin-walled beam open **cross section** is be the result of plate buckling.

Reading assignment

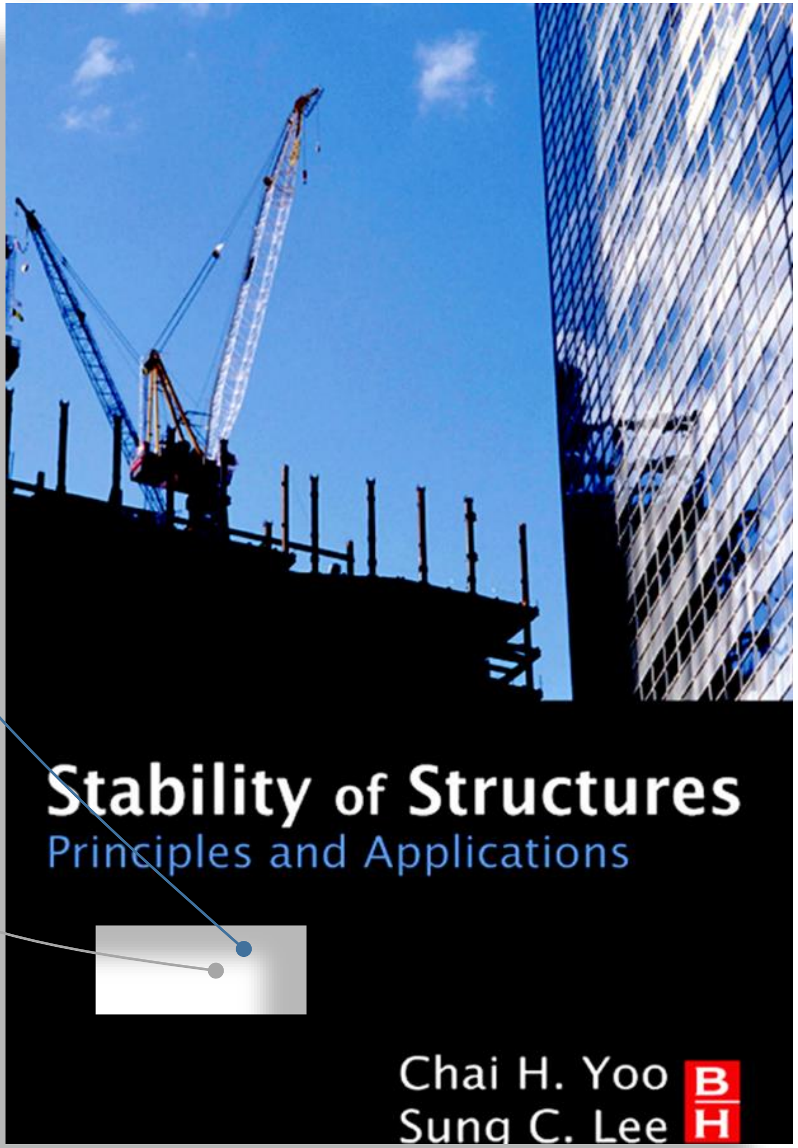
This week

Chapter 8. Buckling of Plate Elements

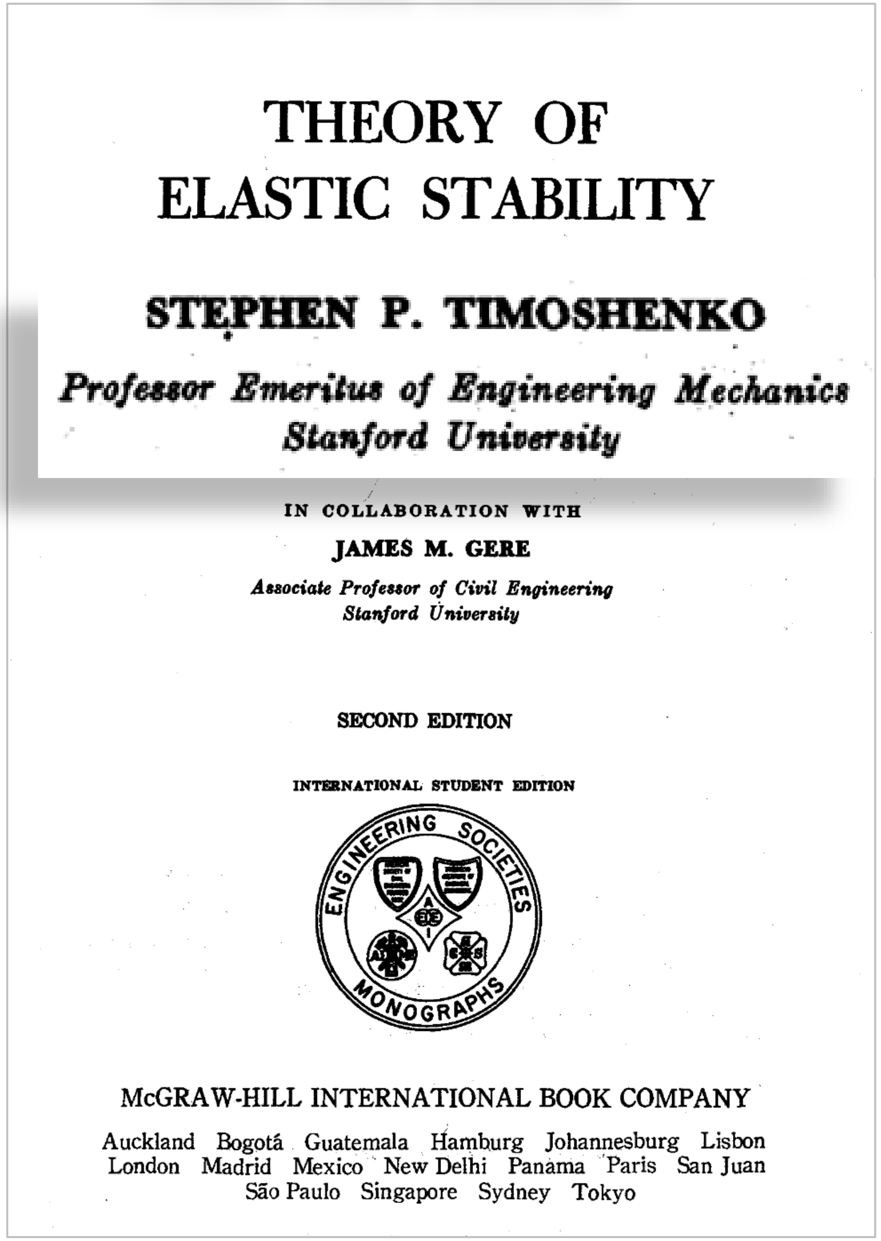
Next week

Chapter 9. Buckling of Thin Cylindrical Elements

This course textbook *e*-book



Must read classics


THEORY OF ELASTIC STABILITY

STEPHEN P. TIMOSHENKO

Professor Emeritus of Engineering Mechanics
Stanford University

IN COLLABORATION WITH

JAMES M. GERE

Associate Professor of Civil Engineering
Stanford University

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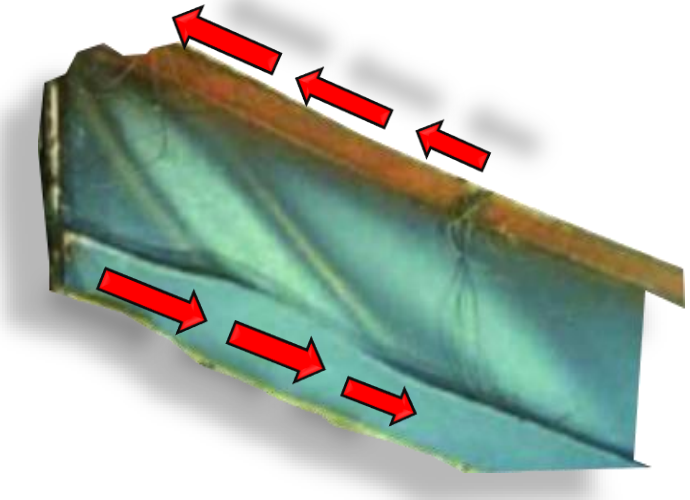


Shear buckling of thin plates
Levyn leikkauslommahdus

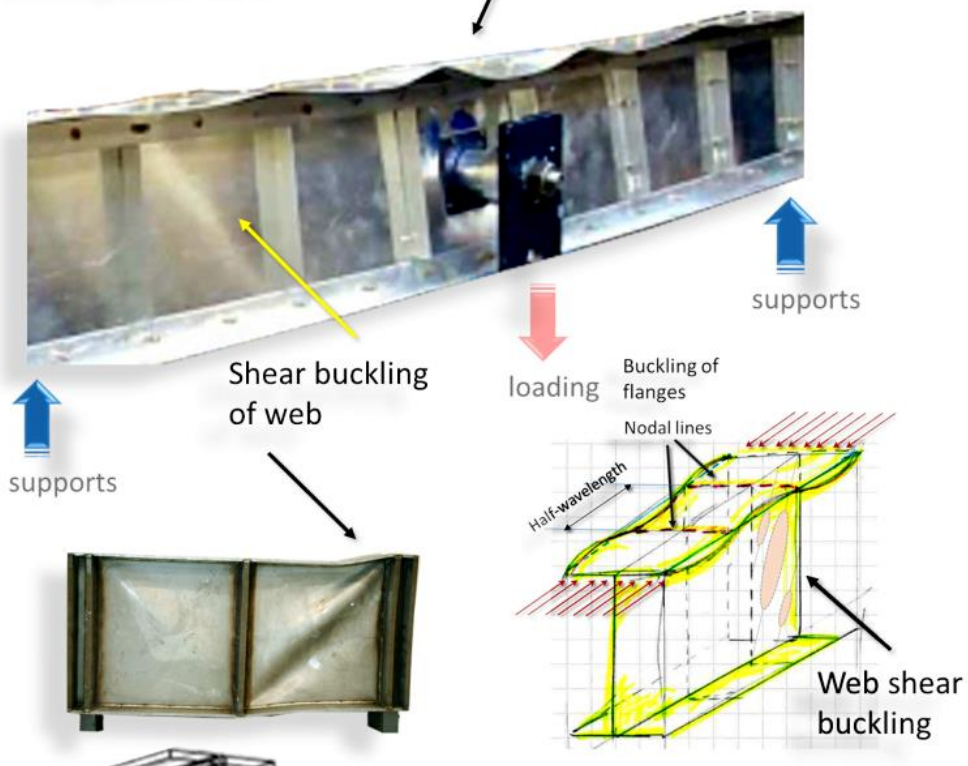
Stability of plates

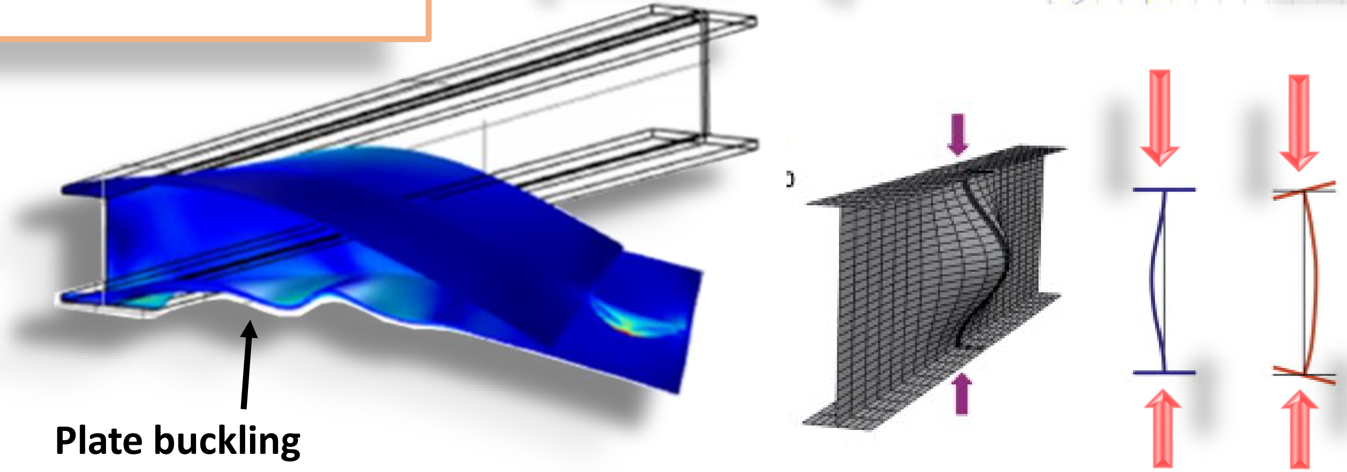
Many **structures** or **structural elements** have **plate elements** or substructures that under certain loading conditions can **buckle**

- flanges and webs of rolled or build-up beams & columns
- hollow metallic bars (rectangular cross-sections)
- shear walls in buildings
- aircraft fuselage panels
- aircraft wings, rudder and wing panels

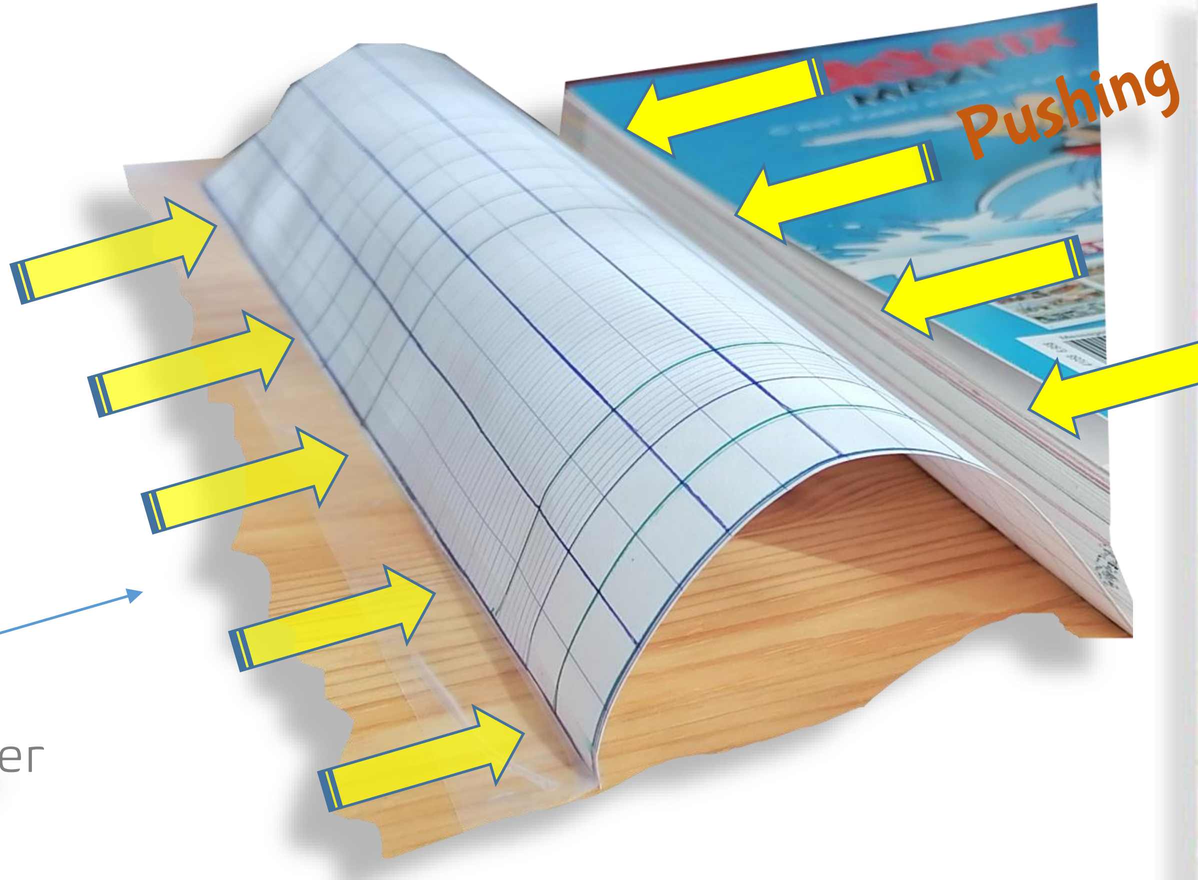
Examples of common structures

Buckling of Plates
Bending experiment

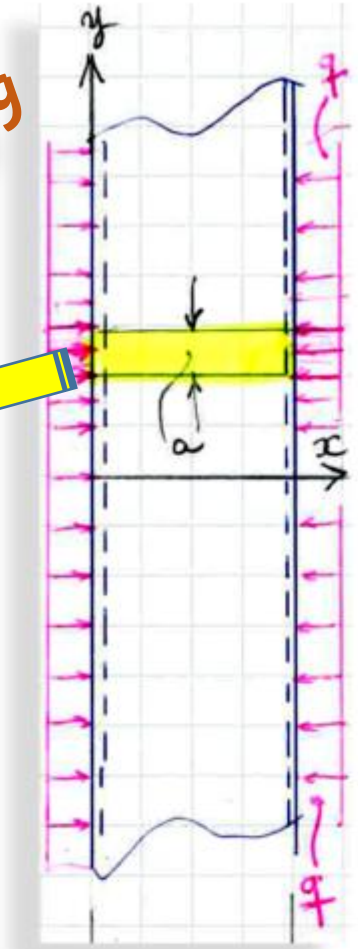




Pushing

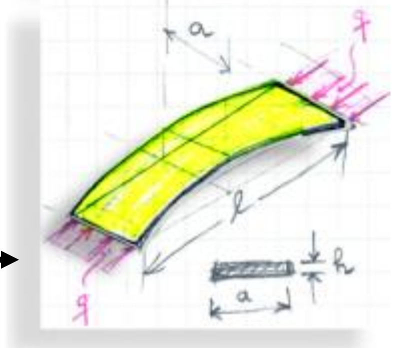


Initially flat paper sheet buckles under in-plane pushing



Cylindrical plate buckling

Analogous to Euler column with unit width *a*



Introductory example
Cylindrical plate buckling

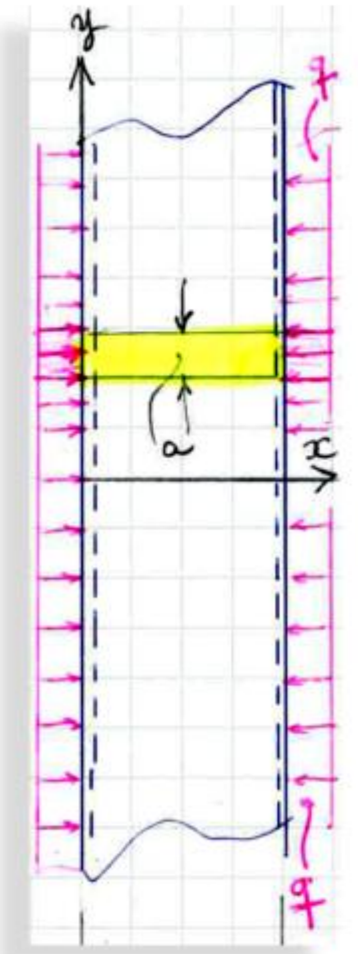
thin plates with relative thickness $h/\ell \leq \; 1/10$.
Kirchhoff–Love theory
von Kármán moderate rotations.

the linearised homogeneous equations of (loss) of stability

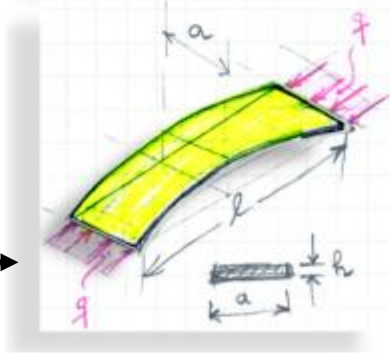
$$D\Delta\Delta w - N^0_{\alpha\beta} w_{,\alpha\beta} = 0,$$

$$D\left[w_{,xxxx} + 2w_{,xxyy} + w_{yyyy}\right] - N^0_{xx} w_{xx} - 2N^0_{xy} w_{xy} - N^0_{yy} w_{yy} = 0.$$

You and I will derive these equation very easily
Assume them, for the moment given.



Analogous to Euler column
with unit width *a*



Cylindrical-type buckling of plate slab

Introduction

the linearised homogeneous equations of (loss) of stability

This stability equation will be derived

$$D\Delta\Delta w - N^0_{\alpha\beta} w_{,\alpha\beta} = 0,$$

$$D\left[w_{,xxxx} + 2w_{,xxyy} + w_{yyyy}\right] - N^0_{xx} w_{xx} - 2N^0_{xy} w_{xy} - N^0_{yy} w_{yy} = 0.$$

$a = 1$ [m] of an 'infinitely' long plate

$$D\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + q\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0,$$

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0,$$

$$EI = a \cdot D, \quad P = a \cdot q$$

The critical load q_{cr} [N/m]

By analogy with Euler–buckling:

$$q_{cr} = \pi^2 D/\ell^2$$

$$P_{cr} = \pi^2 EI/\ell^2$$

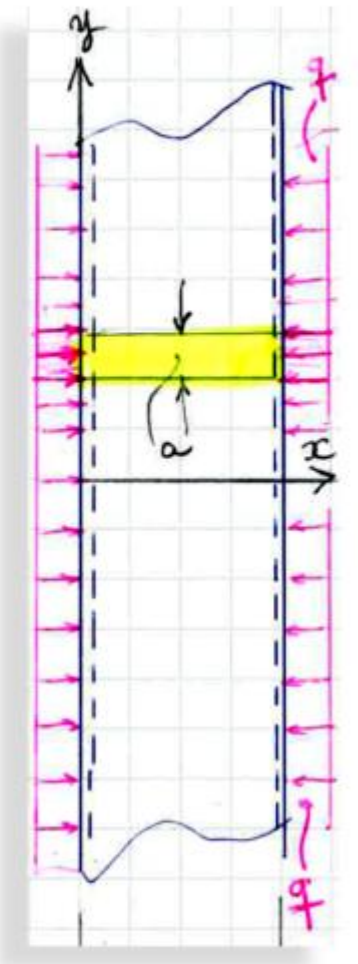
'slenderness'

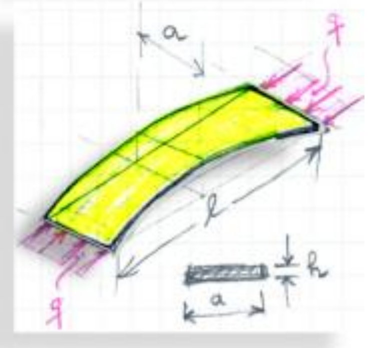
$$\sigma_{cr} \propto [h/\ell]^2$$

Engineering formula

$$\sigma_{cr} = \frac{q_{cr}}{h} = \frac{\pi^2 D}{h\ell^2} = \frac{1}{h} \cdot \frac{\pi^2 E h^3}{12(1-\nu^2)\ell^2} = \frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{\ell}\right]^2$$

What is beautiful is that we now *discovered* this complex looking formula just by understanding the analogy with Euler column buckling. Such analogy is not visible and remains hidden to 'lookers' not having the needed background in structural mechanics





$a = 1$ [m] of an 'infinitely' long plate

Analysis of an Engineering formula

Plate buckling:

$$\sigma_{cr} = K \cdot \underbrace{\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{\ell}\right]^2}_{\text{reference buckling stress}}$$

Buckling coefficient
Lommahduskerroin

The **buckling coefficient** *K* depends on

- Loading
- boundary conditions and
- the aspect **ratio** *length* to *width* $\alpha = a/b$
  (for rectangular plates)

aspect ratio $\alpha = a/b$

Euler column buckling:

$$P_{cr} = \mu \cdot \underbrace{\frac{\pi^2 EI}{\ell^2}}_{\text{reference buckling load}},$$

Effect of boundary conditions

$$\Downarrow$$

$$\sigma_E = \frac{P_{cr}}{A} = \mu\pi^2 E\left[\frac{r}{\ell}\right]^2$$

$r^2 = I/A$

'slenderness'

$$\sigma_{cr} = K \cdot \frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{\ell}\right]^2$$

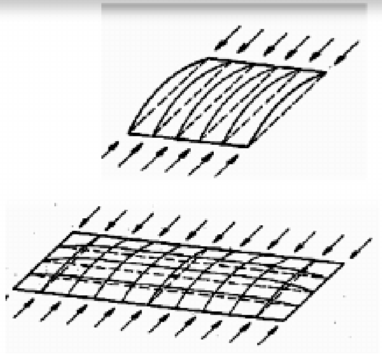
formula from SFS-EN-1993-1-5

4.5.3 Column type buckling

(1) The elastic critical column should be taken as the buckling st

(2) For an unstiffened plate the

$$\sigma_{cr,c} = \frac{\pi^2 E t^2}{12(1-\nu^2)a^2}$$



Basic buckling formula for a column-like buckling mode taken SFS-EN-199 (section 4.5.3). ...ess is denoted t (c... Eq. (1.594)).

SFS-EN-1993-1-5

Deriving the equations of stability

(loss of)

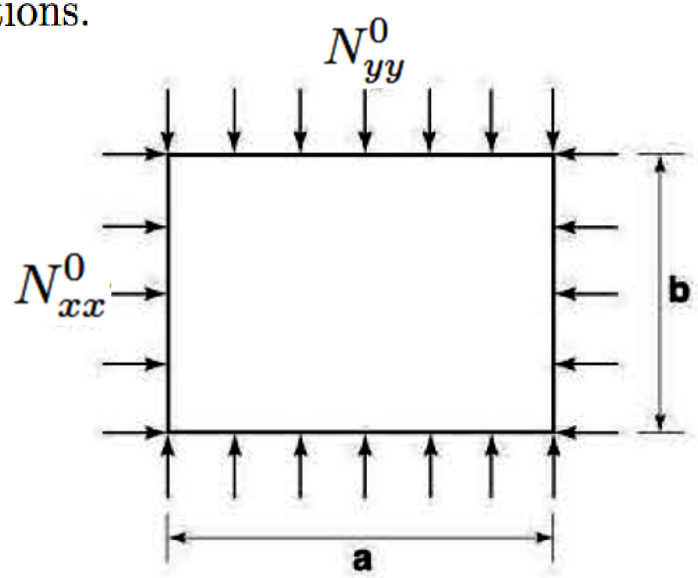
$\delta(\Delta\Pi) = 0$

$\Downarrow$

thin plates with relative thickness $h/\ell \leq 1/10$

Kirchhoff–Love theory

von Kármán moderate rotations.



From where these equations came?

the linearised homogeneous equations of (loss) of stability

$$D\Delta\Delta w - N^0_{\alpha\beta} w_{,\alpha\beta} = 0,$$

$$D\left[w_{,xxxx} + 2w_{,xxyy} + w_{yyyy}\right] - N^0_{xx} w_{xx} - 2N^0_{xy} w_{xy} - N^0_{yy} w_{yy} = 0.$$

Initial mid-plane kinematics, in the reference primary equilibrium state:

$$\begin{cases} u = u^0(x, y) \\ v = v^0(x, y) \\ w = w^0(x, y) \equiv 0 \end{cases}$$

Pre-buckled configuration and intial (membrane = in-plane) stresses

Can be loading also edge shear N^0_{xy}

Mid-plane kinematics increments, in the post-buckled configuration:

$$\begin{cases} \Delta u = u - z w_{,x} \\ \Delta v = v - z w_{,y} \\ \Delta w = w \end{cases}$$

Post-buckled configuration

Huom.! Here, this Δ is not the Laplace operator, it is just a difference = increment

Deriving the equations of stability

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments}}$$

buckling of a column

$$\Delta\Pi = \frac{1}{2}\int_0^\ell \underbrace{EIw_{,xx}^2}_{EI\kappa_x^2}\mathrm{d} + \underbrace{\int_0^\ell N_x^0\left(\frac{1}{2}w_{,x}^2\right)\mathrm{d}x}_{\Delta W(\sigma_x^0, \Delta\epsilon_{x,NL})}$$

Buckling of plate

The initial membrane stress-resultants $N_{xx}^0, N_{yy}^0, N_{xy}^0$ satisfy the equilibrium

The pre-stress resultants $N_{xx}^0, N_{yy}^0, N_{xy}^0$ will work $N_{xx}^0, N_{yy}^0, N_{xy}^0$

Quadratic part of strain increment (at buckling)

$$\epsilon_2 \equiv [\epsilon_{xx}^*,\ \epsilon_{yy}^*,\ \gamma_{xy}^* = 2\epsilon_{xy}^*]^{\mathrm{T}}$$

$$\epsilon_{ij}^* = 1/2\, u_{k,i}u_{k,j},$$

$$\epsilon_{xx}^* = \frac{1}{2}[\underbrace{u_{,x}^2 + v_{,x}^2}_{\approx 0 \ll w_{,x}^2} + w_{,x}^2] \approx \frac{1}{2}w_{,x}^2,$$

$$\epsilon_{yy}^* = \frac{1}{2}[\underbrace{u_{,y}^2 + v_{,y}^2}_{\approx 0} + w_{,y}^2] \approx \frac{1}{2}w_{,y}^2,$$

$$\gamma_{xy}^* \equiv 2\epsilon_{xy}^* = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}$$

At buckling, the derivatives of displacement components of the primary equilibrium are small (can be ignored)

Recall column buckling where *u*,x was ignored since it doesn't change during buckling (in the neighborhood of critical point)

Deriving the equations of stability

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments}}$$

Initial (membrane) stress resultants satisfy equilibrium equation in pre-buckled state N^0_{yy} N^0_{xx} N^0_{xy}

$N^0_{xx}, N^0_{yy}, N^0_{xy}$

$$\epsilon_2 \equiv [\epsilon^*_{xx}, \epsilon^*_{yy}, \gamma^*_{xy} = 2\epsilon^*_{xy}]^{\mathrm{T}}$$

$$\epsilon^*_{ij} = 1/2\, u_{k,i}u_{k,j},$$

$$\epsilon^*_{xx} = \frac{1}{2}[\underbrace{u^2_{,x} + v^2_{,x}}_{\approx 0 \ll w^2_{,x}} + w^2_{,x}] \approx \frac{1}{2}w^2_{,x},$$

$$\epsilon^*_{yy} = \frac{1}{2}[\underbrace{u^2_{,y} + v^2_{,y}}_{\approx 0} + w^2_{,y}] \approx \frac{1}{2}w^2_{,y},$$

$$\gamma^*_{xy} \equiv 2\epsilon^*_{xy} = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}$$

For an isotropic plate

$$\Delta U = \frac{1}{2}\int_V \sigma : \epsilon_1 \mathrm{d}V = \frac{1}{2}\int_V \epsilon_1^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V =$$

$$= \frac{1}{2}D\int_A \left[w^2_{,xx} + w^2_{,yy} + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w^2_{,xy}\right] \mathrm{d}A$$

Refer to you course: Plates and Shells

$$\epsilon_1 = [\partial u/\partial x,\ \partial v/\partial y,\ \partial u/\partial y + \partial v/\partial x]^{\mathrm{T}}$$
$$= -z[w_{,xx},\ w_{,yy},\ w_{,xy} + w_{,yx}]^{\mathrm{T}}$$
$$= -z[\kappa_x,\ \kappa_y,\ 2\kappa_{xy}]^{\mathrm{T}},$$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \underbrace{\frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}}_{\mathbf{E}}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$

Deriving the equations of stability

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments}}$$

Initial (membrane) stress resultants satisfy equilibrium equation in pre-buckled state N_{yy}^0 N_{xx}^0 N_{xy}^0

$N_{xx}^0, N_{yy}^0, N_{xy}^0$

$$\epsilon_2 \equiv [\epsilon_{xx}^*, \epsilon_{yy}^*, \gamma_{xy}^* = 2\epsilon_{xy}^*]^{\mathrm{T}}$$

For an isotropic plate

$$\Delta U = \frac{1}{2}\int_V \sigma : \epsilon_1 \mathrm{d}V = \frac{1}{2}\int_V \epsilon_1^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V =$$

$$= \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right]\mathrm{d}A$$

Refer to you course: Plates and Shells

Linear part of strain increment:

$$\epsilon_1 = [\partial u/\partial x,\ \partial v/\partial y,\ \partial u/\partial y + \partial v/\partial x]^{\mathrm{T}}$$

$$= -z[w_{,xx},\ w_{,yy},\ w_{,xy} + w_{,yx}]^{\mathrm{T}}$$

$$= -z[\kappa_x,\ \kappa_y,\ 2\kappa_{xy}]^{\mathrm{T}},$$

$$M_x = D(\kappa_x + \nu\kappa_s),$$
$$M_s = D(\kappa_s + \nu\kappa_x),$$
$$M_{xs} = D(1-\nu)\kappa_{xs},$$

Non-linear (quadratic) part of strain increment:

$$\epsilon_{xx}^* = \frac{1}{2}[\underbrace{u_{,x}^2 + v_{,x}^2}_{\approx 0 \ll w_{,x}^2} + w_{,x}^2] \approx \frac{1}{2}w_{,x}^2,$$

$$\epsilon_{yy}^* = \frac{1}{2}[\underbrace{u_{,y}^2 + v_{,y}^2}_{\approx 0} + w_{,y}^2] \approx \frac{1}{2}w_{,y}^2,$$

$$\gamma_{xy}^* \equiv 2\epsilon_{xy}^* = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}$$

Deriving the equations of stability

The work increment of pre-stresses on the increment of second order part of strains being

$$\epsilon^*_{xx} = \frac{1}{2}[\underbrace{u^2_{,x} + v^2_{,x}}_{\approx 0 \ll w^2_{,x}} + w^2_{,x}] \approx \frac{1}{2}w^2_{,x},$$

$$\epsilon^*_{yy} = \frac{1}{2}[\underbrace{u^2_{,y} + v^2_{,y}}_{\approx 0} + w^2_{,y}] \approx \frac{1}{2}w^2_{,y},$$

$$\gamma^*_{xy} \equiv 2\epsilon^*_{xy} = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}$$

$$\Delta W(\sigma_0, \Delta\epsilon_{NL}) = D\int_V \sigma^0 : \Delta\epsilon_{NL}\,\mathrm{d}V =$$

$$= \int_A \int_{-h/2}^{+h/2} \left[\sigma^0_{xx}\epsilon^*_{xx} + \sigma^0_{yy}\epsilon^*_{yy} + \tau^0_{xy}\epsilon^*_{xy} + \tau^0_{yx}\epsilon^*_{yx}\right] \mathrm{d}z\mathrm{d}A$$

$$= \int_A \int_{-h/2}^{+h/2} \left[\sigma^0_{xx}\epsilon^*_{xx} + \sigma^0_{yy}\epsilon^*_{yy} + \tau^0_{xy}\gamma^*_{xy}\right] \mathrm{d}z\mathrm{d}A$$

$$= \int_A \left[N^0_{xx}\frac{1}{2}w^2_{,x} + N^0_{yy}\frac{1}{2}w^2_{,y} + N^0_{xy}w_{,x}w_{,y}\right] \mathrm{d}A$$

Initial membrane stresses work with their conjugate NL-strain increments (they are work-conjugate)

$$N^0_{xx}, N^0_{yy}, N^0_{xy}$$

$$\epsilon_2 \equiv [\epsilon^*_{xx}, \epsilon^*_{yy}, \gamma^*_{xy} = 2\epsilon^*_{xy}]^{\mathrm{T}}$$

The increment of total potential energy is now

$$\Delta\Pi = \frac{1}{2}D\int_A \left[w^2_{,xx} + w^2_{,yy} + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w^2_{,xy}\right] \mathrm{d}A +$$

$$+\frac{1}{2}\int_A \left[N^0_{xx}w^2_{,x} + N^0_{yy}w^2_{,y} + 2N^0_{xy}w_{,x}w_{,y}\right] \mathrm{d}A$$

$$\delta(\Delta\Pi) = 0 \quad \Longrightarrow$$

Deriving the equations of stability

$$\delta(\Delta\Pi) = 0$$

Variation of the strain energy increment term will result in the classical part known for bending of plate (ignoring the boundary terms)

$$\Longrightarrow \quad \delta(\Delta U) = \int_A \left[w_{,xxxx} + 2w_{,xx}w_{,yy} + w_{,yyyy}\right]\delta w \mathrm{d}A$$

Variation of increment of work of pre-stresses on the increment of deformations (ignoring boundary terms) will be

Gives the mechanical boundary conditions

$$\Longrightarrow \quad \delta(\Delta W) = -\frac{1}{D}\int_A \left[N^0_{xx}w_{,xx} + N^0_{yy}w_{,yy} + 2N^0_{xy}w_{,xy}\right]\delta w \mathrm{d}A$$

$$\Longrightarrow \quad \delta(\Delta\Pi) = \int_A D\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right]\delta w + -\left[N^0_{xx}w_{,xx} + N^0_{yy}w_{,yy} + 2N^0_{xy}w_{,xy}\right]\delta w \mathrm{d}A = 0, \ \forall \delta w$$

Equations of loss of stability:

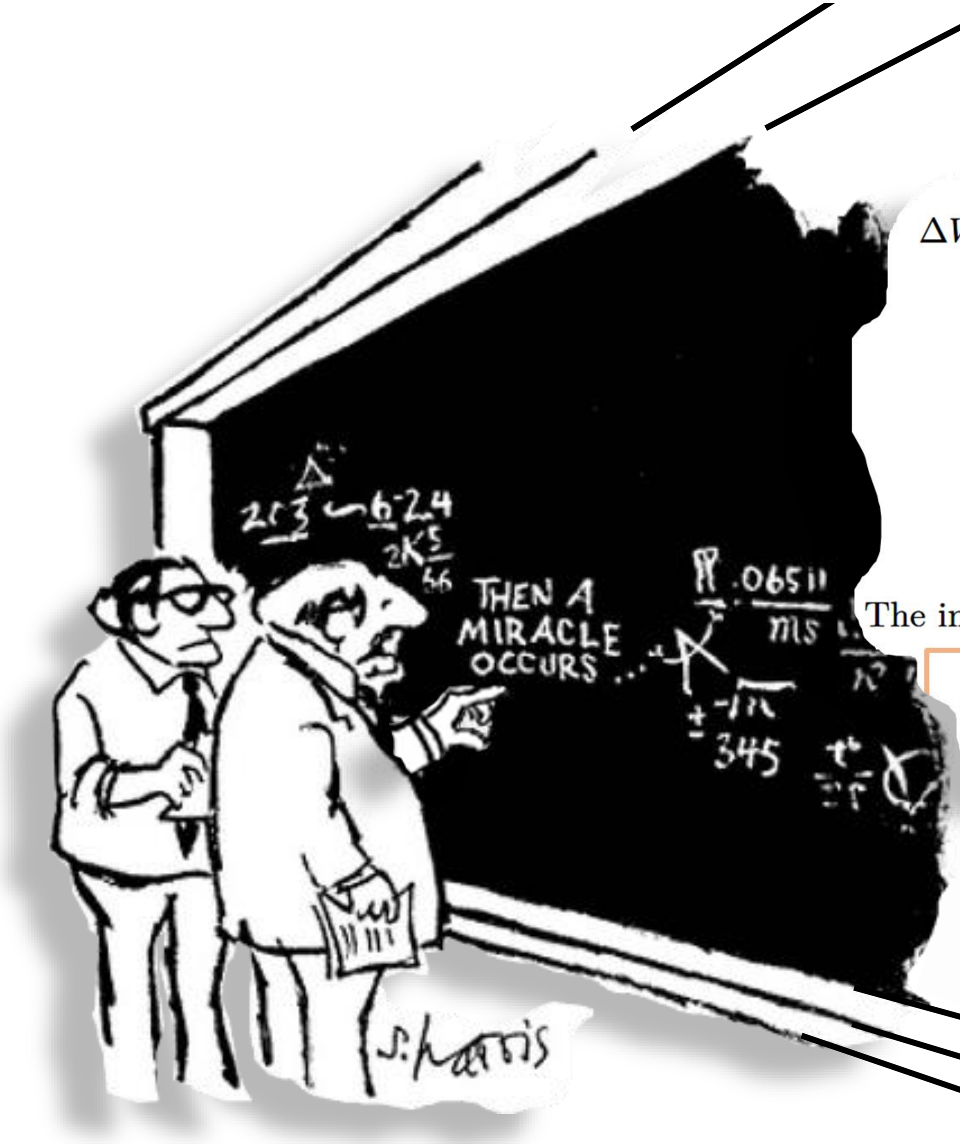
$$D\underbrace{\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right]}_{\Delta\Delta w} - \underbrace{\left[N^0_{xx}w_{,xx} + N^0_{yy}w_{,yy} + 2N^0_{xy}w_{,xy}\right]}_{N^0_{\alpha\beta}w_{,\alpha\beta}} = 0$$

The membrane stresses have to equilibrium equations in the pre-buckled state

$$D\Delta\Delta w - N^0_{\alpha\beta}w_{,\alpha\beta} = 0, \quad (x,y) \in S \times [-h/2, h/2], \ (+\text{BCs})$$

$\delta(\mathrm{d}f/\mathrm{d}x) = \mathrm{d}(\delta f)/\mathrm{d}x = \delta(f_{,x}) = (\delta f)_{,x}$

Deriving the equations of stability



$$\Delta W(\sigma_0, \Delta\epsilon_{NL}) = D\int_V \sigma^0 : \Delta\epsilon_{NL}\,\mathrm{d}V =$$

$$= \int_A \int_{-h/2}^{+h/2} \left[\sigma^0_{xx}\epsilon^*_{xx} + \sigma^0_{yy}\epsilon^*_{yy} + \tau^0_{xy}\epsilon^*_{xy} + \tau^0_{yx}\epsilon^*_{yx}\right] \mathrm{d}z\mathrm{d}A$$

$$= \int_A \int_{-h/2}^{+h/2} \left[\sigma^0_{xx}\epsilon^*_{xx} + \sigma^0_{yy}\epsilon^*_{yy} + \tau^0_{xy}\gamma^*_{xy}\right] \mathrm{d}z\mathrm{d}A$$

$$= \int_A \left[N^0_{xx}\frac{1}{2}w^2_{,x} + N^0_{yy}\frac{1}{2}w^2_{,y} + N^0_{xy}w_{,x}w_{,y}\right] \mathrm{d}A$$

The increment of total potential energy is now

$$\Delta\Pi = \frac{1}{2}D\int_A \left[w^2_{,xx} + w^2_{,yy} + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w^2_{,xy}\right]\mathrm{d}A +$$

$$+\frac{1}{2}\int_A \left[N^0_{xx}w^2_{,x} + N^0_{yy}w^2_{,y} + 2N^0_{xy}w_{,x}w_{,y}\right]\mathrm{d}A$$

$\epsilon_2 \equiv$

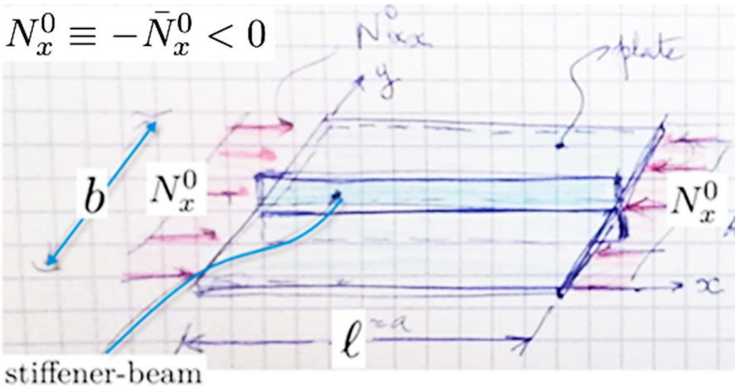
$$\delta(\Delta\Pi) = 0 \implies$$

$$D\underbrace{\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right]}_{\Delta\Delta w} - \underbrace{\left[N^0_{xx}w_{,xx} + N^0_{yy}w_{,yy} + 2N^0_{xy}w_{,xy}\right]}_{g\,w_{,\alpha\beta}} = 0$$

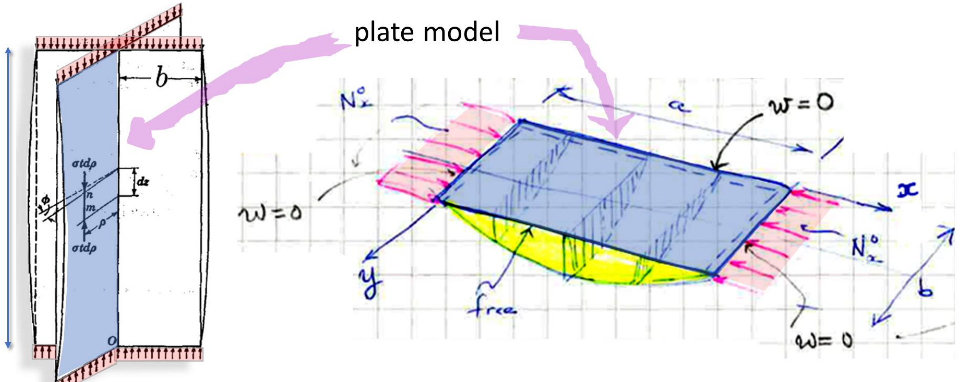
"I think that you should be more explicit here in step two."

Energy principle to estimate buckling loads with examples

1. **Buckling of a stiffened plate** ↔

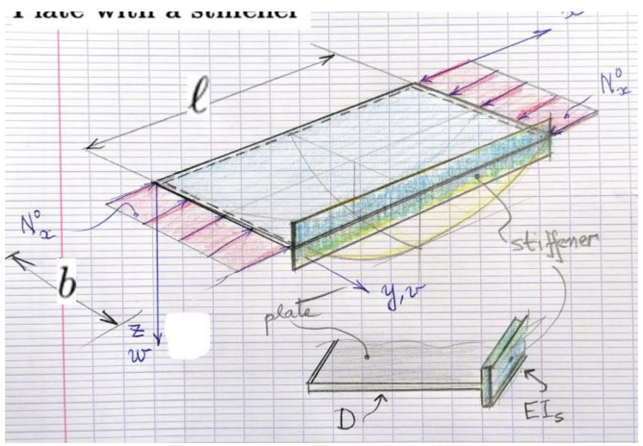


2. *2-D versus 1-D:* Plate model *versus* beam-model for torsional buckling ↔



a) b)

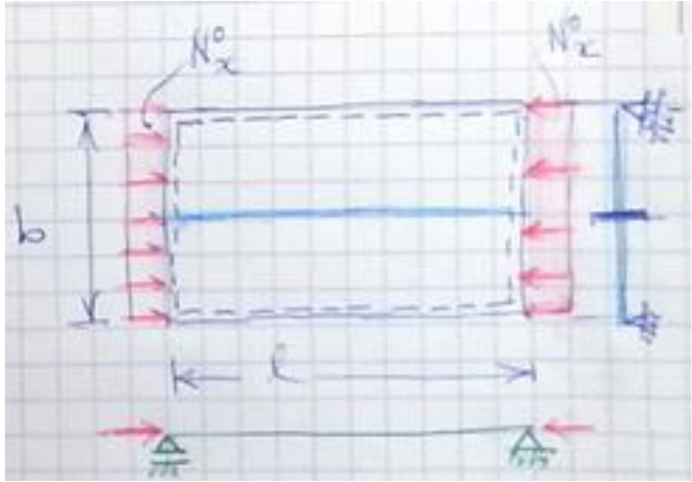
3. **Plate with a stiffener** ↔



Energy principles to estimate buckling load

Buckling of a stiffened plate

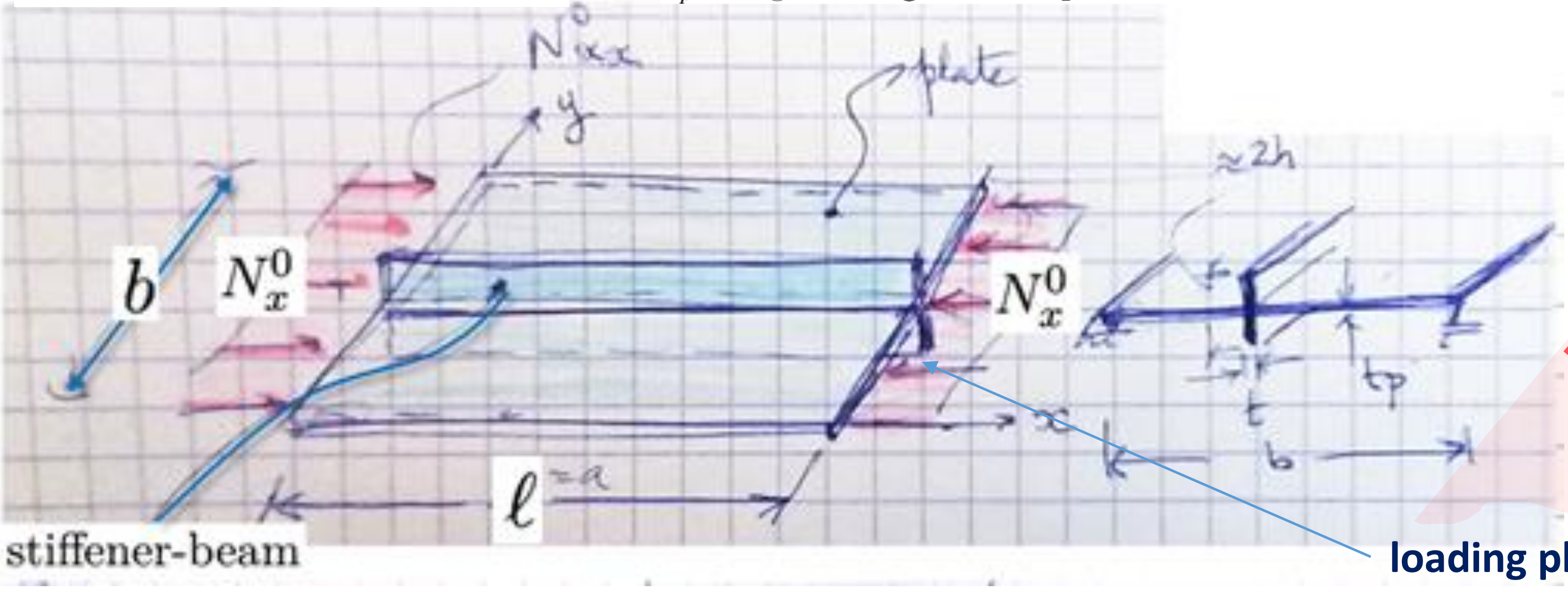
Consider a thin elastic plate (Fig. 1.202) of thickness t_p having length ℓ and width b. The effective bending rigidity of the plate being D. The external compressive load is applied along the lines $x = 0$ and $x = \ell$ with intensity $N_x^0 < 0$ (N/m). A stiffener[381] is designed along the loading direction. The bending rigidity of the stiffener-beam being EI_s. In this example, we assume that ratio b/ℓ is such that the first buckling mode of the plate is in form of a bubble. The stiffener has half-height h_s and thickness t_s. Notice that the relevant inertia moment of the stiffener is

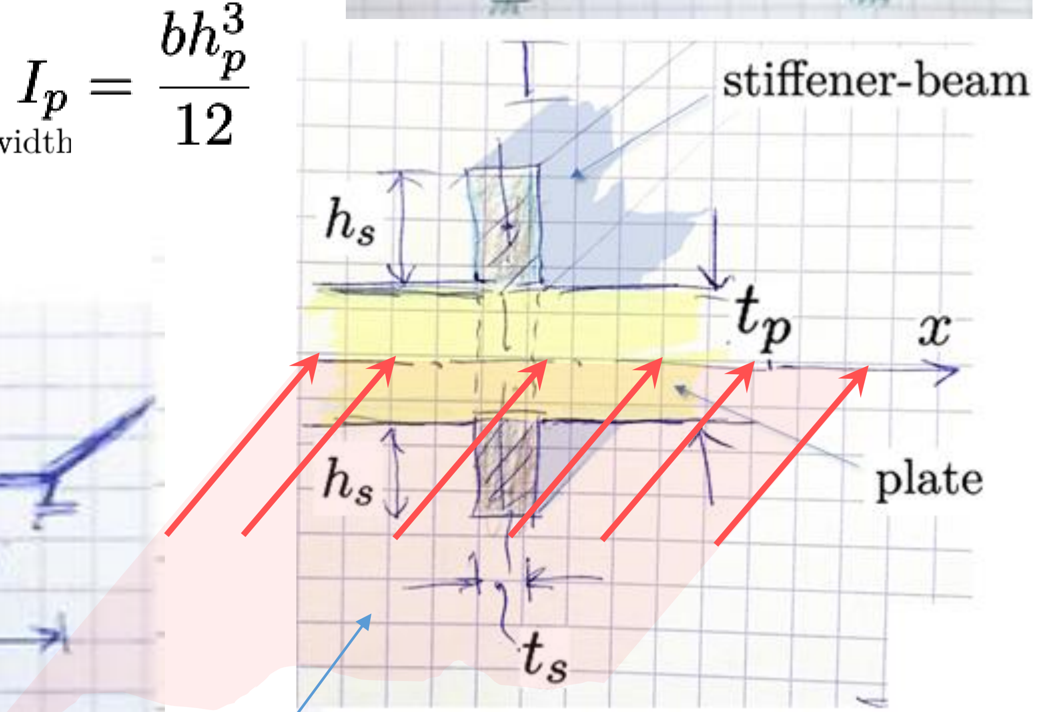


$$I_s = \frac{t_s \cdot (2h_s + t_p)^3}{12} - \frac{t_s t_p^3}{12}$$

effective inertia moment for the plate $I_p = \dfrac{b h_p^3}{12}$

h_p being the height of the plate and b its loaded width





loading plane

Energy principles to estimate buckling load

Trial bucling modes

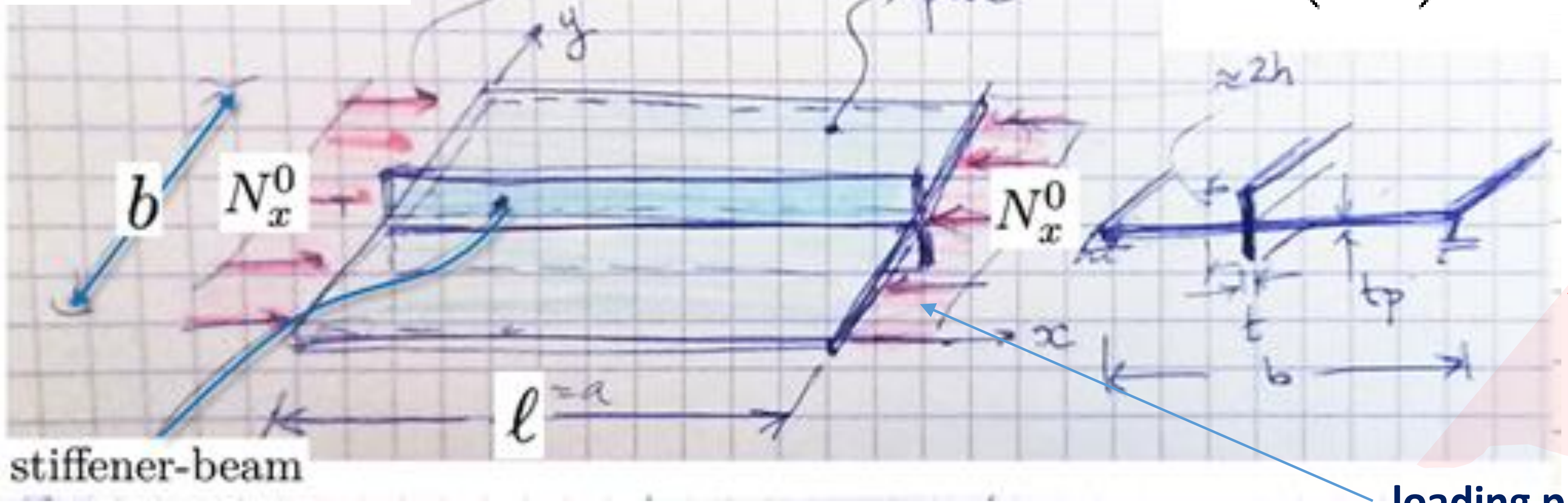
$$w(x,y) = w_0 \cdot \sin\left(\frac{\pi x}{\ell}\right) \cdot \sin\left(\frac{\pi y}{b}\right)$$

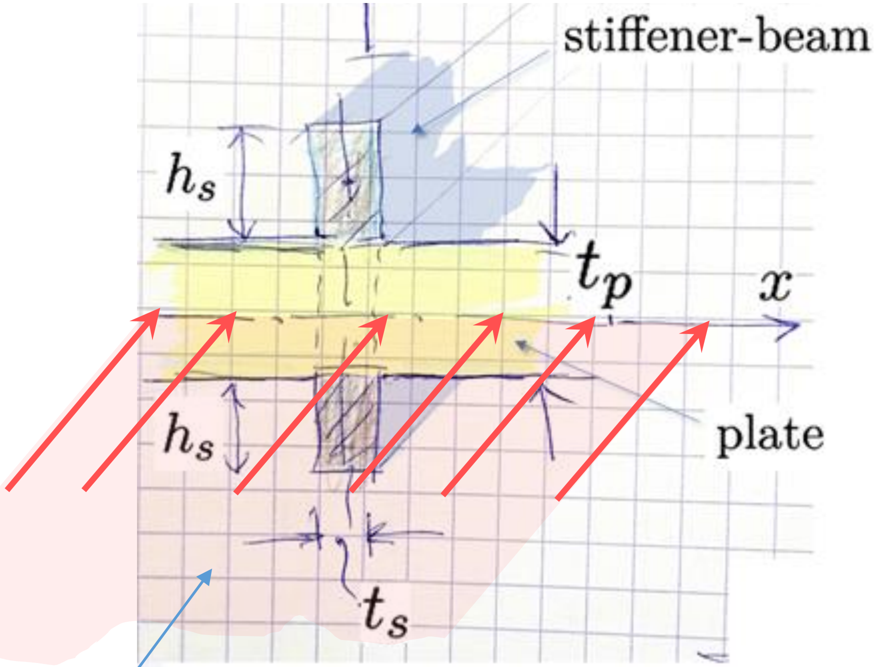
$$w(x,y) = w_0 \cdot x(\ell - x)/\ell^2 \cdot y(b-y)/b^2$$

$$\Delta\Pi = \frac{1}{2}\int_A D\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right]\mathrm{d}A + \frac{1}{2}\int_A \left[\underbrace{N_{xx}^0}_{\equiv -\bar{N}_{xx}^0 < 0} \cdot w_{,x}^2 + \underbrace{N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x}w_{,y}}_{\text{now, this term } = 0}\right]\mathrm{d}A + \frac{1}{2}\int_0^\ell EI_s w_{,xx}^2(x, y=b/2)\mathrm{d}x.$$

$$N_x^0 \equiv -\bar{N}_x^0 < 0$$

$$\delta(\Delta\Pi) = 0$$





loading plane

Energy principles to estimate buckling load

$$\Delta\Pi = \frac{1}{2}\int_A D\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right]\mathrm{d}A +$$

$$+\frac{1}{2}\int_A \left[\underbrace{N_{xx}^0}_{\equiv -\bar{N}_{xx}^0<0} \cdot w_{,x}^2 + \underbrace{N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x}w_{,y}}_{\text{now, this term } = 0}\right]\mathrm{d}A +$$

$$+\frac{1}{2}\int_0^\ell EI_s w_{,xx}^2(x, y=b/2)\mathrm{d}x.$$

Trigonometric trial:

$$w(x,y) = w_0 \cdot \sin\left(\frac{\pi x}{\ell}\right)\cdot \sin\left(\frac{\pi y}{b}\right)$$

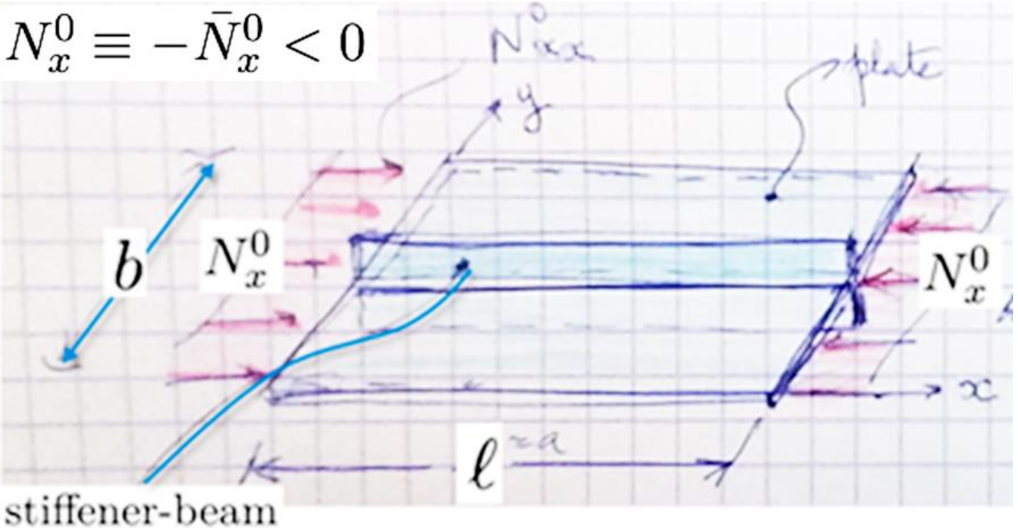
$$\delta(\Delta\Pi) = \frac{\partial(\Delta\Pi)}{\partial w_0}\cdot \delta w_0 = 0,\ \forall \delta w_0,$$

$$\Delta\Pi(w_0; N_x^0) = \frac{w_0^2\pi^2}{8\ell^3 b^3} = \frac{w_0^2\pi^2}{8\ell^3 b^3}\cdot\left[2EI\pi^2 b^3 + D\pi^2[\ell^2+b^2]^2 - \bar{N}_x^0\ell^2 b^4\right]$$

$$\Longrightarrow (\bar{N}_x^0)_{\mathrm{cr}} = \frac{2EI\pi^2}{\ell^2 b} + \frac{D\pi^2[\ell^2+b^2]^2}{\ell^2 b^4}$$

$$= \frac{2\pi^2 EI}{\ell^2 b} + \underbrace{\frac{D\pi^2}{b^2}\cdot \underbrace{\left(2 + \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2\right)}_{=4,\ \text{for}\ ,\ \ell=b}}_{=4\pi^2 D/b^2 = \text{ exact analytical, for } \ell=b},$$

$$(\bar{N}_x^0)_{\mathrm{cr}} = \frac{2\pi^2 EI}{\ell^2 b} + \frac{D\pi^2}{b^2}\cdot\left(\frac{\ell}{b}+\frac{b}{\ell}\right)^2$$



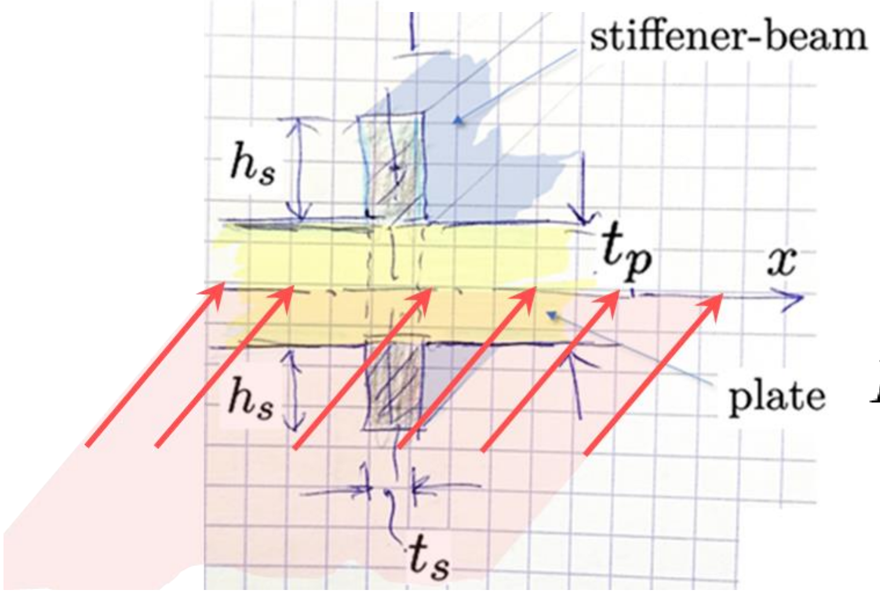
Trigonometric trial:

$$w(x,y) = w_0 \cdot \sin\left(\frac{\pi x}{\ell}\right) \cdot \sin\left(\frac{\pi y}{b}\right)$$

$$\implies (\bar{N}_x^0)_{cr} = \frac{2EI\pi^2}{\ell^2 b} + \frac{D\pi^2[\ell^2+b^2]^2}{\ell^2 b^4}$$

$$= \frac{2\pi^2 EI}{\ell^2 b} + \underbrace{\frac{D\pi^2}{b^2} \cdot \underbrace{\left(2 + \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2\right)}_{=4,\text{ for },\ \ell=b}}_{=4\pi^2 D/b^2 = \text{ exact analytical, for } \ell=b},$$

$$I_s = \frac{t_s \cdot (2h_s + t_p)^3}{12} - \frac{t_s t_p^3}{12}$$



$$I_p = \frac{bh_p^3}{12}$$

$$(\bar{N}_x^0)_{cr} = \frac{2\pi^2 EI_s}{\ell^2 b} + \frac{D\pi^2}{b^2} \cdot \underbrace{\left(2 + \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2\right)}_{\equiv k_c}$$

$$= k_c \cdot \frac{D\pi^2}{b^2} \cdot \left[1 + \frac{2\pi^2 EI_s}{\ell^2 b} \cdot \frac{b^2}{k_c D\pi^2}\right]$$

$$= k_c \cdot \frac{D\pi^2}{b^2} \cdot \left[1 + \frac{2\pi^2 EI_s}{\ell^2 b} \cdot \frac{b^2}{k_c \frac{Eh^3}{12(1-\nu)}\pi^2}\right]$$

$$= k_c \cdot \frac{D\pi^2}{b^2} \cdot \left[1 + \frac{2I_s}{\ell^2} \cdot \frac{b^2}{k_c \frac{bh^3}{12(1-\nu)}}\right]$$

$$= \underbrace{k_c \cdot \frac{D\pi^2}{b^2}}_{\text{plate alone}} \cdot \left[1 + \underbrace{\left[\frac{b}{\ell}\right]^2 \cdot \frac{2(1-\nu)}{k_c} \cdot \frac{I_s}{I_p}}_{\text{stiffner contribution}}\right]$$

Parabolic trial: $w(x,y) = w_0 \cdot x(\ell - x)/\ell^2 \cdot y(b-y)/b^2$

Less accurate ... but good

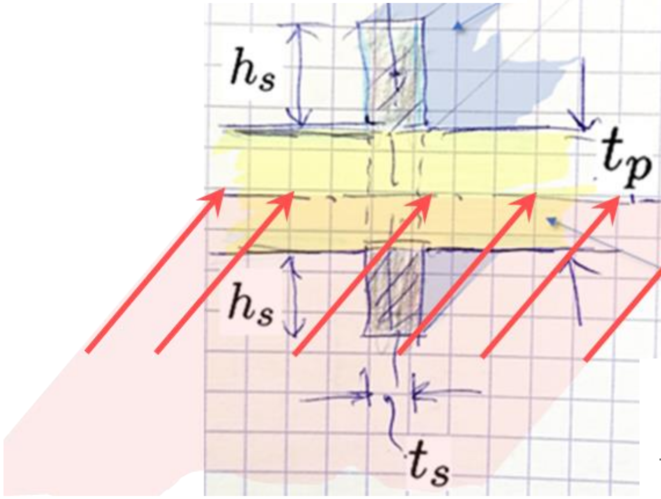
$$\Delta\Pi(w_0; N_x^0) = \frac{w_0^2}{360\ell^3 b^3} \cdot (24D\ell^4 + 2N_x^0 \ell^2 b^4 + 40D\ell^2 b^2 + 24Db^4 + 45EI_s b^3)$$

$$\forall \delta w_0, \quad \delta(\Delta\Pi) = \frac{\partial(\Delta\Pi)}{\partial w_0} \cdot \delta w_0 = 0$$

$$\ell^2 b^4 (\bar{N}_x^0)_{\mathrm{cr}} = 12D\ell^4 + 20D\ell^2 b^2 + 12Db^4 + 45/2 \cdot EI_s b^3 \implies$$

$$(\bar{N}_x^0)_{\mathrm{cr}} \approx 2.3\frac{\pi^2 EI}{\ell^2 b} + \frac{D\pi^2}{b^2} \cdot \underbrace{\left(2.0 + 1.2\left[\frac{\ell}{b}\right]^2 + 1.2\left[\frac{b}{\ell}\right]^2\right)}_{\approx k_c}$$

$$I_s = \frac{t_s \cdot (2h_s + t_p)^3}{12} - \frac{t_s t_p^3}{12}$$ stiffener-beam



$$I_p = \frac{bh_p^3}{12}$$

$$\implies (\bar{N}_x^0)_{\mathrm{cr}} = \frac{2EI\pi^2}{\ell^2 b} + \frac{D\pi^2[\ell^2 + b^2]^2}{\ell^2 b^4}$$

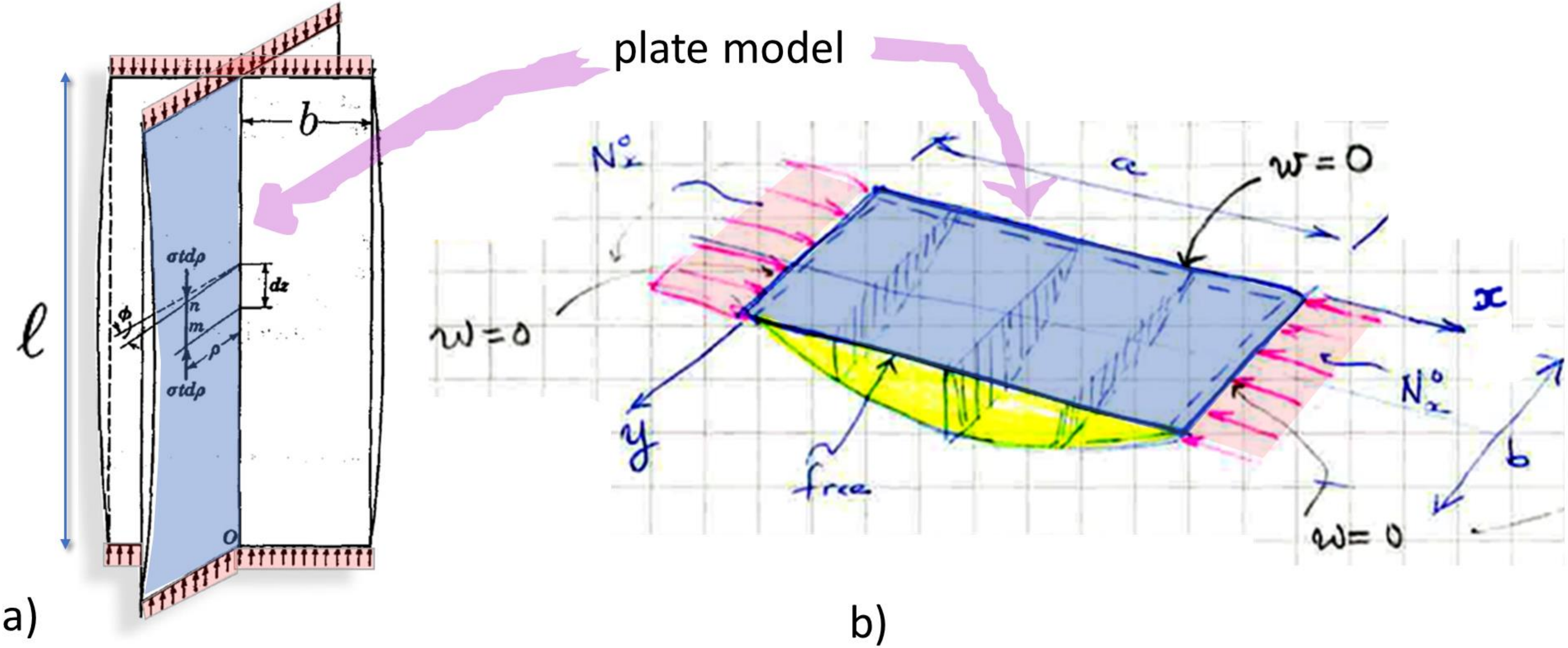
$$= \frac{2\pi^2 EI}{\ell^2 b} + \underbrace{\frac{D\pi^2}{b^2} \cdot \underbrace{\left(2 + \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2\right)}_{=4,\ \text{for}\ ,\ \ell=b}}_{=4\pi^2 D/b^2 \,=\, \text{exact analytical, for } \ell=b},$$

Trigonometric trial:

```
% Energy method to approximate buckling load
% for a stiffened thn plate with in-plane compressive Nox along one side
% the Poisson expansion is not restrained by the supports.
% -----------------------------------------------------
%    Author: Baroudi D. 2021
% ----------------------------------------------------
syms x y
syms delta_P   delta_W

syms w w0
syms L b  D  EI nu
syms N0x
% ----------------------------------------
% Displacement approximation  (you can use better approximations)
% -----------------------------------
%% w(x, y, w0, L)    =  w0 / ((L^2) * (b^2)) * x * ( L - x) * y * (b - y)  %
less good than the trigonometric
 w(x, y, w0, L)    =  w0 *  sin(pi* x /L) * sin(pi* y /b) % best one

d1x_w(x, y, w0, L, b)  = simplify( diff(w, x) )
d2xy_w(x, y, w0, L, b) = simplify( diff(d1x_w, y) )
d2x_w(x, y, w0, L, b)  = simplify( diff(d1x_w, x) )

d1y_w(x, y, w0, L, b)  = simplify( diff(w, y) )
d2yx_w(x, y, w0, L, b) = simplify( diff(d1y_w, x) )
d2y_w(x, y, w0, L, b)  = simplify( diff(d1y_w, y) )

% Strain energy (plate alone)
delta_U(L, b, D, w0) = 0.5* D * int( int(d2x_w * d2x_w, x, [0 L]) , y, [0 b]) +
...
                       0.5* D * int( int(d2y_w * d2y_w, x, [0 L]) , y, [0 b]) +
...
                       nu * D * int( int(d2x_w * d2y_w, x, [0 L]) , y, [0 b]) +
...
                  (1 - nu)* D * int( int(d2xy_w * d2xy_w, x, [0 L]) , y, [0 b])

% Strain energy (stiffner beam alone)
d2x_w_beam = d2x_w(x, b/2, w0, L, b)
%% delta_U_beam(L, b, D, w0) = 0.5* EI * int( int(d2x_w_beam * d2x_w_beam, x,
[0 L]) , y, [0 b])
delta_U_beam(L, b, D, w0) = 0.5* EI * int( d2x_w_beam * d2x_w_beam, x, [0 L]  )

% Work increment of initial stresses (applied N0x along the boundaries x=0
% and x=L
%--------------------------------
delta_W(L, b, w0, N0x) = 0.5 * int( int(N0x * d1x_w * d1x_w, x, [0 L]) , y,
[0 b]    )
%% Note that here Nox is negative
% ------------------------------------
% Total increment of potential energy
% -----------------------------------
delta_Pi = delta_U(L, b, D, w0) + delta_W(L, b, w0, N0x) + delta_U_beam(L, b,
D, w0)
delta_Pi = simplify( delta_Pi)
% Equations of neutral equilibrium
% ------------------------------------
delta_Pi_w   = simplify ( diff(delta_Pi, w0) )
% ---------------------------------------------
texti = 'Remember Nox is now negative : = - Nox_ref'
```

$$\Delta\Pi = \frac{1}{2}\int_A D\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right]\mathrm{d}A + \frac{1}{2}\int_A \left[\underbrace{N_{xx}^0}_{\equiv -\bar{N}_{xx}^0 < 0} \cdot w_{,x}^2 + \underbrace{N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x}w_{,y}}_{\text{now, this term } = 0}\right]\mathrm{d}A + \frac{1}{2}\int_0^{\ell} EI_s w_{,xx}^2(x, y=b/2)\mathrm{d}x.$$

2-D versus 1-D: Plate model *versus* beam-model for torsional buckling

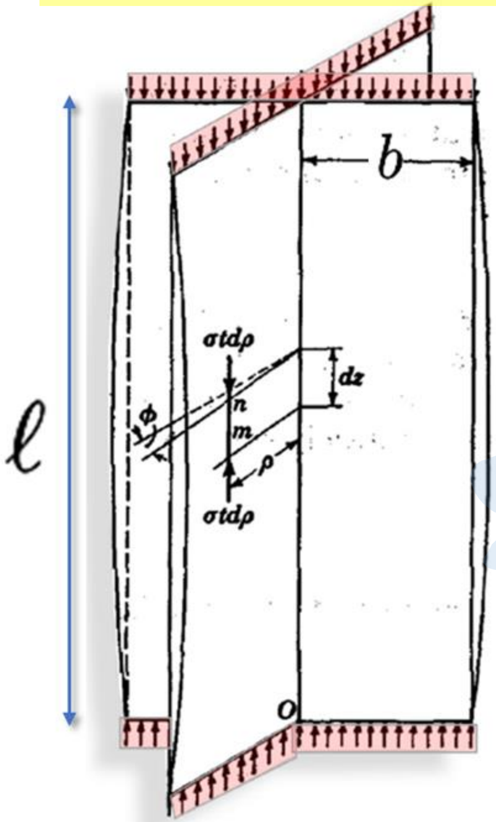


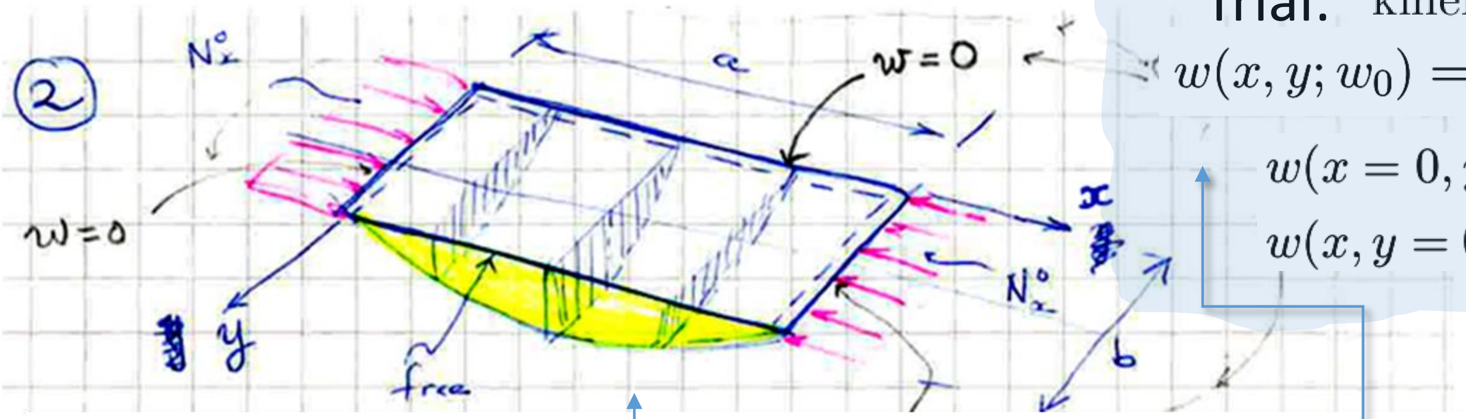
a) b)

Task: use sationary total potential energy principle and estimate buckling load

2-D versus 1-D: Plate model *versus* beam-model for torsional buckling 1(3)

Task: use sationary pot. energy principle and estimate buckling load





Trial: kinematicaly admissible

$$w(x, y; w_0) = w_0 \cdot \sin(\pi x/\ell) \cdot y^2/\ell^2$$

$$w(x=0, y) = w(x=\ell, y) = 0,$$
$$w(x, y=0) = 0.$$

This is why we keep the term y^2 in the trial . Keeping only y will give $w_{,yy} \equiv 0$ and this is not a good thing. These 'not good trials' correspond to the so called *zero energy modes* or *rigid body motion modes*.

$$\Delta\Pi = D \cdot \pi^2 b w_0^2 \cdot (960\ell^4/\pi^2 + 48b^4\pi^2 + 640\ell^2 b^2 + 48\ell^2 b^4 \cdot N_x^0/D - 960\ell^2 b^2 \nu)/960\ell^7$$

Energy-approximation: $(\bar{N}_x^0)_{\rm cr} \approx \dfrac{\pi^2 D}{b^2} \cdot \left(0.20\left[\dfrac{\ell}{b}\right]^2 + \left[\dfrac{b}{\ell}\right]^2 + 1.35 - 2.03\nu\right)$

Analytical: torsional buckling (1-D model)

$$\sigma_{\rm cr} = \frac{Gt^2}{b^2}$$ (Timoshenko)

buckling stress in pure torsional buckling for an angle cross-section

compare

This is a good analytical approximation:

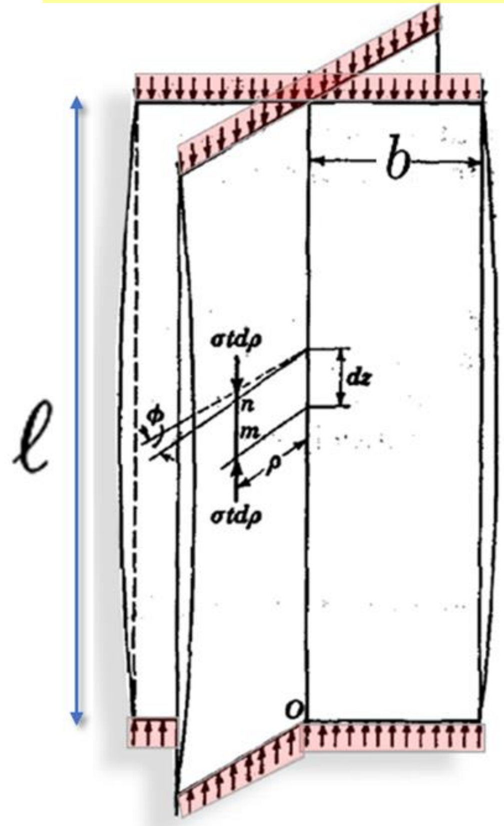
critical buckling stress (Timoshenko)

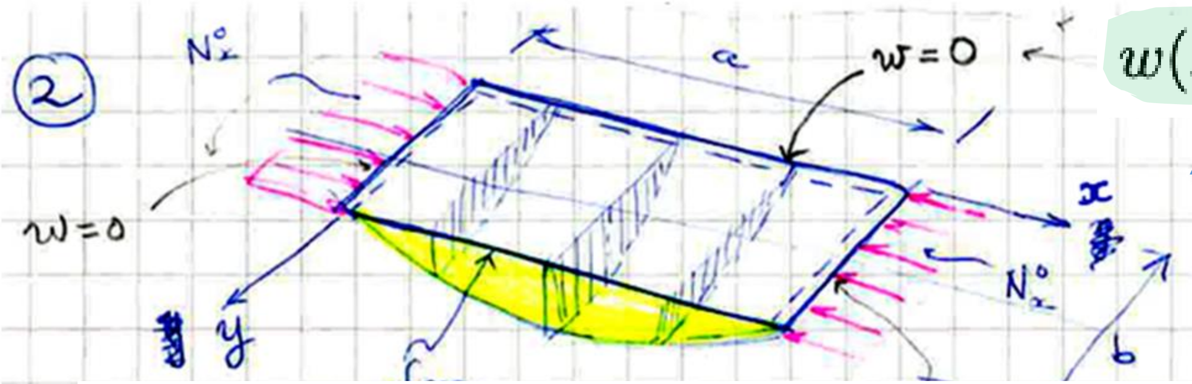
$$\sigma_{\rm cr} = \underbrace{\left(0.456 + b^2/\ell^2\right) \frac{\pi^2}{6(1-\nu)}}_{\text{Length effect}} \cdot \frac{Gt^2}{b^2}$$ Buckling stress for this plate

$$\Longrightarrow (\bar{N}_x^0)_{\rm cr} = \sigma_{\rm cr} \cdot t = \left(0.456 + b^2/\ell^2\right) \frac{\pi^2 D}{b^2}$$

2-D versus 1-D: Plate model *versus* beam-model for torsional buckling 2(3)

Task: use sationary pot. energy principle and estimate buckling load





Trial: kinematicaly admissible

$$w(x,y;w_0) = w_0 \cdot \sin(\pi x/\ell)\cdot y/\ell$$

$$w(x=0,y) = w(x=\ell,y) = 0,$$
$$w(x,y=0) = 0.$$

$$\Delta\Pi = w_0^2 \cdot \frac{b\pi^2}{12\ell^5}[6D\ell^2 + Db^2\pi^2 + \ell^2 N_0^x b^2 + 3EI_s b\pi^2 - 6D\ell^2\nu)]$$

$$6D\ell^2 + Db^2\pi^2 + \ell^2 N_x^0 b^2 + 3EI_s b\pi^2 - 6D\ell^2\nu = 0 \implies$$

$$\bar{N}_x^0 = \frac{\pi^2 D}{b^2}\left(\left[\frac{b}{\ell}\right]^2 + \frac{6(1-\nu)}{\pi^2} + 3\cdot\frac{EI_s}{Db}\left[\frac{b}{\ell}\right]^2\right).$$

Analytical: torsional buckling (1-D mod...)

$$\sigma_{\mathrm{cr}} = \frac{Gt^2}{b^2}$$ (Timoshenko)

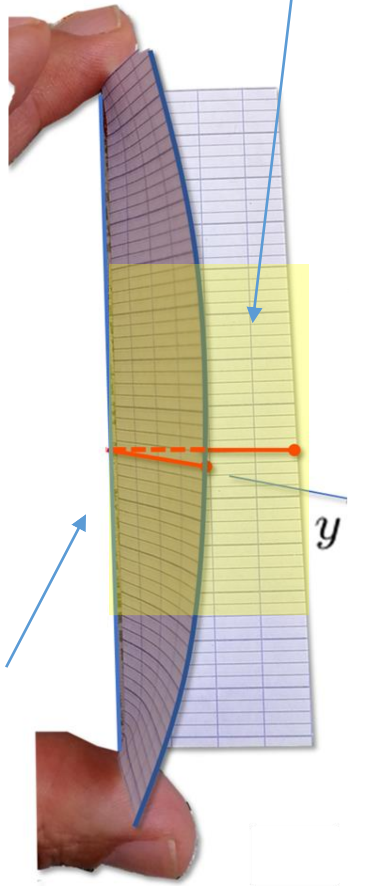
$$\bar{N}_x^0 = \frac{\pi^2 D}{b^2}\left(0.456 + \left[\frac{b}{\ell}\right]^2\right), \quad \text{Timoshenko}$$

$$\bar{N}_x^0 = \frac{\pi^2 D}{b^2}\left(\underbrace{6(1-\underbrace{\nu}_{=0.25})/\pi^2}_{\approx 0.45597} + \left[\frac{b}{\ell}\right]^2\right), \quad \text{our result}$$

This a better result than with the quadratic approximation in *y*.

Note that in this region, the cross-section **rotates** almost **as a rigid body**

$$w_{,yy} \approx 0$$

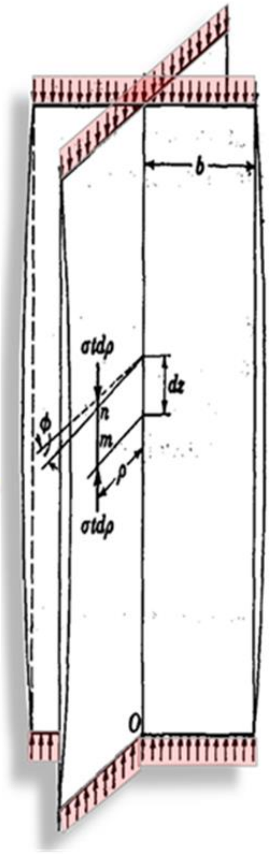


2-D versus 1-D: Plate model *versus* beam-model for torsional buckling 3(3)

Task: use sationary pot. energy principle and estimate buckling load

Trial: kinematicaly admissible

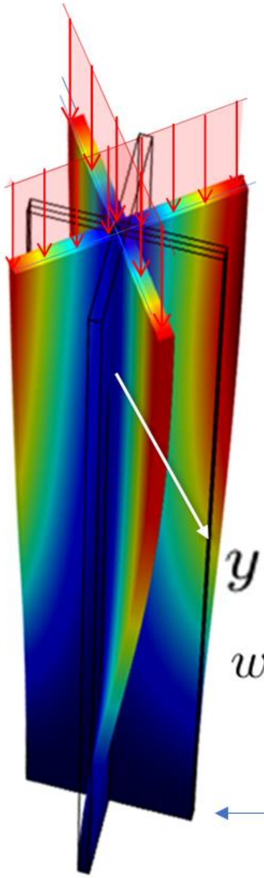
$$w(x, y; w_0) = w_0 \cdot \sin(\pi x/\ell) \cdot y/\ell$$



y

$w_{,yy} \approx 0$

Centric load with doubly symmetric X-section



3D-FEM model

Note that in this region, the cross-section **rotates** almost **as a rigid body**

so no distortion = no bending of the plate in the normal direction toward the free edge

y

$w_{,yy} \approx 0$

this support is clamped in the simulation

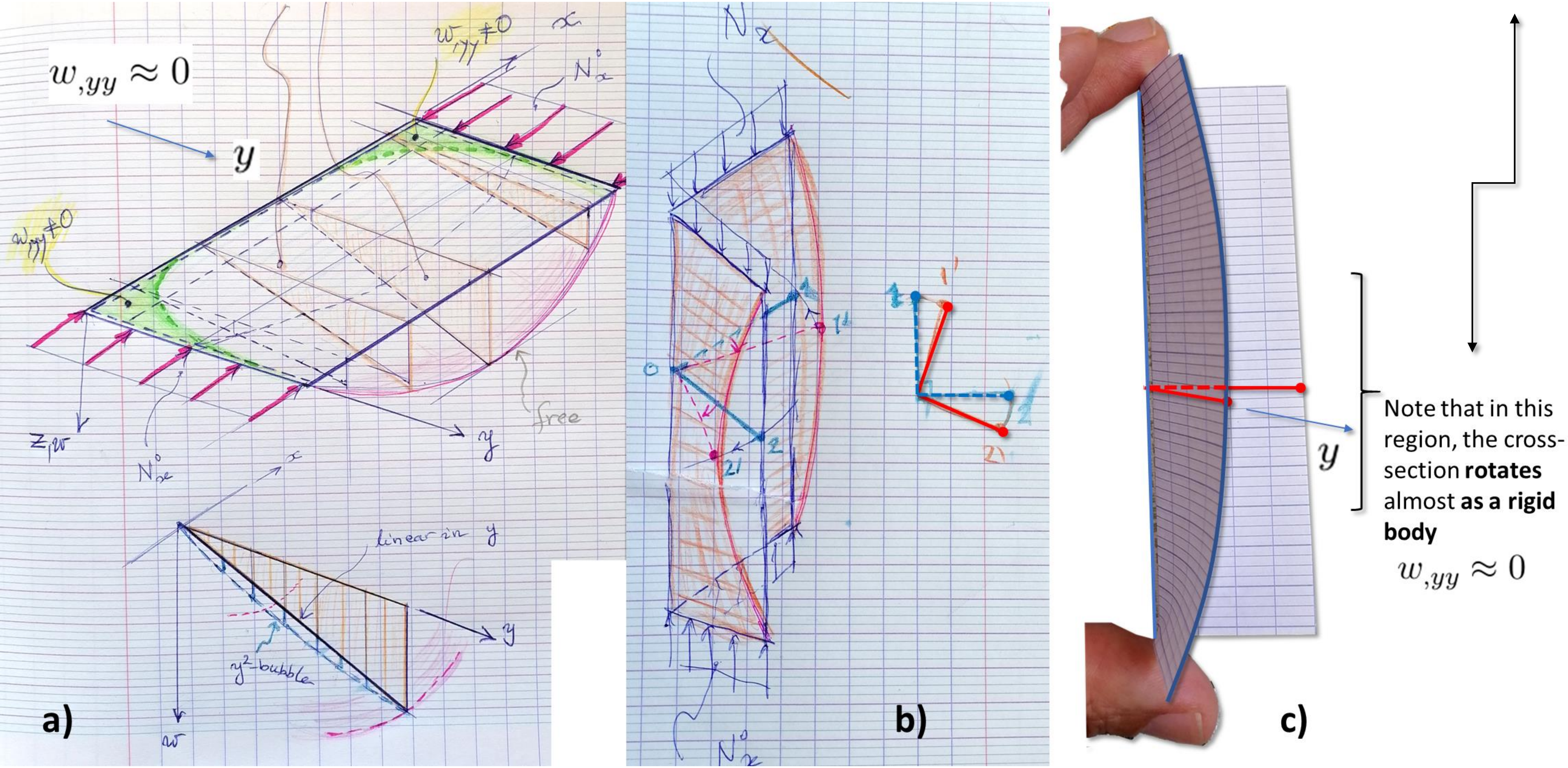
This a better result than with the quadratic approximation in *y*. There is a reason.

$$\bar{N}_x^0 = \frac{\pi^2 D}{b^2}\left(0.456 + \left[\frac{b}{\ell}\right]^2\right), \quad \text{Timoshenko}$$

$$\bar{N}_x^0 = \frac{\pi^2 D}{b^2}\left(\underbrace{6(1 - \underbrace{\nu}_{=0.25})/\pi^2}_{\approx 0.45597} + \left[\frac{b}{\ell}\right]^2\right), \quad \text{our result.}$$

Reation between plate bucklng and pure torsional buckling of thin plates

$$w(x, y; w_0) = w_0 \cdot \sin(\pi x/\ell) \cdot y/\ell$$



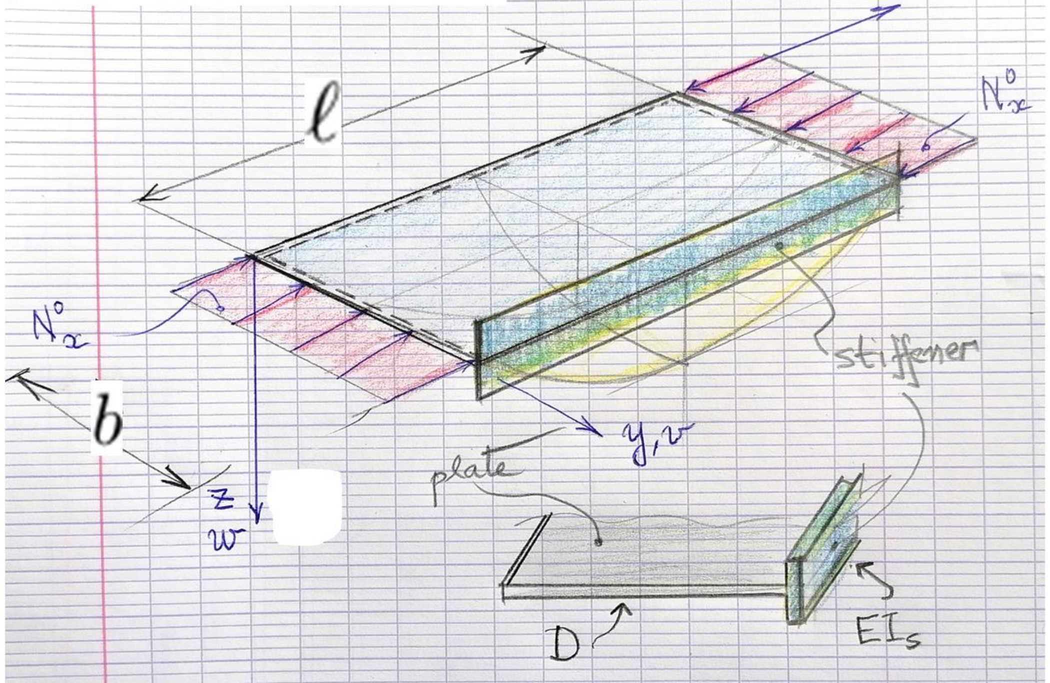
Why we obtained a very good approximation?

Energy method to approximate buckling load of plates

Task: Figure shows a stiffened thin plate under edge compressive loading. Estimate the buckling load of such stiffened plate. What is the benefit given by such stiffener? Express that by a *formula* to find out what are the key design parameters.

Such problem can be treated analytically. It is tractable by solving the relevant differential equations for both the thin plate and the beam, and by imposing continuity and equilibrium at the interface plate-beam. This approach is very demanding in work hours. An example of a Longitudinally Stiffened Plate is treated by this approach in our course textbook by CHAI H. YOO and SUNG C. LEE. Our approach will be different. We will use the energy method to solve approximately but correctly this problem in few minutes. Bellow follow the examples.

Plate with a stiffener



$$\Delta\Pi = \frac{1}{2} D \int_A \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2 \right] \mathrm{d}A + $$
$$+ \frac{1}{2} \int_A \left[N_{xx}^0 w_{,x}^2 + N_{yy}^0 w_{,y}^2 + 2 N_{xy}^0 w_{,x} w_{,y} \right] \mathrm{d}A + $$
$$+ \frac{1}{2} \int_0^\ell E I_s {w_{xx}}^2 (x, y=b) \mathrm{d}x.$$

$$w(x,y) = \sum_{n=0}^{\infty} f(y) \sin(n\pi x/\ell)$$

$\Downarrow$

Trial: kinematicaly admissible

$$w(x,y;w_0) = w_0 \cdot \sin(\pi x/\ell) \cdot y^2/\ell^2$$

$$w(x=0,y) = w(x=\ell,y) = 0,$$
$$w(x,y=0) = 0.$$

$\Downarrow$

$$\Delta\Pi = \frac{b w_0^2 D \pi^2}{60\ell^7} \left(\frac{60\ell^4}{\pi^2} + 3b^4\pi^2 + \frac{15EI_s}{Db} b^4 \pi^2 + 40\ell^2 b^2 + 3\ell^2 b^2 \frac{N_x^0 b^2}{D} - 60\nu \ell^2 b^2 \right)$$

$\Downarrow$

$$\delta(\Delta\Pi) = 0 \implies \frac{\partial(\Delta\Pi)}{\partial w_0} \cdot \delta w_0 = 0, \ \forall \delta w_0$$

$\Downarrow$

$$(\bar{N}_x^0)_{\mathrm{cr}} = \frac{\pi^2 D}{b^2} \cdot \left(\frac{20}{\pi^2} \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2 + \frac{13.33 - 20\nu}{\pi^2} + \frac{5EI_s}{\underbrace{Db}_{=EI_\mathrm{p}}} \left[\frac{b}{\ell}\right]^2 \right)$$

$$= \frac{\pi^2 D}{b^2} \cdot \left(\frac{20}{\pi^2} \left[\frac{\ell}{b}\right]^2 + \left[\frac{b}{\ell}\right]^2 + \frac{13.33 - 20\nu}{\pi^2} + \underbrace{5 \cdot \frac{I_s}{I_\mathrm{p}} \left[\frac{b}{\ell}\right]^2}_{\text{stiffener effect}} \right)$$

The Matlab code that produced the previous result

```
% Energy method to approximate buckling load
% for a stiffened thn plate with at y=b (free end) in-plane
compressive Nox along one side
% the Poisson expansion is not restrained by the supports.
% ---------------------------------------------------------
%    Author: Baroudi D. 2021
% ---------------------------------------------------------
clear all

syms x y
syms delta_P   delta_W

syms w w0
syms L b  D  EI nu
syms N0x
syms n m

% ---------------------------------------
% Displacement approximation  (you can use better
approximations)
% ----------------------------------
%% w(x, y, w0, L)    =  w0 / ((L^2) * (b^2)) * x * ( L - x) *
y * (b - y)  % less good than the trigonometric
%% w(x, y, w0, L)    =  w0 *  sin(pi* x /L) * sin(pi* y /b) %
best one
 w(x, y, w0, L)    =  w0 *  sin(pi* x /L) * (y*y) / (L*L)

d1x_w(x, y, w0, L, b)  = simplify( diff(w, x) )
d2xy_w(x, y, w0, L, b) = simplify( diff(d1x_w, y) )
d2x_w(x, y, w0, L, b)  = simplify( diff(d1x_w, x) )

d1y_w(x, y, w0, L, b)  = simplify( diff(w, y) )
d2yx_w(x, y, w0, L, b) = simplify( diff(d1y_w, x) )
d2y_w(x, y, w0, L, b)  = simplify( diff(d1y_w, y) )
```

$$\Delta\Pi = \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right] dA +$$
$$+\frac{1}{2}\int_A \left[N_{xx}^0 w_{,x}^2 + N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x}w_{,y}\right] dA +$$
$$+\frac{1}{2}\int_0^\ell EI_s {w_{xx}}^2(x, y=b)\mathrm{d}x.$$

```
% Strain energy (plate alone)
delta_U(L, b, D, w0) = 0.5* D * int( int(d2x_w * d2x_w, x, [0
L]) , y, [0 b]) + ...
                       0.5* D * int( int(d2y_w * d2y_w, x, [0
L]) , y, [0 b]) + ...
                       nu * D * int( int(d2x_w * d2y_w, x, [0
L]) , y, [0 b]) + ...
                  (1 - nu)* D * int( int(d2xy_w * d2xy_w, x, [0
L]) , y, [0 b])

% Strain energy (stiffner beam alone)
ystiff = b;
d2x_w_beam = d2x_w(x, ystiff, w0, L, b)
delta_U_beam(L, b, D, w0) = 0.5* EI * int( d2x_w_beam *
d2x_w_beam, x, [0 L]  )

% Work increment of initial stresses (applied N0x along the
boundaries x=0
% and x=L
%---------------------------------
delta_W(L, b, w0, N0x) = 0.5 * int( int(N0x * d1x_w * d1x_w, x,
[0 L]) , y, [0 b]   )
%% Note that here Nox is negative
% ----------------------------------
% Total increment of potential energy
% ----------------------------------
delta_Pi = delta_U(L, b, D, w0) + delta_W(L, b, w0, N0x) +
delta_U_beam(L, b, D, w0);
delta_Pi = simplify( delta_Pi)
% Equations of neutral equilibrium
% ----------------------------------
delta_Pi_w   = simplify ( diff(delta_Pi, w0) )
% ----------------------------------------
texti = 'Remember Nox is now negative : = - Nox_ref'
```

Some classical analytical solutions of the partial differential equations of buckling

Some classical cases

Buckling of a simply supported rectangular plate - on-side compression

$$D\,\underbrace{[w_{,xxxx}+2w_{,xxyy}+w_{,yyyy}]}_{\Delta\Delta w}-\underbrace{[N^0_{xx}w_{,xx}+N^0_{yy}w_{,yy}+2N^0_{xy}w_{,xy}]}_{N^0_{\alpha\beta}w_{,\alpha\beta}}=0$$

simplifies $\Downarrow$

The buckling of the plate is described by the Eigen-value problem

$$w_{,xxxx}+2w_{,xxyy}+w_{,yyyy}=\frac{N^0_{xx}}{D}w_{,xx}$$

$$w(0,y)=w(a,y)=0,\quad w(x,0)=w(x,b)=0$$

$$M_x(0,y)=M_x(a,y)=0,\quad M_y(x,0)=M_y(x,b)=0$$

trial solution

It is known to you from your previous course on *thin Plates*

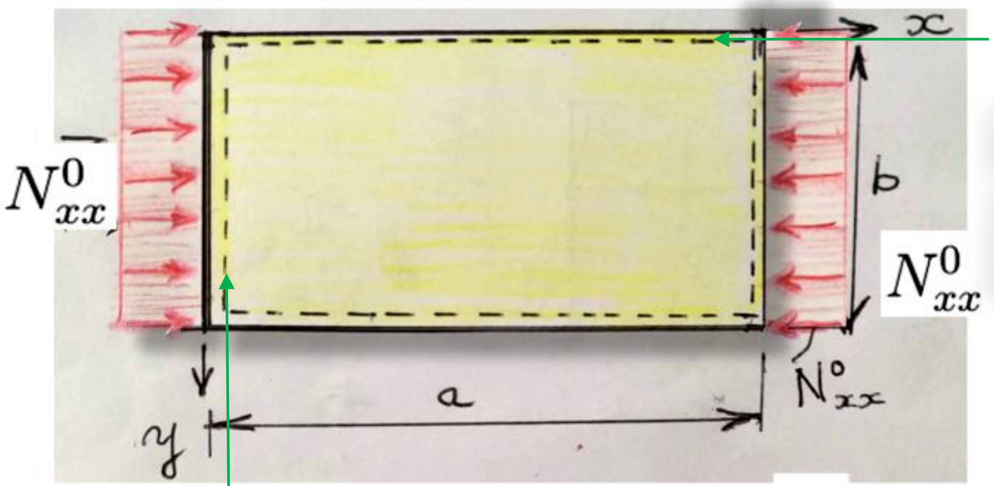
$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\alpha_m x\sin\beta_n y$$

We look for a solution in this form

$$\alpha_m=m\pi/a$$

$$\beta_n=n\pi/b$$

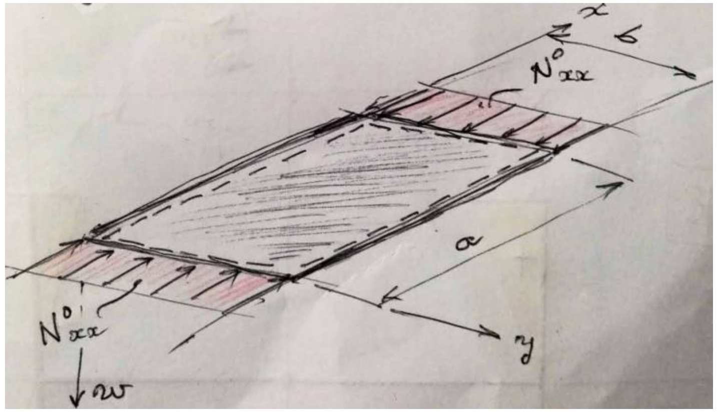
compressive load $-N$ $(N>0)$ [N/m]



Can move

Can move

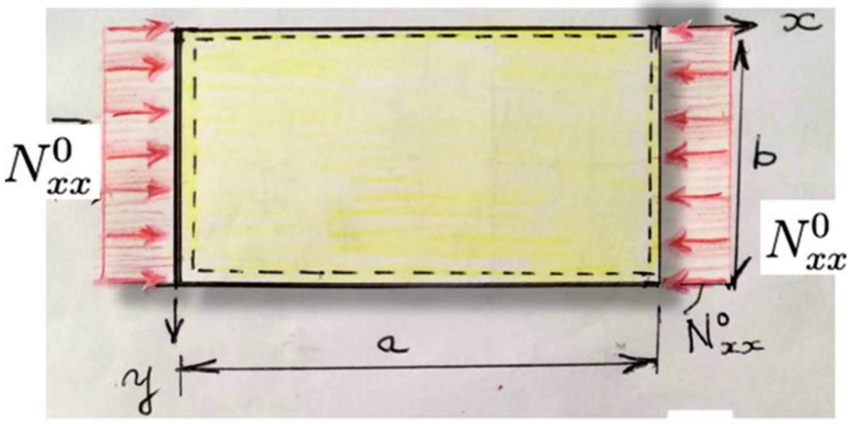
thrust $N_{xx}=-N<0$



Some classical cases

Buckling of a simply supported rectangular plate - on-side compression

compressive load $-N$ $(N > 0)$ [N/m]



thrust $N_{xx} = -N < 0$.

$$w_{,xxxx} + 2w_{,xx}w_{,yy} + w_{,yyyy} = \frac{N^0_{xx}}{D} w_{,xx}$$

$$w(0,y) = w(a,y) = 0, \quad w(x,0) = w(x,b) = 0$$

$$M_x(0,y) = M_x(a,y) = 0, \quad M_y(x,0) = M_y(x,b) = 0$$

It is known to you from your previous course on *thin Plates*

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\alpha_m x \sin\beta_n y \qquad \begin{cases} \alpha_m = m\pi/a \\ \beta_n = n\pi/b \end{cases}$$

$$\Downarrow$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \underbrace{\left[\pi^4\left(\frac{m^4}{a^4} + 2\frac{m^2}{a^2}\frac{n^2}{b^2} + \frac{n^4}{b^4}\right) + \frac{N^0_{xx}}{D}\pi^2\frac{m^2}{a^2}\right]}_{=0} \sin\alpha_m x \sin\beta_n y = 0$$

non-trivial $a_{mn} \neq 0, \forall m, n$

N.B. Stability loss Criteria: here it is asked from the trial to fulfill the eigenvalue problem

$$\Downarrow$$

Buckling occurs when:

$$N^0_{xx} = -D\pi^2 \frac{a^2}{m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$$

$n = 1$ gives the smallest critical (the **buckling**) stress-resultant

$$\Longrightarrow$$

Buckling stress-resultant [N/m]

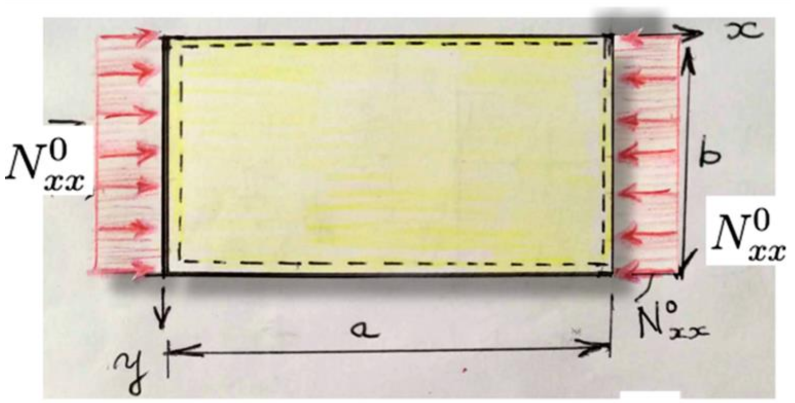
$$N = D\frac{\pi^2}{b^2}\left(\frac{m}{\eta} + \frac{\eta}{m}\right)^2 \equiv k_c D \frac{\pi^2}{b^2}$$

ratio of the sides, $\eta = a/b$ (for $n = 1$)

Some classical cases

Buckling of a simply supported rectangular plate - on-side compression

compressive load $-N$ $(N > 0)$ [N/m]



thrust $N_{xx} = -N < 0.$

$\Downarrow$ Buckling occurs when:

$$N^0_{xx} = -D\pi^2 \frac{a^2}{m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$$

n = 1 gives the smallest critical (the **buckling**) stress-resultant

Buckling stress-resultant [N/m]

$$N = D\frac{\pi^2}{b^2}\left(\frac{m}{\eta} + \frac{\eta}{m}\right)^2 \equiv k_c D\frac{\pi^2}{b^2}$$

Buckling coefficient

ratio of the sides, $\eta = a/b$ (for $n = 1$)

Stationary condition for a minimum $\mathrm{d}N/\mathrm{d}\eta = 0$

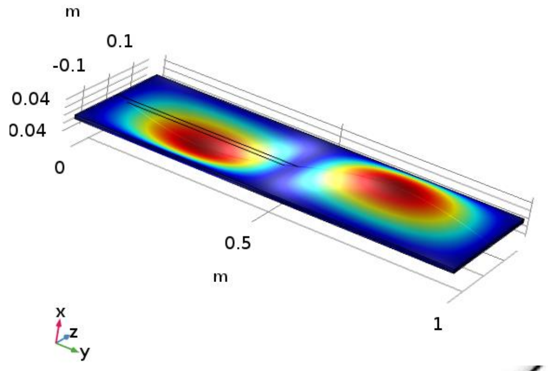
$\Longrightarrow \quad \eta = m$ take the closest integer *m*

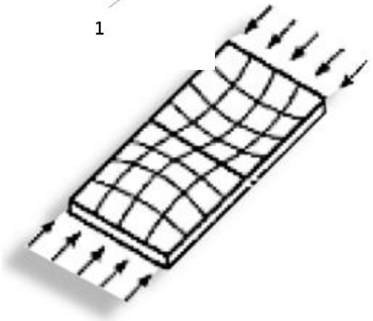
$\Downarrow$

$$|N_{xx,cr}| = 4\frac{\pi^2 D}{b^2}$$

Buckling stress-resultant [N/m]

$$k_c = \left(\frac{m}{\eta} + \frac{\eta}{m}\right)^2$$



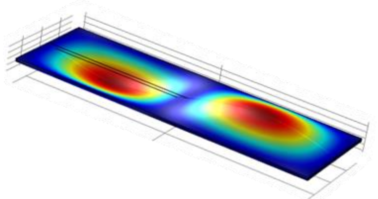


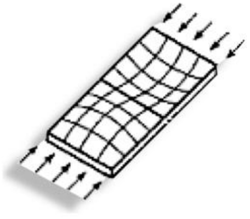
Buckling mode, $m = 2$, $n = 1$.

Let's make a graph to see how many buckles (half-waves) we have depending on *a*/*b*

Buckling of a simply supported rectangular plate - on-side compression

Buckling coefficient





Buckling mode,

Buckling stress-resultant [N/m]

$$N = D\frac{\pi^2}{b^2}\left(\frac{m}{\eta}+\frac{\eta}{m}\right)^2 \equiv k_c D\frac{\pi^2}{b^2}$$

ratio of the sides, $\eta = a/b$

(for $n = 1$)

Buckling coefficient

$$k = \frac{\bar{N}_x b^2}{\pi^2 D}$$

Buckling stress-resultant [N/m]

$$|N_{xx,cr}| = 4\frac{\pi^2 D}{b^2}$$

Buckling stress-resultant [N/m]

(for $m = \eta$)

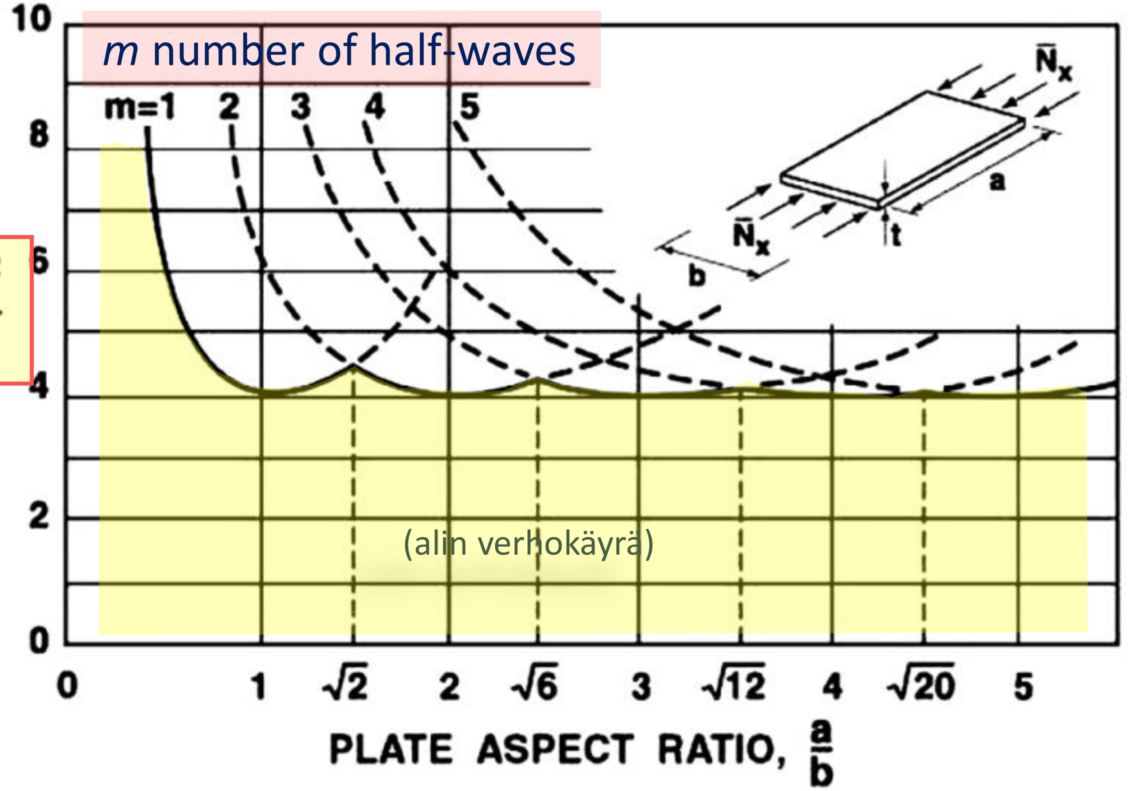
$$k_c = \left(\frac{m}{\eta}+\frac{\eta}{m}\right)^2$$

At the limit kc = 4

Let's make a graph to see how many buckles (m half-waves) we have depending on a/b

$$\sigma_{cr} = K \cdot \underbrace{\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{\ell}\right]^2}_{\text{reference buckling stress}}$$

m number of half-waves



Normalized buckling load k for simply supported rectangular plates with Various Plate Aspect Ratios

(Ref. *Robert M. Jones. Buckling of bars, plates and shells. Bull Ridge Publishing, 2006.*)

Buckling of a simply supported rectangular plate - on-side compression

Buckling coefficient

Lommahdus kerroin

Buckling stress-resultant [N/m]

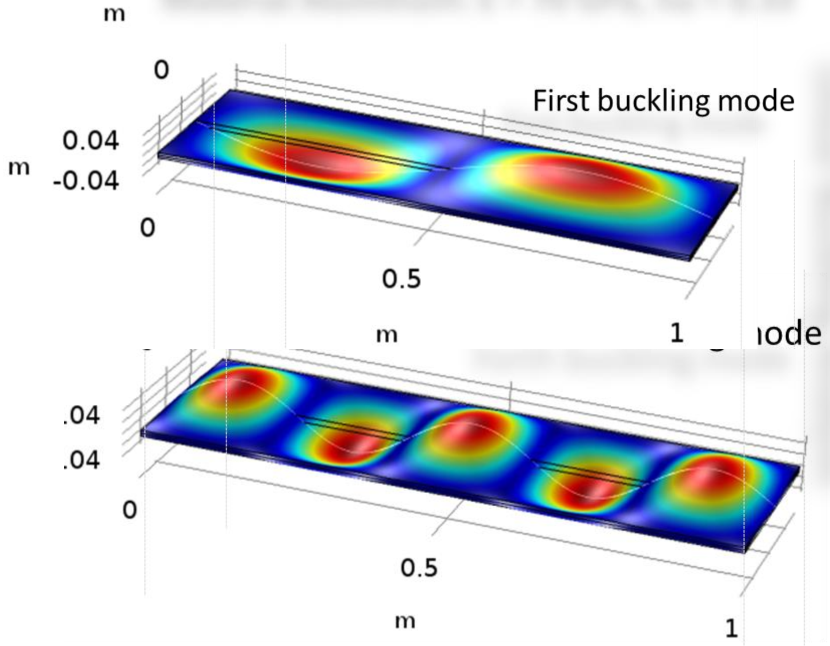
$$N = D\frac{\pi^2}{b^2}\left(\frac{m}{\eta} + \frac{\eta}{m}\right)^2 \equiv k_c D\frac{\pi^2}{b^2}$$

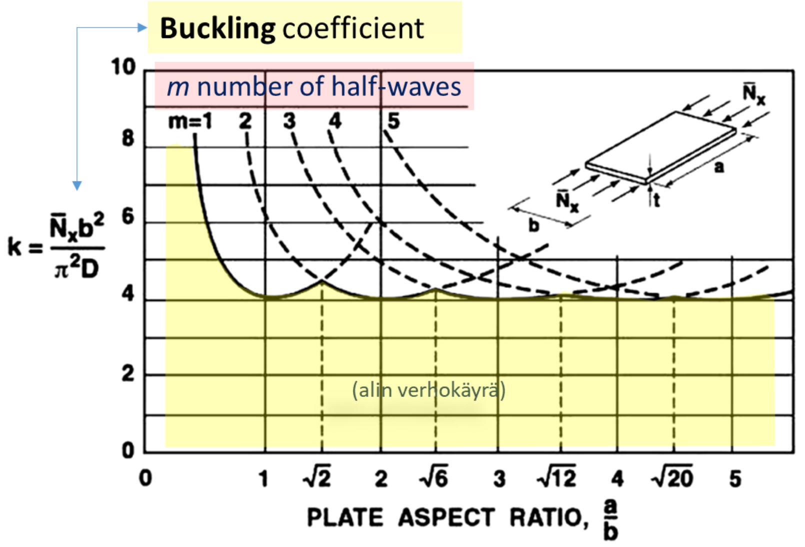
ratio of the sides, $\eta = a/b$

Can be written in this canonical form

$$\sigma_{cr} = K \cdot \underbrace{\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{\ell}\right]^2}_{\text{reference buckling stress}}$$

Material Aluminum: E = 70 GPa, nu = 0.33





Normalized buckling load *k* for simply supported rectangular plates with Various Plate Aspect Ratios

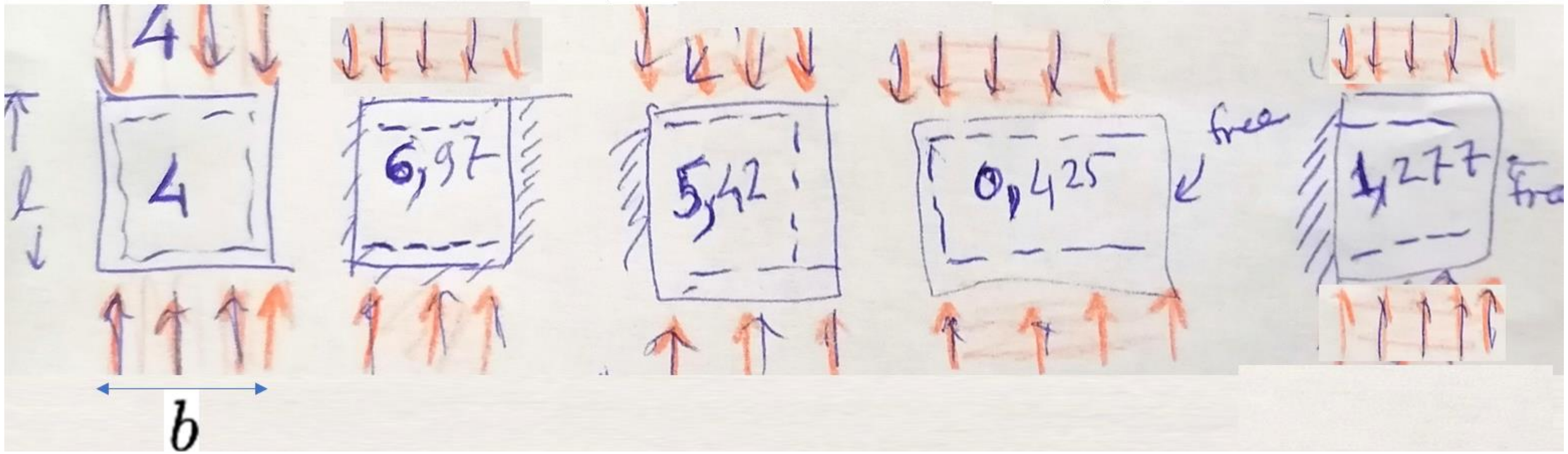
(Ref. *Robert M. Jones. Buckling of bars, plates and shells. Bull Ridge Publishing, 2006.*)

Linear Buckling Analysis

Buckling coefficients forsome boundary conditions and axial edge load

$$\sigma_{\mathrm{cr}} = k_\sigma \cdot \frac{\pi^2 D}{b^2 h}$$

$$k_\sigma \;=$$



Buckling of a simply supported rectangular plate with constrained compression

$$k_c = \frac{\left[\left(\frac{mb}{a}\right)^2 + n^2\right]^2}{\left(\frac{mb}{a}\right)^2 + \nu n^2}$$

$a/b = 2,$

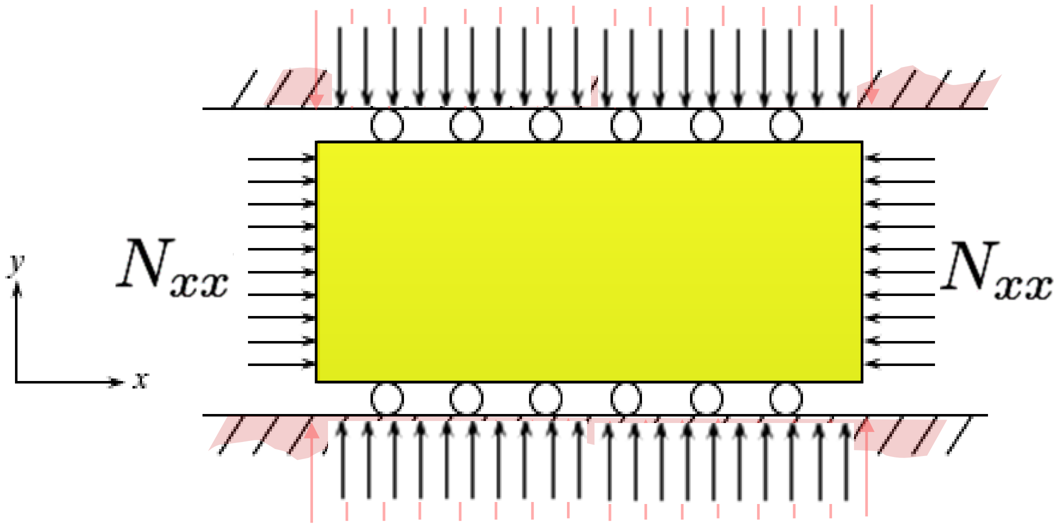
n \ m	1	2	3
1	10.7	3	4.09
2	3.8	10.7	10.9
3	26	25	24.1

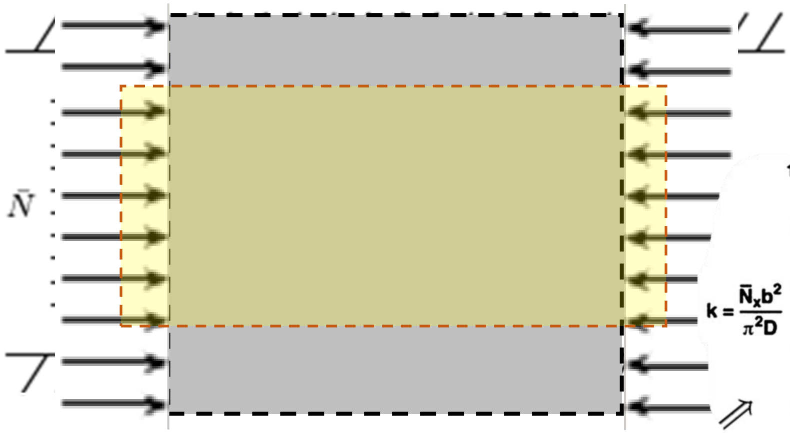
Buckling stress reduces to 3/4

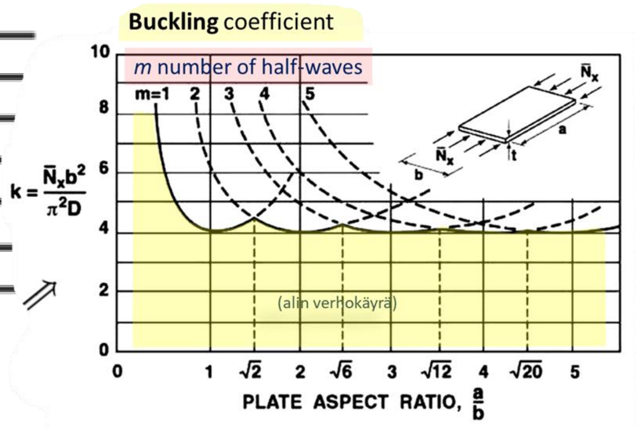
No restraints:

$$|N_{xx,cr}| = 4\frac{\pi^2 D}{b^2}$$

Reaction stress resultant: $N_{yy} = \nu N_{xx}$





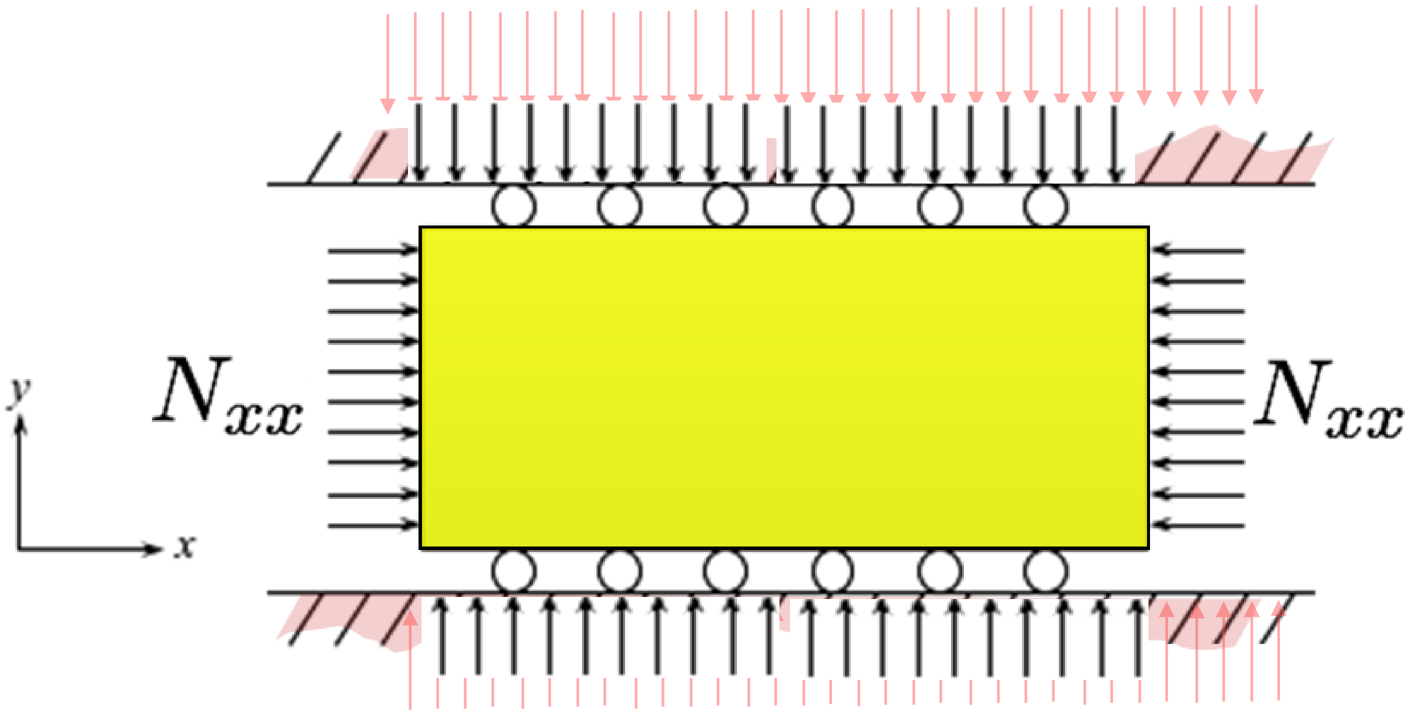


Normalized buckling load *k* for simply supported rectangular plates with Various Plate Aspect Ratios

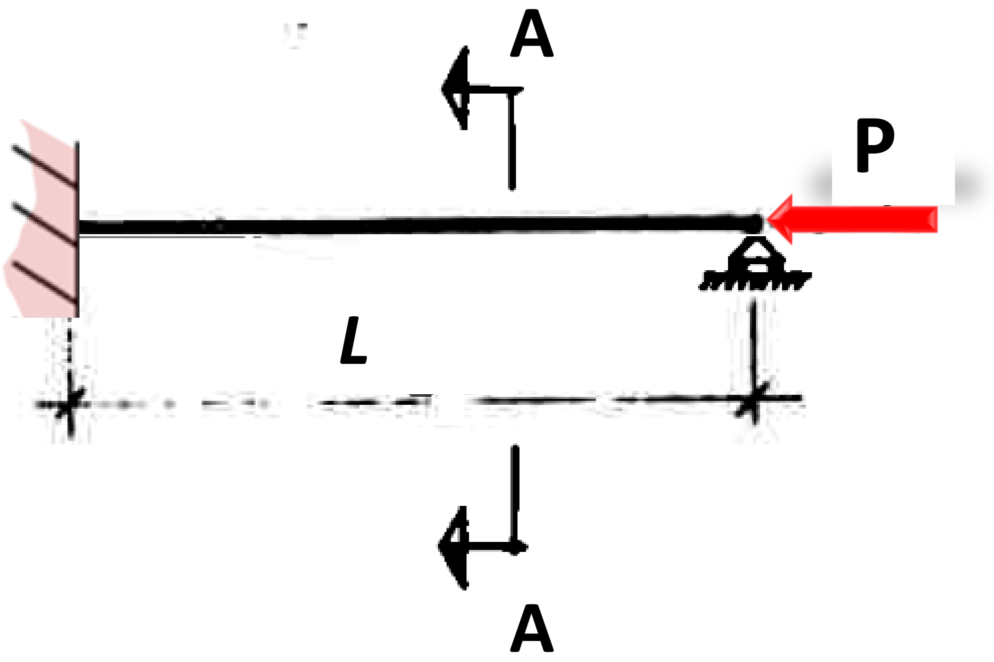
(Ref. Robert M. Jones. Buckling of bars, plates and shells. Bull Ridge Publishing, 2006.)

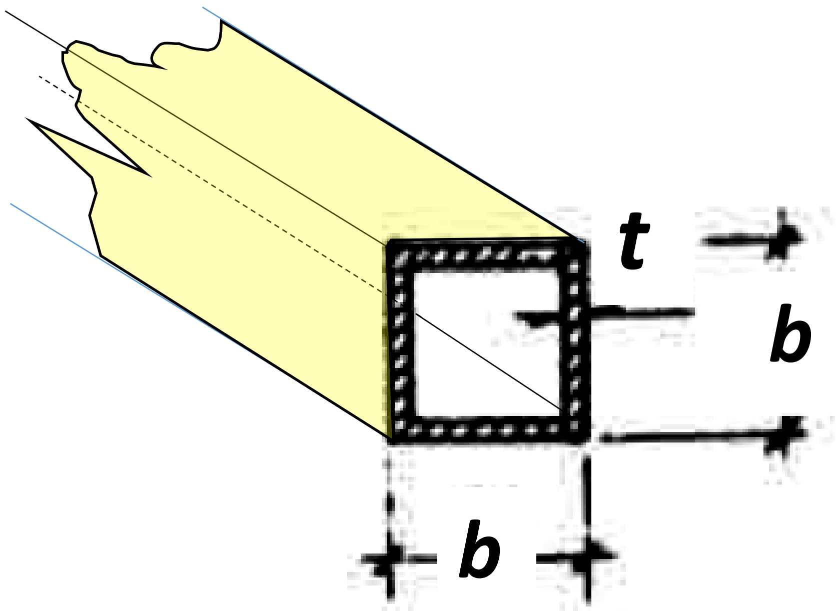
$$|N_{xx,cr}| = 4\frac{\pi^2 D}{b^2}$$

Reaction stress resultant: $N_{yy} = \nu N_{xx}$



$$N_{yy} = \nu N_{xx}$$





A - A

Example from exam 11.4.2019

Solution

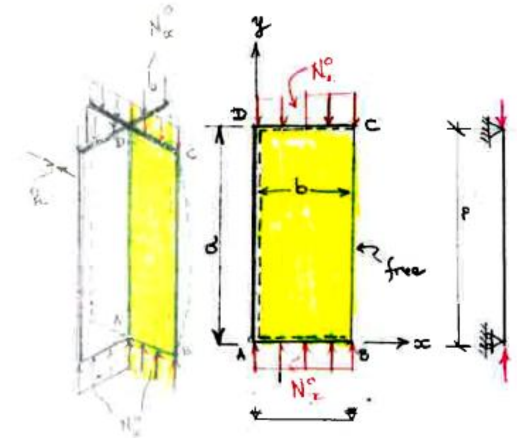
Energy methods

Exercise 2: Buckling of plates [5 points]

Consider only the buckling of the simple plate model in left figure 2b). The plate is loaded by a constant in-plane distributed edge-load N_x^0.

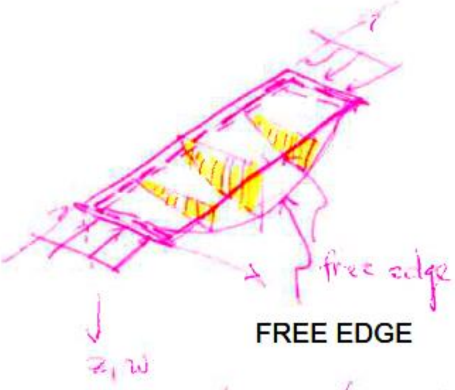
1. Write down all the boundary conditions
2. Specify which boundary conditions are the *kinematic* ones
3. Determine an upper-bound estimate for the buckling edge load $N_{x,cr}^0$ of the plate in figure 2b). Use Rayleigh-Ritz method.

} 0,5p } 5p
} 4,5p



a) Local plate-buckling of thin plates in a crucifix column

b) A plate buckling model



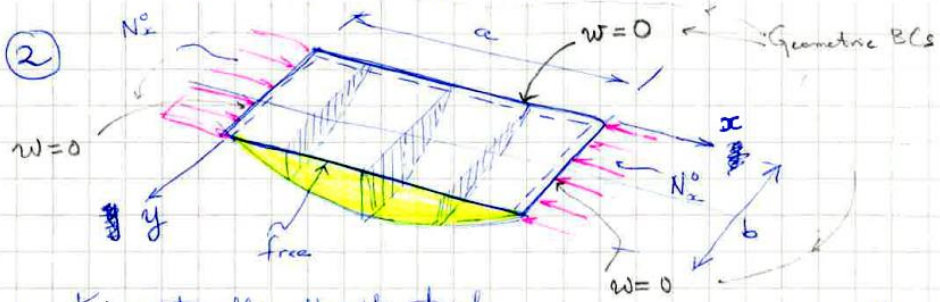
N.B. $\sin\left(\frac{n\pi x}{b}\right)\left(\sin\frac{m\pi y}{a}\right)$ is not good, since it gives zero deflection for the free edge!!

Figure 2: In b) Sides $BADC$ are freely supported and BC is free. The plate is isotropic elastic with thickness h (thin). The data of the problem E, ν and bending rigidity D are given. [the rollers are only on the ends and direction of sides AB and DC to allow free expansion]

The energy functional: The increment of total potential energy[1]:

$$\Delta\Pi = \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}^2\right] dA +$$
$$+\frac{1}{2}\int_A \left[N_{xx}^0 w_{,x}^2 + N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x}w_{,y}\right] dA$$

NB. there is one FREE EDGE and the trial functions should not be such the displacement along this edge being constrained to be ZERO. In such case, you are solving another problem than the one stated in the exam question.



- Kinematically admissible trial $w = w_0 \cdot y \sin\left(\frac{\pi x}{a}\right)$ [or any $w = \sin\left(\frac{n\pi x}{a}\right)\sum_{k=1}^{N} w_k y^k$ & ... for the approximation we take n=1 and k=1]
- Kinematic BCs: $w|_{x=a} = w|_{x=0} = 0$, $w|_{y=0} = 0$; $w_n(x,y) = f(y)\sin\left[\frac{n\pi x}{a}\right]$
- However the trial should be physically well-sounding and should not force the free-boundary to have zero deflection.

$w_{,x} = \frac{\pi}{a} w_0 y \cos\left[\frac{\pi x}{a}\right] \Rightarrow w_{,xy} = \frac{\pi}{a} w_0 \cos\left[\frac{\pi x}{a}\right]$, $w_{,xx} = -\left[\frac{\pi}{a}\right]^2 w_0 y \left(\sin\frac{\pi x}{a}\right)$

$w_{,y} = w_0 \sin\left[\frac{\pi x}{a}\right] \rightarrow w_{,yy} = 0$

etc...

$$\Delta\Pi = w_0^2\frac{ab}{4}\left(D\left[\frac{b^2}{3}\left[\frac{\pi}{a}\right]^4 + 2(1-\nu)\left[\frac{\pi}{a}\right]^2\right] - N_x^0\left[\frac{\pi}{a}\right]^2 b^2\cdot\frac{1}{3}\right)$$

$\delta(\Delta\Pi) = 0 \Rightarrow \frac{\partial(\Delta\Pi)}{\partial w_0} = 0 \Rightarrow$

$$N_{x,cr}^0 = \left[\frac{b^2}{a^2} + \frac{6(1-\nu)}{\pi^2}\right]\cdot\frac{\pi^2 D}{b^2}$$

the very good approximation already ~~exact solution~~ is the same because the trial corresponded to first term of the analytical solution.

now: $K_\sigma = \frac{1}{(a/b)^2} + \frac{6(1-\nu)}{\pi^2}$ | for $\nu = 0{,}25 \Rightarrow \frac{b^2}{a^2} + 0{,}456$ |

Timoshenko (exact) $\rightarrow K_\sigma = 0{,}456 + \left(\frac{b}{a}\right)^2$, $\nu = 0{,}25$

NB*): OF COURSE, YOU CAN USE OTHER TRIALS THAN

Mathematica script

NB. there is one FREE EDGE and the trial functions should not be such the displacement along this edge being constrained to be ZERO. In such case, you are solving another problem than the one stated in the exam question.

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```
In[146]:= w[x_, y_, w0_, L_] := w0 Sin[π x/L] y
wx[x_, y_, w0_, a_] := D[w[x, y, w0, a], x];
wxx[x_, y_, w0_, a_] := D[D[w[x, y, w0, a], x], x];

wy[x_, y_, w0_, a_] := D[w[x, y, w0, a], y];
wyy[x_, y_, w0_, a_] := D[D[w[x, y, w0, a], y], y];

wxy[x_, y_, w0_, a_] := D[D[w[x, y, w0, a], x], y];
wyx[x_, y_, w0_, a_] := D[D[w[x, y, w0, a], y], x];

UplateBending1 = 2*nu*wxx[x, y, w0, a]*wyy[x, y, w0, a] + 2*(1 - nu)*wxy[x, y, w0, a]^2;
UplateBending2 = wxx[x, y, w0, a]^2 + wyy[x, y, w0, a]^2 ;
U = Integrate [1/2*D*(UplateBending1 + UplateBending2) , {x, 0, a}, {y, 0, b}]
Nxx0 = -P;
V = 1/2* Integrate[Nxx0* wx[x, y, w0, a]^2, {x, 0, a}, {y, 0, b}]
PI = U + V

Dcr = FullSimplify[D[PI, w0]]
FullSimplify[Solve[Dcr == 0, P, Reals]]
```

$= w_0 \cdot y \cdot \sin\left(\pi \frac{x}{a}\right)$; $= w_{,x}$; $= w_{,xx}$; $\leftrightarrow w_{,y}$; $\leftrightarrow w_{,yy}$; $\leftrightarrow w_{,xy}$; $\leftrightarrow w_{,yx}$; $\leftrightarrow \Delta U$ (Bending); $= \Delta U_{\text{bending}}$; $\leftrightarrow \Delta V$; $\leftrightarrow \Delta\Pi = \Delta U + \Delta V$; $\frac{\partial}{\partial w_0}(\Delta\Pi) = 0 \Rightarrow N_{cr} = ?$; solution

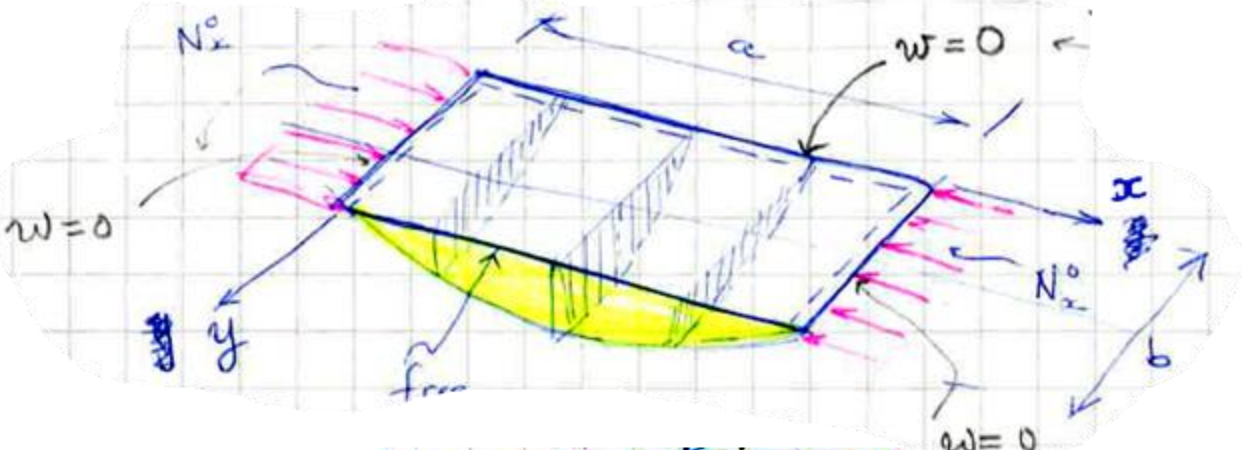
Out[155]= $\frac{b D \pi^2 (-6 a^2 (-1 + nu) + b^2 \pi^2) w0^2}{12 a^3}$

Out[157]= $-\frac{b^3 P \pi^2 w0^2}{12 a}$

Out[158]= $-\frac{b^3 P \pi^2 w0^2}{12 a} + \frac{b D \pi^2 (-6 a^2 (-1 + nu) + b^2 \pi^2) w0^2}{12 a^3}$

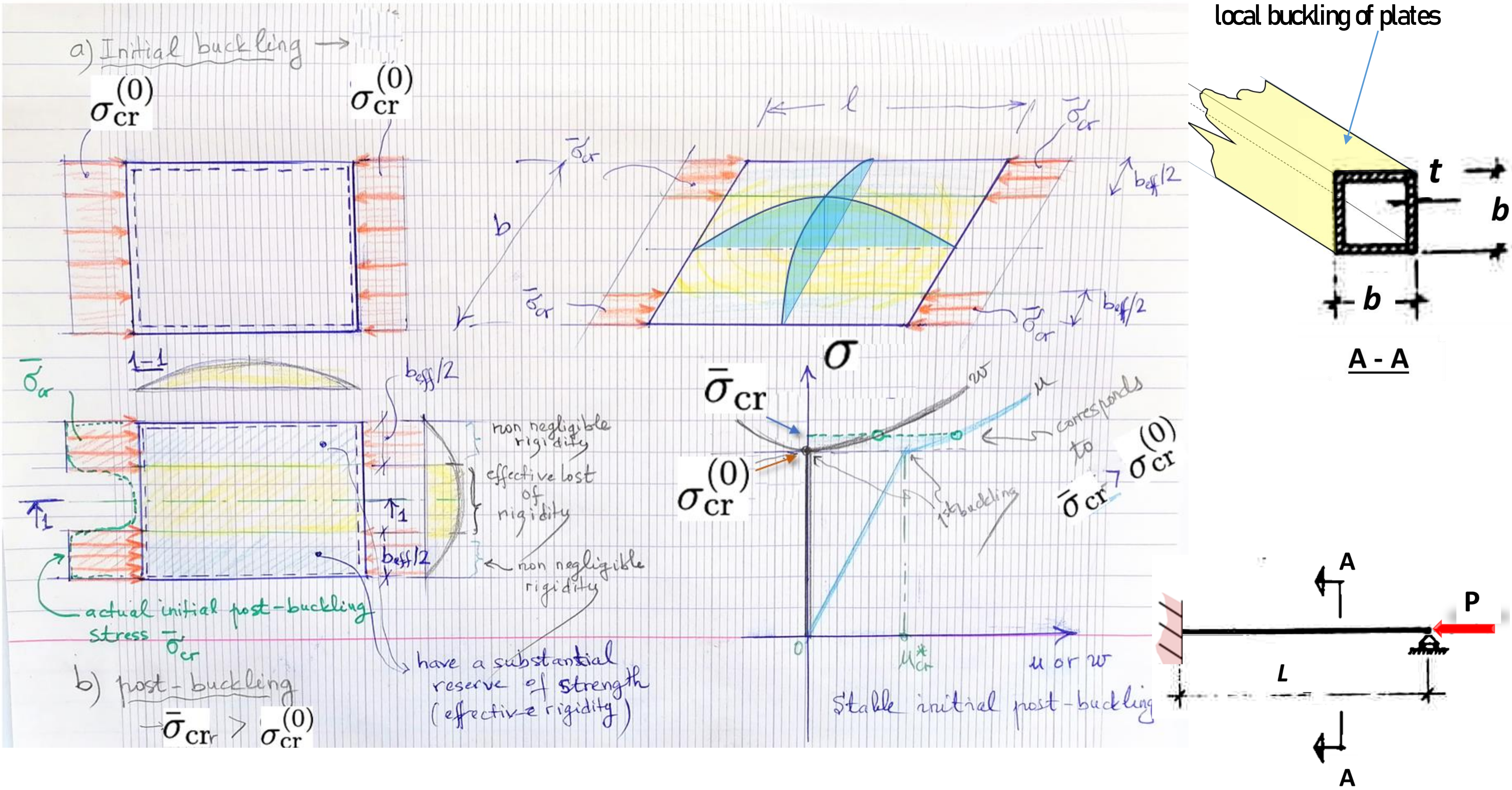
Out[159]= $\frac{\pi^2 (-a^2 b (6 D (-1 + nu) + b^2 P) + b^3 D \pi^2) w0}{6 a^3}$

Out[160]= $\{\{P \to D \left(\frac{6 - 6 nu}{b^2} + \frac{\pi^2}{a^2}\right)\}\}$ $= \frac{D\pi^2}{b^2}\left[\frac{b^2}{a^2} + \frac{6(1-\nu)}{\pi^2}\right] = N^0_{cr}$, $\left[\ldots\right] \equiv K_\sigma$

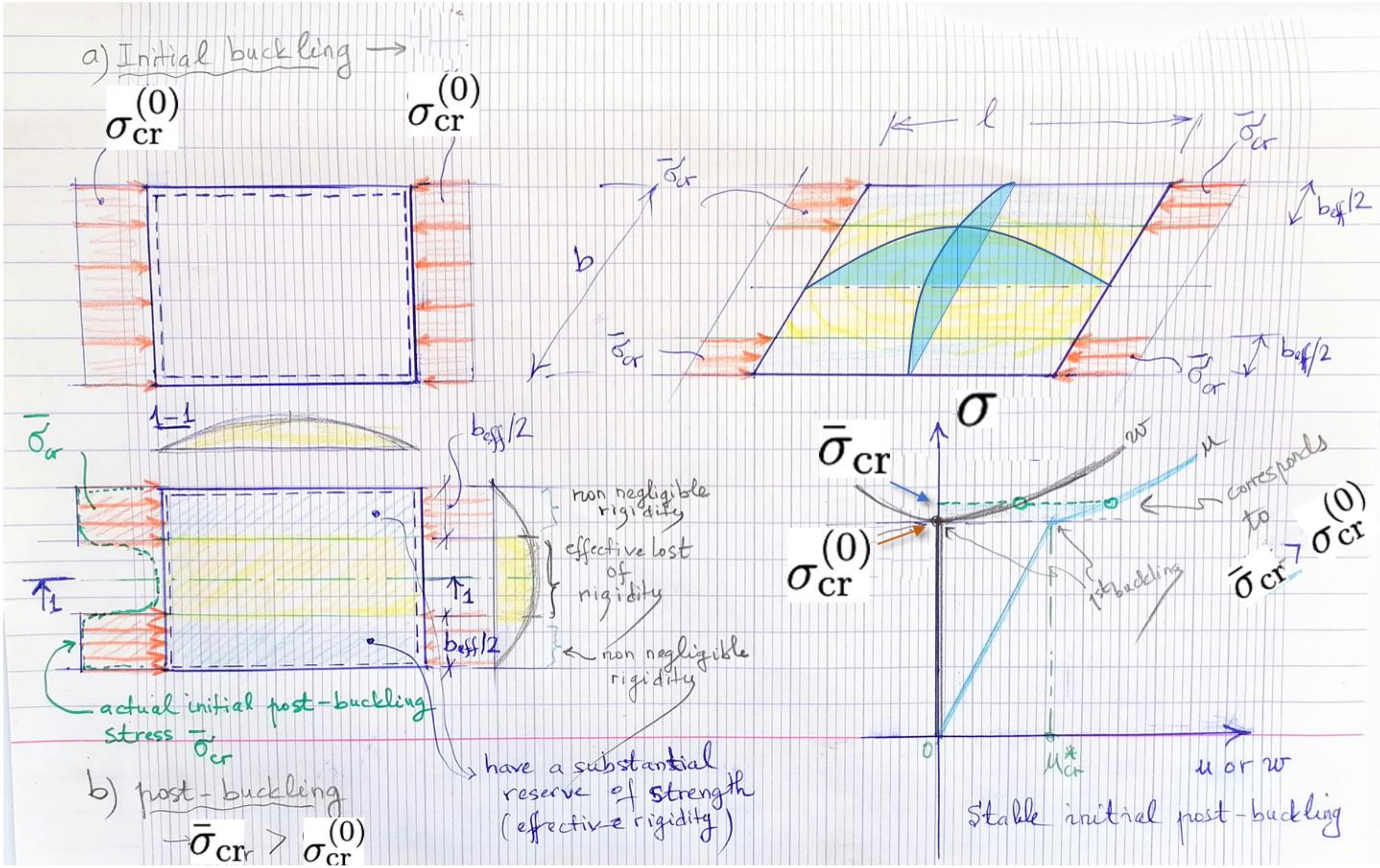


$$N^0_{x,cr} = \left[\frac{b^2}{a^2} + \frac{6(1-\nu)}{\pi^2}\right] \cdot \frac{\pi^2 D}{b^2}$$

The concept of effective-width in local buckling resistance of plates



The concept of effective-width in local buckling resistance of plates



$$\bar{\sigma}_{cr} = \underbrace{\bar{k}_\sigma \cdot \frac{\pi^2 D}{b_{eff}^2 h}}_{\text{buckling}} = \sigma_y$$

$$\approx \frac{1}{\beta^2} \cdot \underbrace{k_\sigma \cdot \frac{\pi^2 D}{b^2 h}}_{=\sigma_{cr}^{(0)}} = \sigma_y$$

$$\Longrightarrow \quad \sigma_{cr}^{(0)} = \beta^2 \sigma_y$$

$$\Longrightarrow \quad \frac{b_{eff}}{b} \equiv \beta = \sqrt{\frac{\sigma_{cr}^{(0)}}{\sigma_y}}$$

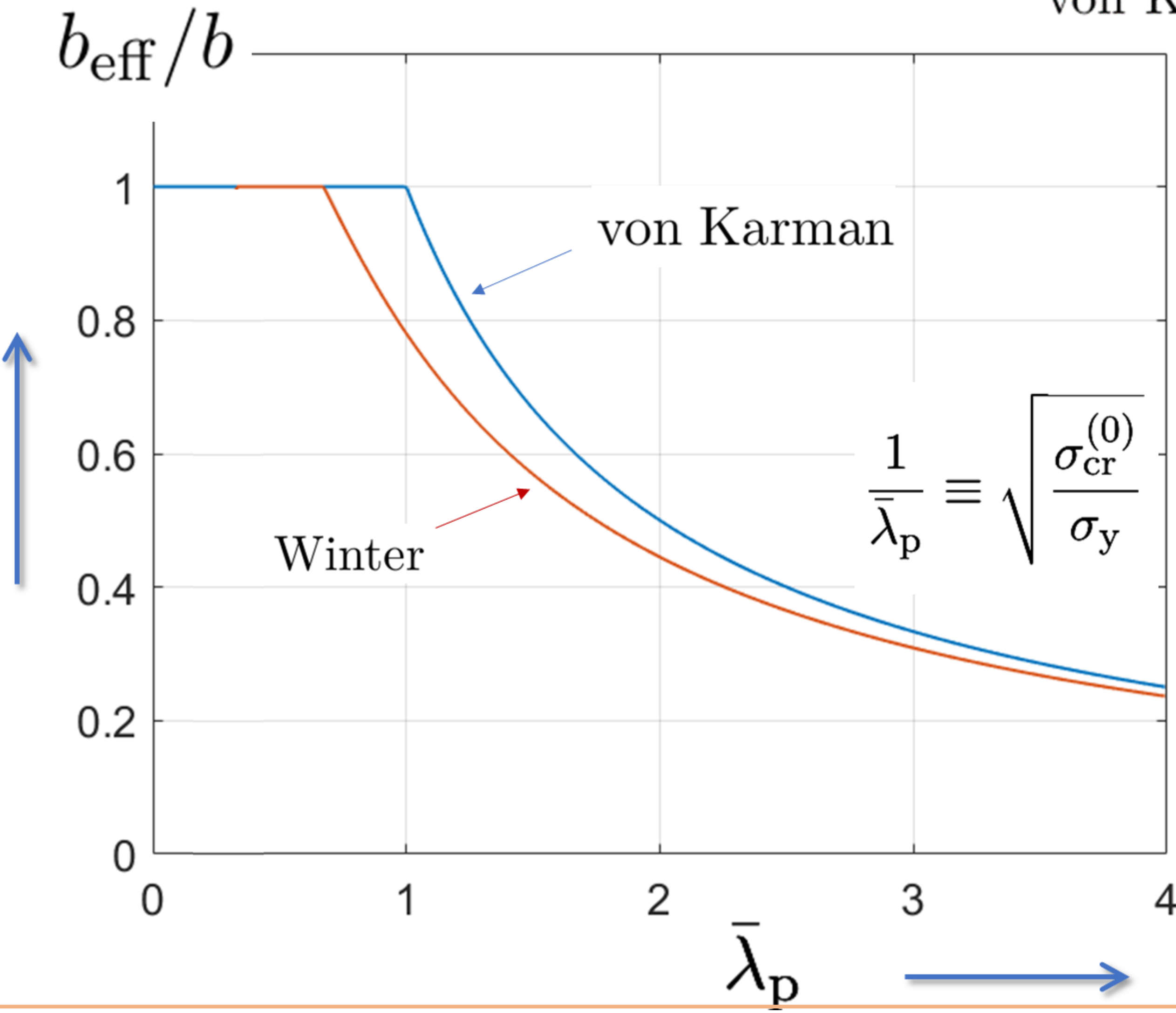
von Karman's equation

Winter's

$$\frac{b_{eff}}{b} = \sqrt{\frac{\sigma_{cr}^{(0)}}{\sigma_y}}\left(1 - 0.22\sqrt{\frac{\sigma_{cr}^{(0)}}{\sigma_y}}\right)$$

EC3-part 1.3

account for all kind of initial imperfections present in cold-steel members based on a large series of experiments on CF steel members.



von Karman $\bar{\sigma}_{\text{cr}} = \sigma_{\text{y}}$

$b_{\text{eff}} = \beta b,$

$0 < \beta < 1$

$$\bar{\sigma}_{\text{cr}} = \underbrace{\bar{k}_{\sigma} \cdot \frac{\pi^2 D}{b_{\text{eff}}^2 h}}_{\text{buckling}} = \sigma_{\text{y}}$$

$$\approx \frac{1}{\beta^2} \cdot \underbrace{k_{\sigma} \cdot \frac{\pi^2 D}{b^2 h}}_{=\sigma_{\text{cr}}^{(0)}} = \sigma_{\text{y}}$$

$$\Longrightarrow \quad \sigma_{\text{cr}}^{(0)} = \beta^2 \sigma_{\text{y}}$$

von Karman's equation

$$\Longrightarrow \quad \frac{b_{\text{eff}}}{b} \equiv \beta = \sqrt{\frac{\sigma_{\text{cr}}^{(0)}}{\sigma_{\text{y}}}}$$

Winter

later, adjusted to accounts for all kind of initial imperfections present in cold-formed steel members based on a large series of experiments on CF steel members.

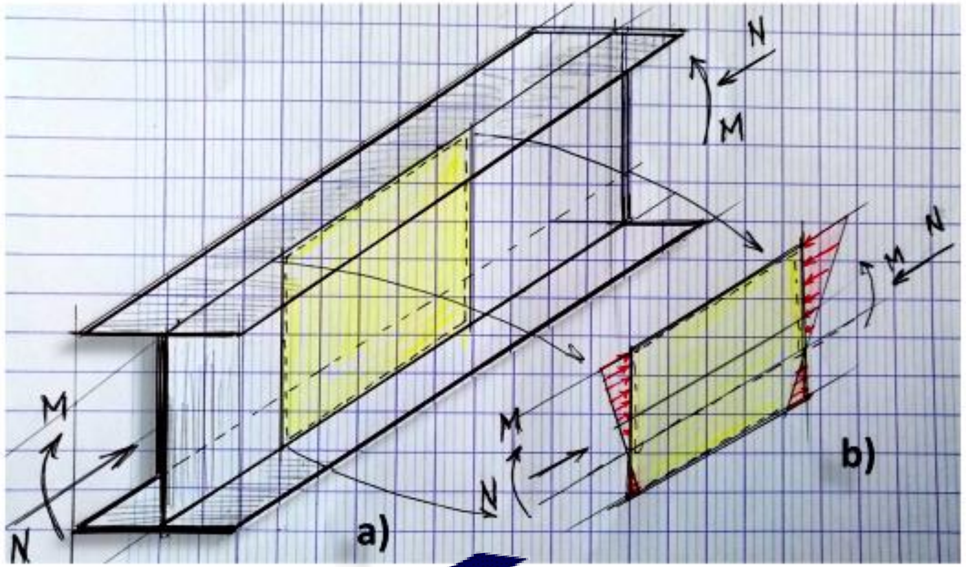
Winter

$$\frac{b_{\text{eff}}}{b} = \sqrt{\frac{\sigma_{\text{cr}}^{(0)}}{\sigma_{\text{y}}}} \left(1 - 0.22\sqrt{\frac{\sigma_{\text{cr}}^{(0)}}{\sigma_{\text{y}}}}\right)$$

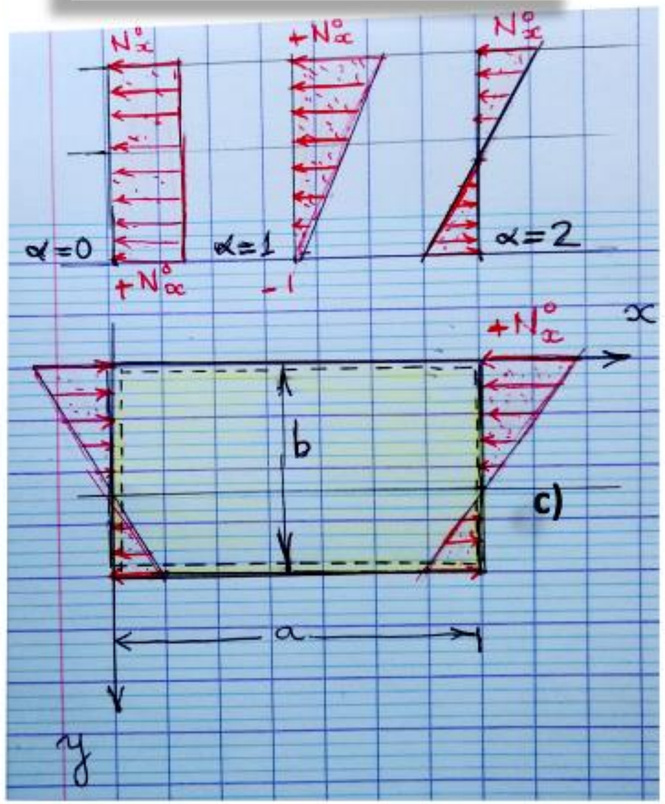
EC3-part 1.3

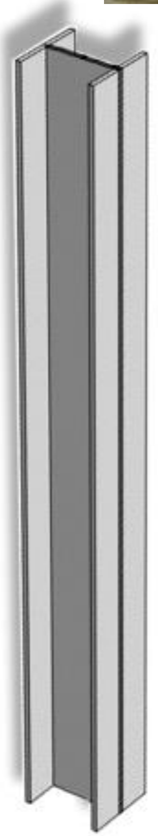
Buckling of a simply supported rectangular plate - in-plane bending and compression

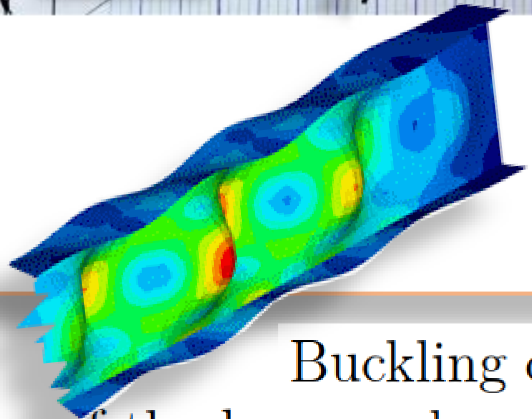
Plate buckling



$$N^0_{xx}(x, y) = N_0 \left[1 - \alpha \frac{y}{b}\right]$$







Buckling of a web plate section due to bending and compression of the beam-column.

Buckling of a simply supported rectangular plate - in-plane bending and compression

This problem can be found in **Timoshenko's** textbook

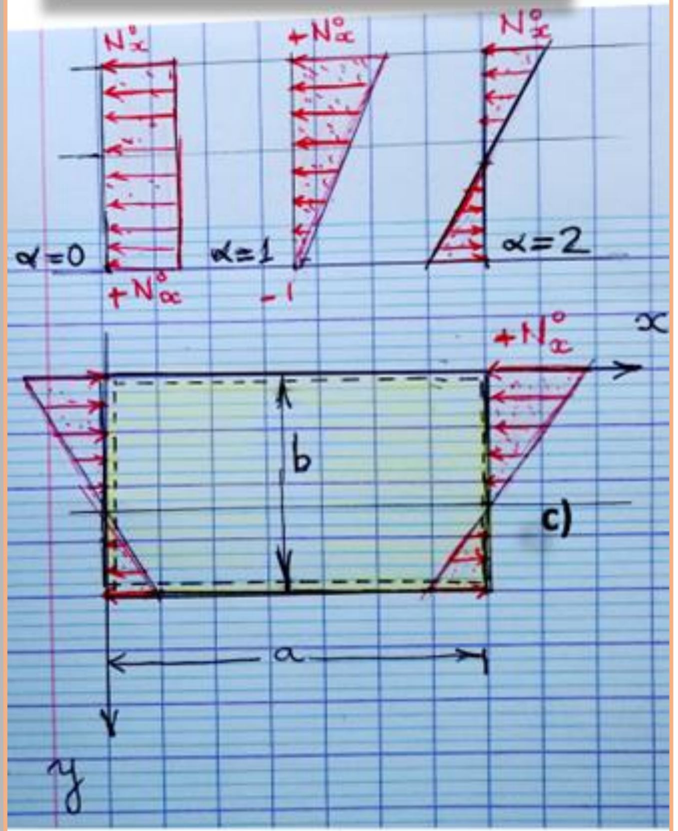
Problem setting

$$N^0_{xx}(x,y) = N_0\left[1-\alpha\frac{y}{b}\right]$$

$$N^0_{yy} = N^0_{xy} = 0$$

$$D\underbrace{\left[w_{,xxxx}+2w_{,xxyy}+w_{,yyyy}\right]}_{\Delta\Delta w}-\underbrace{\left[N^0_{xx}w_{,xx}+N^0_{yy}w_{,yy}+2N^0_{xy}w_{,xy}\right]}_{N^0_{\alpha\beta}w_{,\alpha\beta}}=0$$

$$N^0_{xx}(x,y) = N_0\left[1-\alpha\frac{y}{b}\right]$$



Boundary conditions:

- simplification: insulated panel *a* x *b* simply supported at all 4 edges

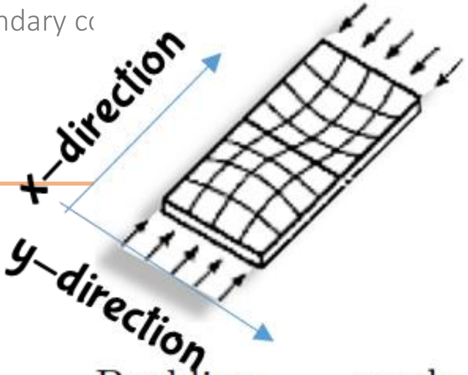
1. This is an approximation leading to a lower-bound for the critical buckling load
2. In reality, the upper and lower edge-connections of the web-plate to the flanges correspond to those of rotational springs due to their rotational rigidity while the web buckles. This type of boundary c
is not impossible to address even theoretically.

Trial solution:

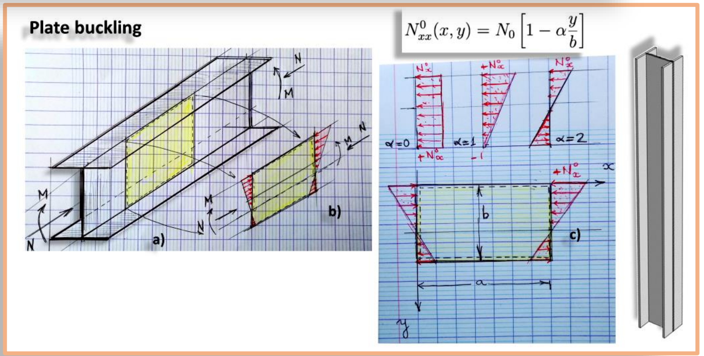
$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\alpha_m x\sin\beta_n y$$

non-trivial $a_{mn} \neq 0, \forall m, n$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$,



Buckling mode, $m = 2$, $n = 1$.



Buckling of a simply supported rectangular plate - in-plane bending and compression

The change in strain energy is simply

$$\Delta U = \frac{D}{2}\frac{ab\pi^4}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

and the increment of the work of initial stresses

$$\Delta W = \frac{1}{2}\int_0^a\int_0^b N_0(1-\alpha\frac{y}{b})(w_{,x})^2 dxdy$$

After performing careful integrations and using the criterion $\Delta\Pi = 0$ one
the critical load $N_{0,cr}(m,n)$ as[221]

$$N_{0,cr} = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 /$$

$$\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2\frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[\sum_{n=1}^{\infty} a_{mn}^2 - \frac{32}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}^{\infty}\frac{a_{mn}a_{mi}\, ni}{(n^2-i^2)^2}\right]\}.$$

index i is such that $n \pm i$ is always an odd number.

$$Da_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 = N_{0,cr}\left(a_{mn}^2\frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\frac{m^2\pi^2}{a^2}\left[a_{mn}^2 - \frac{16}{\pi^2}\sum_{i}^{\infty}\frac{a_{mi}\, ni}{(n^2-i^2)^2}\right]\right)$$

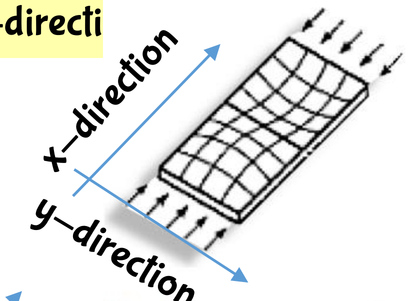
non-trivial $a_{mn} \neq 0, \forall m, n$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\alpha_m x\sin\beta_n y$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$,

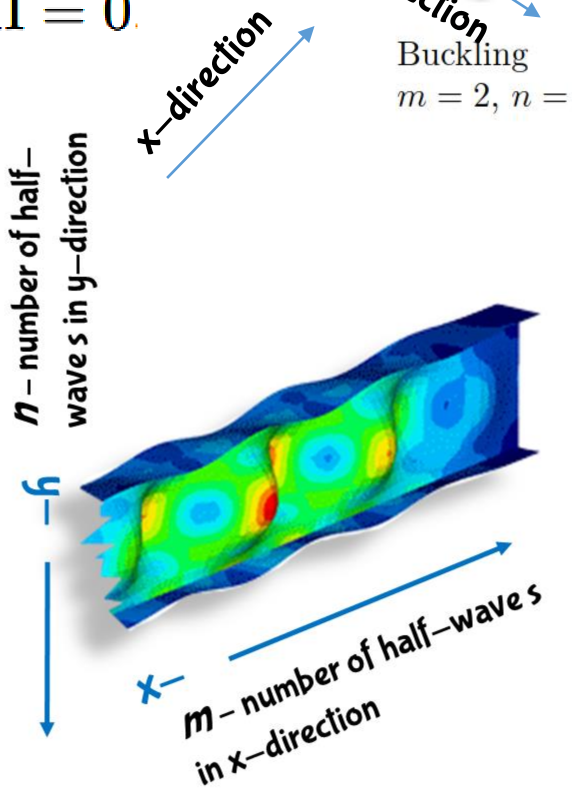
n – number of half–wave s in x–direction
m – number of half–wave s in y–directi

Example:



Buckling mode, $m = 2$, $n = 1$.

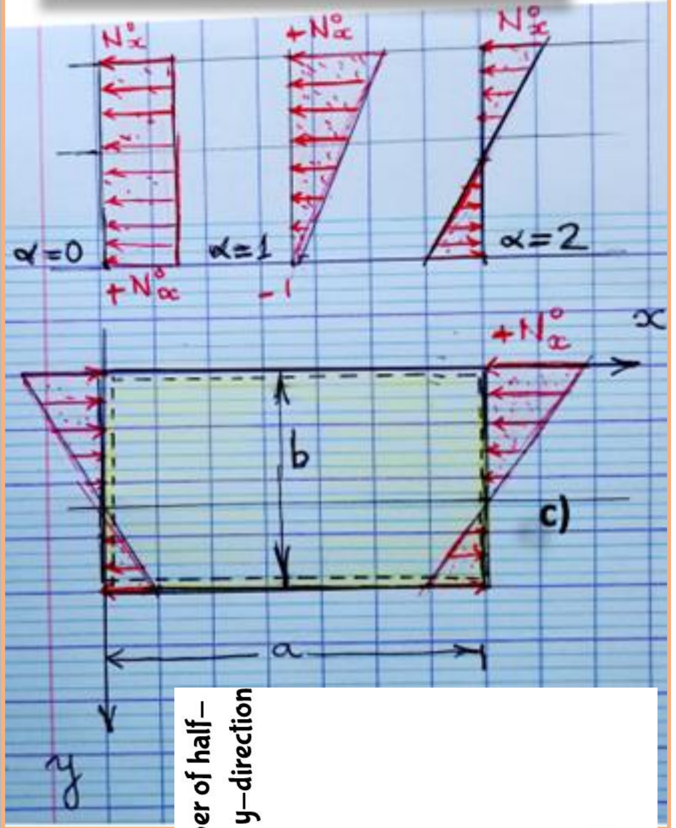
N.B. Stability loss Criteria: here it is now energetic; $\Delta\Pi = 0$



Timoshenko, *Theory of Elastic Stability*, 2nd Ed. p.375.

Buckling of a simply supported rectangular plate - in-plane bending and compression

$$N^0_{xx}(x,y) = N_0\left[1-\alpha\frac{y}{b}\right]$$

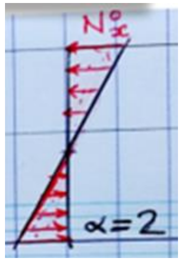


In the following an illustrative example of approximate solution for the above system within less than 1% or relative error in the buckling load by taking only three equations (Timoshenko). For instance, for $\alpha = 2$ (pure bending) fixing $m = 1$ – **number of half–wave s in y–direction**

$$\underbrace{\begin{bmatrix} (1+\frac{a^2}{b^2})^2 & -16\sigma_{cr}\frac{2}{9}\frac{a^2h}{\pi^4 D} & 0 \\ -16\sigma_{cr}\cdot\frac{2}{9}\frac{a^2h}{\pi^4 D} & (1+4\frac{a^2}{b^2})^2 & -16\sigma_{cr}\cdot\frac{6}{25}\frac{a^2h}{\pi^4 D} \\ 0 & -16\sigma_{cr}\cdot\frac{6}{25}\frac{a^2h}{\pi^4 D} & (1+9\frac{a^2}{b^2})^2 \end{bmatrix}}_{\det \mathbf{A}(\sigma_{\mathbf{cr}})=0} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first zero of the determinant provides

$$\sigma_{cr} = 25.6\frac{\pi^2 D}{b^2 h},$$ for $a = b$, $\alpha = 2$



Timoshenko, *Theory of Elastic Stability*, 2nd Ed. p.375.

In general, the critical buckling stresses for various combined loading factor α and ratio a/b is expressed in the canonical form

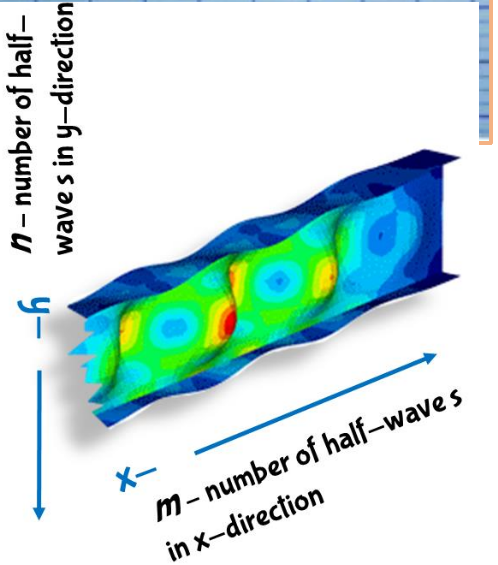
Buckling stress

$$\sigma_{cr} = k\frac{\pi^2 D}{b^2 h} = k\cdot\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{b}\right]^2,$$

Lommahdus kerroin

k - **Buckling coefficient** depends of

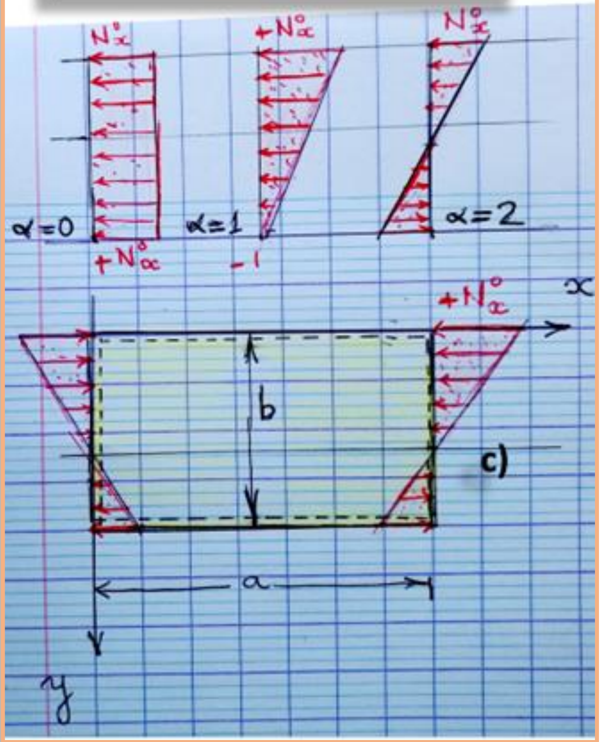
- ration a/b,
- boundary conditions and
- loading case



Buckling of a simply supported rectangular plate - in-plane bending and compression

Lommahdus kerroin

$$N^0_{xx}(x,y) = N_0\left[1-\alpha\frac{y}{b}\right]$$



Buckling stress

$$\sigma_{cr} = k\frac{\pi^2 D}{b^2 h} = k\cdot\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{b}\right]^2,$$

Buckling of simply supported thin plate

$$\sigma_{cr} = k\frac{\pi^2 D}{b^2 h}$$

TABLE . NUMERICAL VALUES OF THE FACTOR k

α \ a/b	0.4	0.5	0.6	0.667	0.75	0.8	0.9	1.0	1.5
2	29.1	25.6	24.1	23.9	24.1	24.4	25.6	25.6	24.1
$\frac{4}{3}$	18.7	....	12.9	....	11.5	11.2	....	11.0	11.5
1	15.1	....	9.7	....	8.4	8.1	....	7.8	8.4
$\frac{4}{5}$	13.3	....	8.3	....	7.1	6.9	....	6.6	7.1
$\frac{2}{3}$	10.8	....	7.1	....	6.1	6.0	....	5.8	6.1

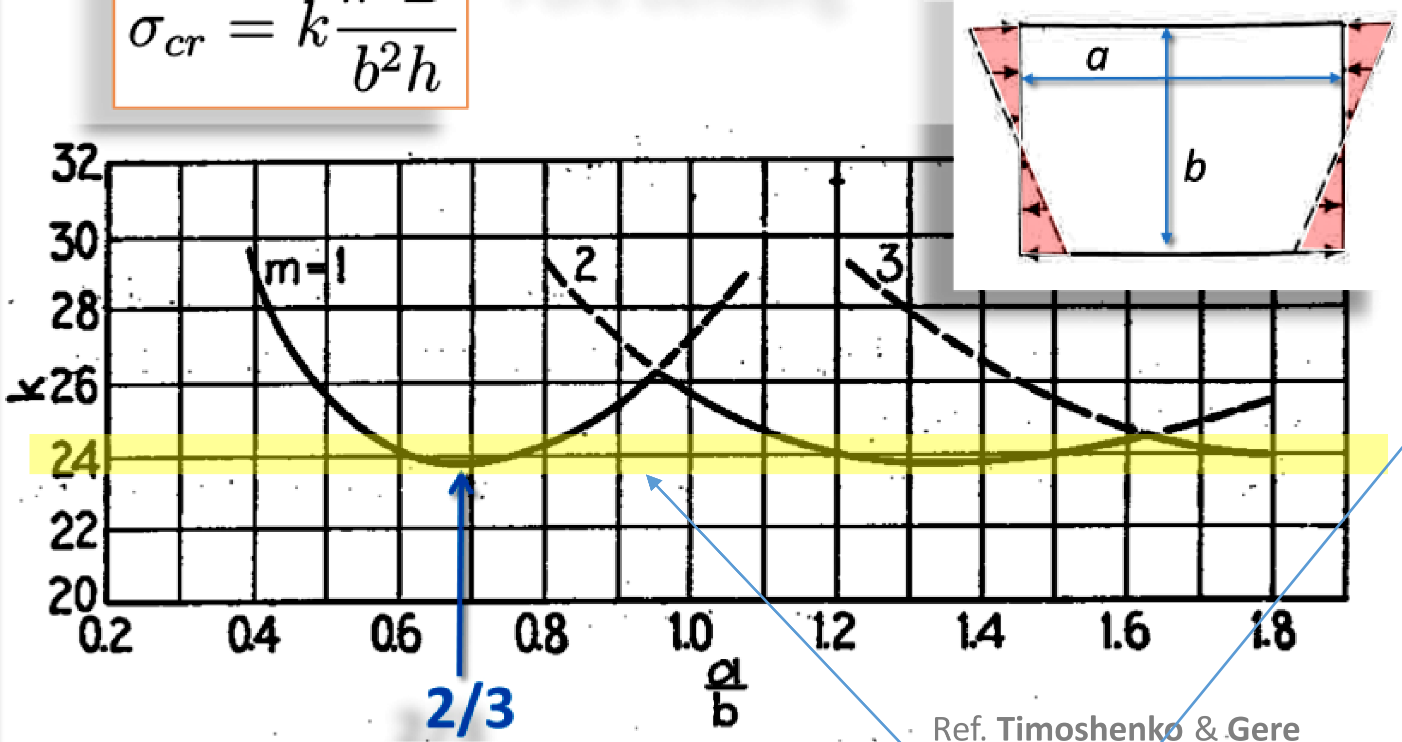
Timoshenko

Buckling of simply supported thin rectangular plate in pure bending

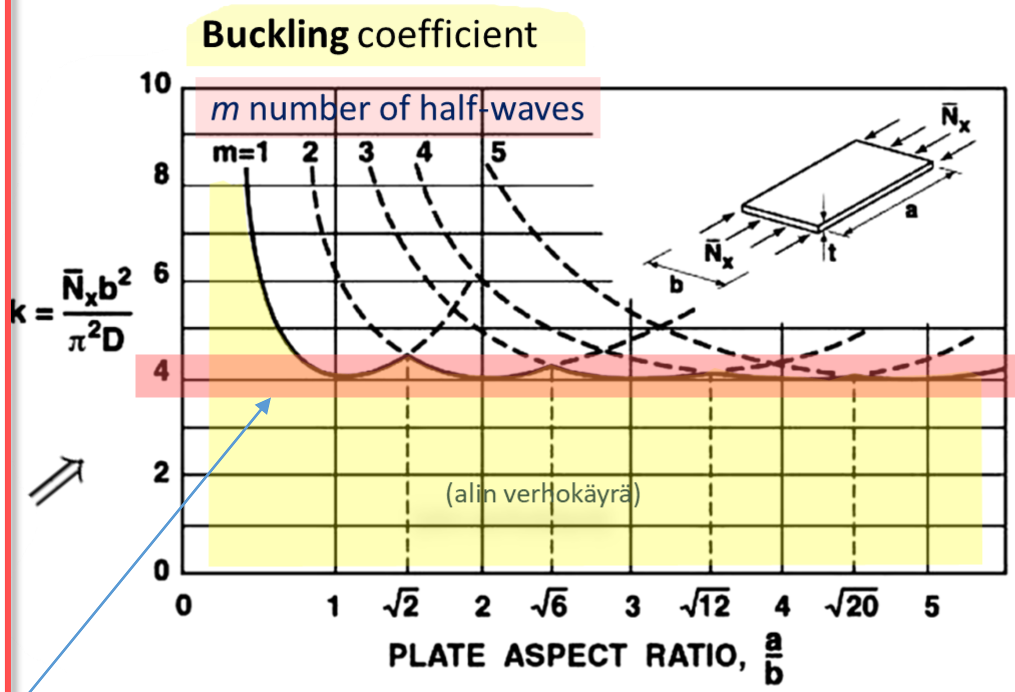
Buckling stress

$$\sigma_{cr} = k\frac{\pi^2 D}{b^2 h}$$

Pure bending $\alpha = 2$



Ref. **Timoshenko** & **Gere**



Normalized buckling load *k* for simply supported rectangular plates with Various Plate Aspect Ratios

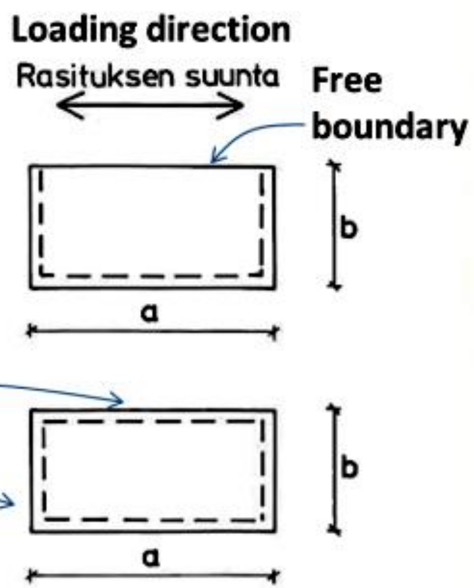
(Ref. *Robert M. Jones. Buckling of bars, plates and shells. Bull Ridge Publishing, 2006.*)

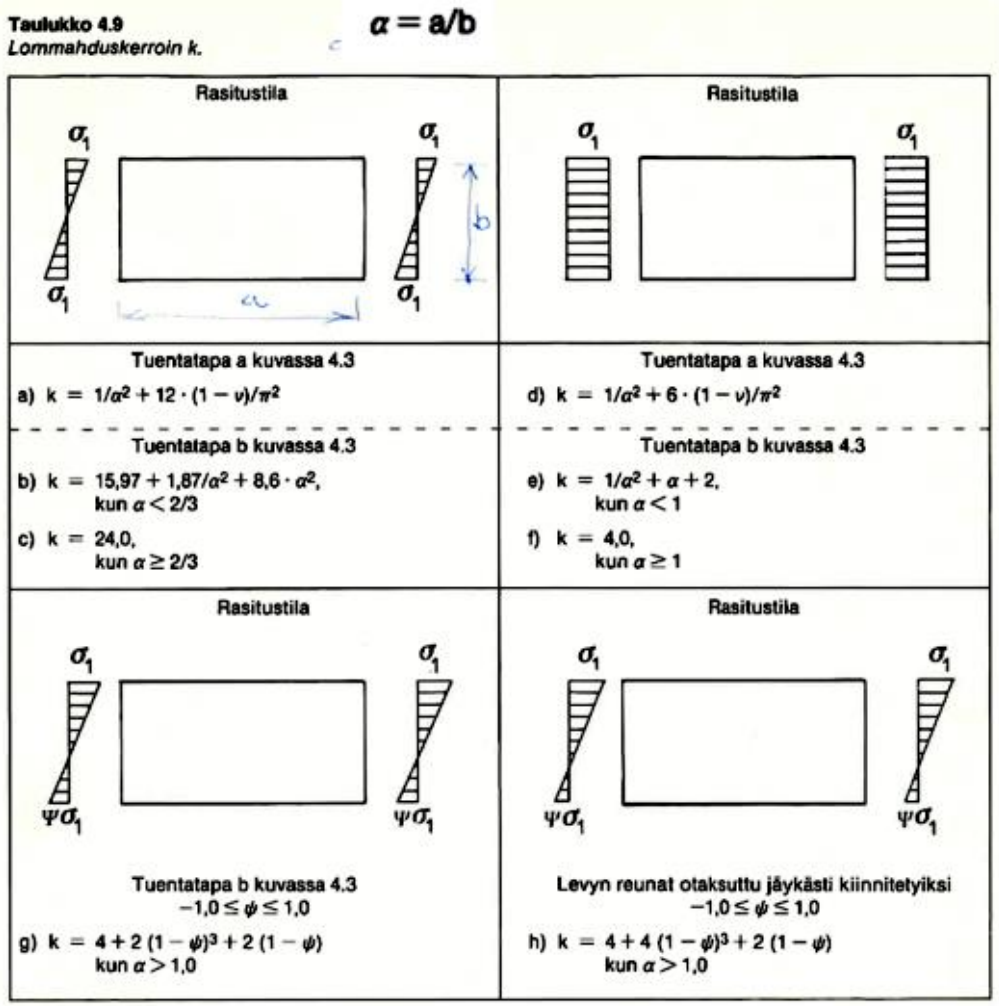
$$|N_{xx,cr}| = 4\frac{\pi^2 D}{b^2}$$

NB. **About 6 times higher buckling load** (buckling strength) **than for compresses rectangular plate**

- Similar tables are provided our-days for practical design in standards related to structural design of metallic structures
- For instance, the standard **EN-1993-1-5**, **Table 6 (2006)**, provides similar buckling coefficients *k* tables **for** combined compression and bending of thin plates for **various boundary and loading conditions**

$$\sigma_{el} = \frac{k \cdot \pi^2 \cdot E}{12 \cdot (1 - \nu^2) \left(\frac{b}{t}\right)^2}$$





Taulukko 4.9
Lommahduskerroin k.

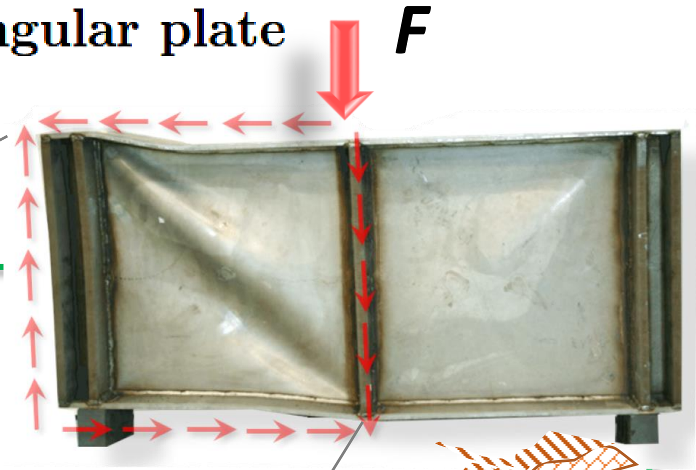
$\alpha = a/b$

Rasitustila	Rasitustila
σ_1 … σ_1 (linear bending)	σ_1 … σ_1 (uniform compression)
Tuentatapa a kuvassa 4.3 a) $k = 1/\alpha^2 + 12 \cdot (1-\nu)/\pi^2$	Tuentatapa a kuvassa 4.3 d) $k = 1/\alpha^2 + 6 \cdot (1-\nu)/\pi^2$
Tuentatapa b kuvassa 4.3 b) $k = 15{,}97 + 1{,}87/\alpha^2 + 8{,}6 \cdot \alpha^2$, kun $\alpha < 2/3$ c) $k = 24{,}0$, kun $\alpha \geq 2/3$	Tuentatapa b kuvassa 4.3 e) $k = 1/\alpha^2 + \alpha + 2$, kun $\alpha < 1$ f) $k = 4{,}0$, kun $\alpha \geq 1$
Rasitustila σ_1 … $\psi\sigma_1$	Rasitustila σ_1 … $\psi\sigma_1$
Tuentatapa b kuvassa 4.3 $-1{,}0 \leq \psi \leq 1{,}0$ g) $k = 4 + 2(1-\psi)^3 + 2(1-\psi)$ kun $\alpha > 1{,}0$	Levyn reunat otaksuttu jäykästi kiinnitetyiksi $-1{,}0 \leq \psi \leq 1{,}0$ h) $k = 4 + 4(1-\psi)^3 + 2(1-\psi)$ kun $\alpha > 1{,}0$

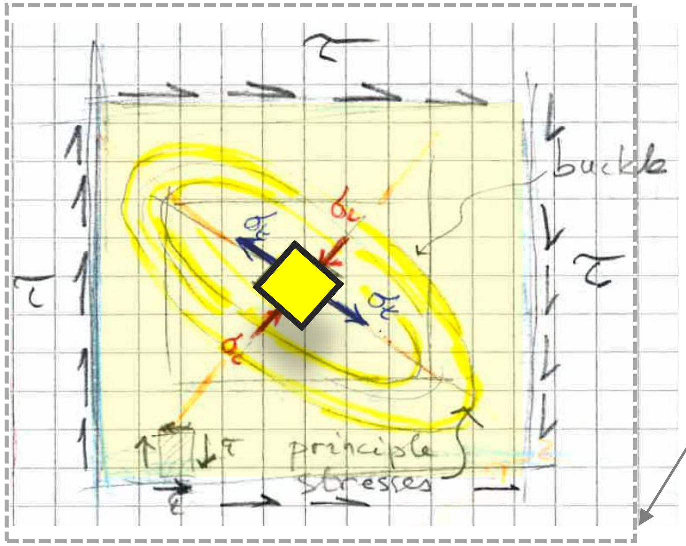
old Finnish standard B7

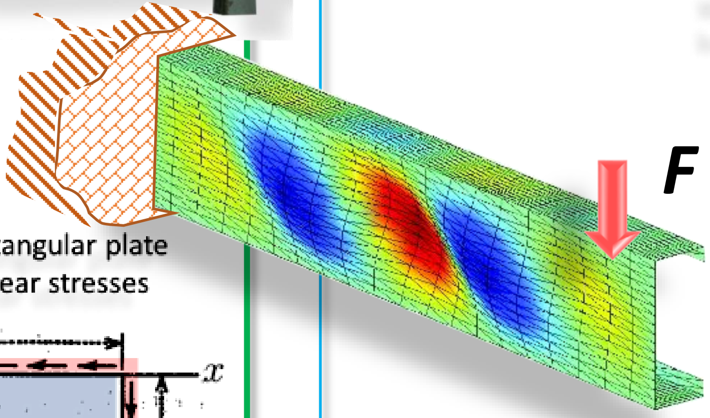
I still admire the clarity and non-ambiguity of this table from old B7

Shear buckling of a rectangular plate

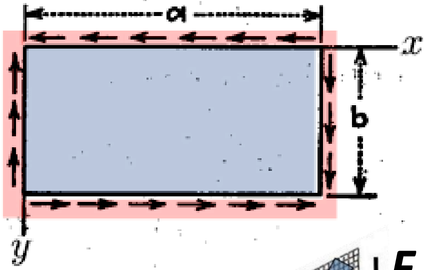


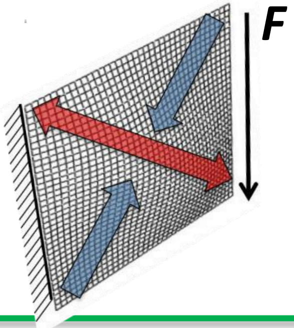
Physics of the phenomena



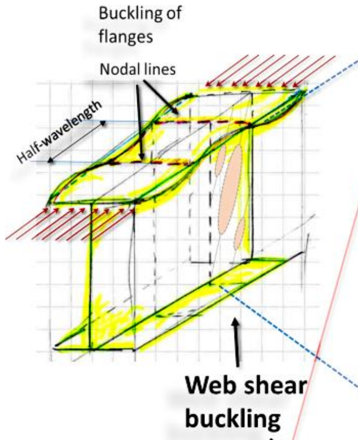


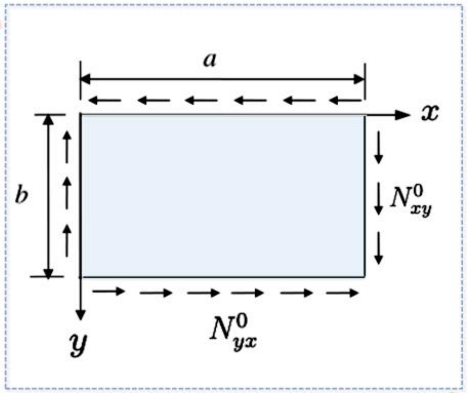
Bulking of rectangular plate under edge shear stresses

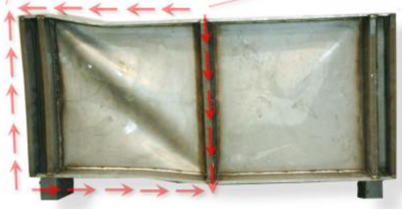




Principle stresses under edge sharing:
tension weaken the buckling strength in the orthogonal **compressive** direction







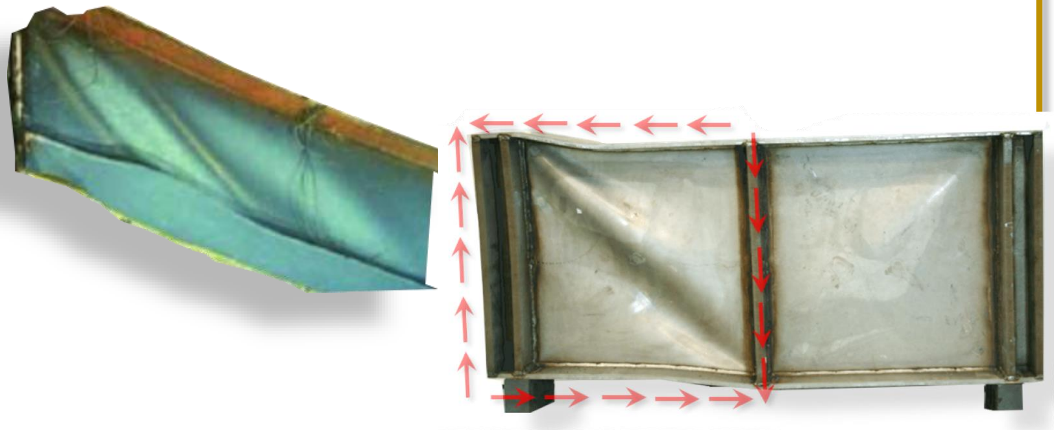
Shear buckling of rectangular plates under in-plane shear edge-loading

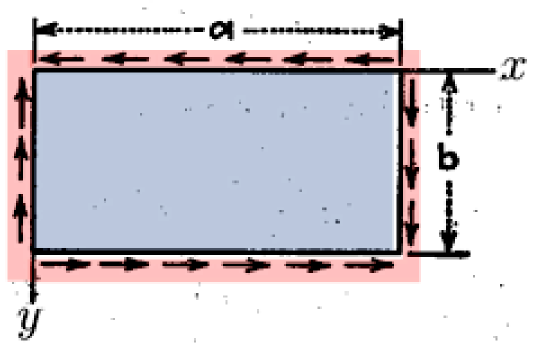
$$D\underbrace{\left[w_{,xxxx}+2w_{,xxyy}+w_{,yyyy}\right]}_{\Delta\Delta w}-\underbrace{\left[N^0_{xx}w_{,xx}+N^0_{yy}w_{,yy}+2N^0_{xy}w_{,xy}\right]}_{N^0_{\alpha\beta}w_{,\alpha\beta}}=0$$

$$D\Delta\Delta w-2N^0_{xy}w_{,xy}=0.$$

$$(N_{xy})_{cr}=k_s\frac{\pi^2 D}{b^2}\implies(\tau_{xy})_{cr}=k_s\cdot\frac{\pi^2 E}{12(1-\nu^2)}\left[\frac{h}{b}\right]^2$$

Exact solution is available only for an infinitely long strip (Brush & Almroth (1975)).





$\Longrightarrow$ **For infinitely long strip:**

$$\begin{cases} k_s &= 5.34, \text{ simply supported} \\ k_s &= 8.98, \text{ clamped support.}\end{cases}$$

Southwell & Skan (1924)

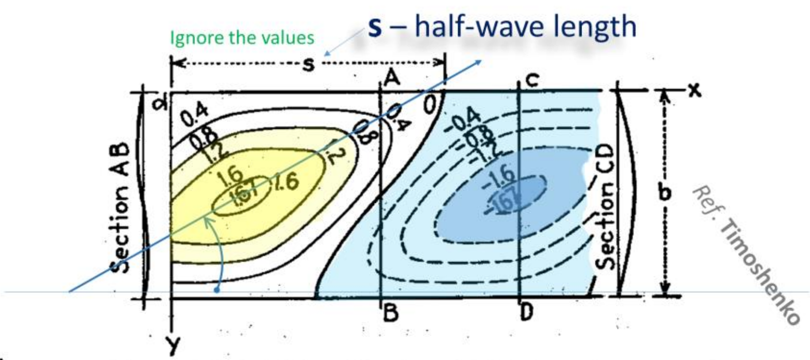
$\Downarrow$

For finite strips: aspect ratio $\alpha = a/b$.

$$\begin{cases} k_s &= 4.00 + 5.34/\alpha^2, \text{ for } \alpha \le 1; \text{ four edges simply supported,} \\ k_s &= 5.34 + 4.00/\alpha^2, \text{ for } \alpha \ge 1; \text{ four edges clamped.}\end{cases}$$

[224] Galambos, T. V. (Ed.). 1998. *Guide to stability design criteria for metal structures.* New York: John & Sons.

Shear buckling of thin plates

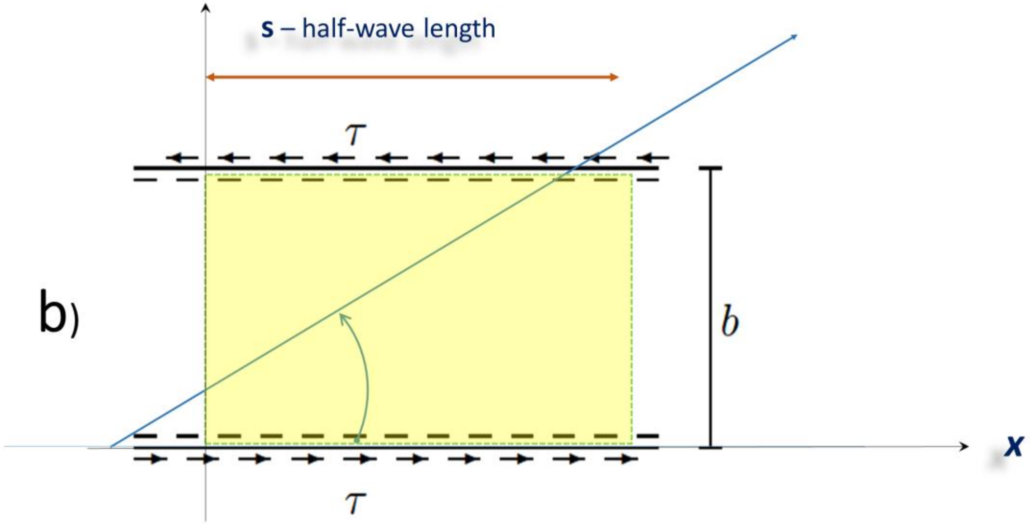


a)

The exact solution of the problem for an infinitely long strip with simply supported edges gives

Exact for infinitely long slab: $\tau_{cr} = 5.35 \frac{\pi^2 D}{b^2 h}$

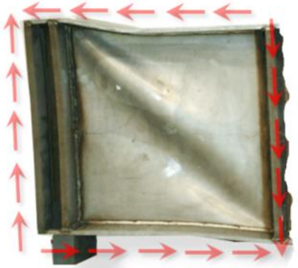
y Shear buckling of thin plates of infinite strip

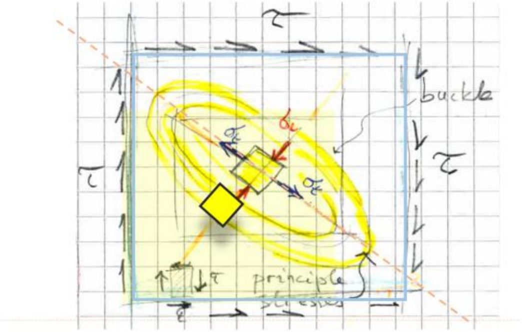


b)

Shear buckling of thin plates of finite length a

Approximate for finite slab: $k = 5.35 + 4(b/a)^2$



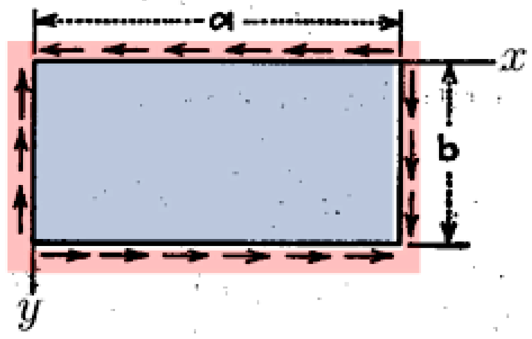


Principle stresses under edge sharing: tension weaken the buckling strength in the orthogonal **compressive** direction

c)

Shear buckling of rectangular plates under in-plane shear edge-loading

Bulking of rectangular plate under edge shear stresses



$$(N_{xy})_{cr} = k_s \frac{\pi^2 D}{b^2} \implies (\tau_{xy})_{cr} = k_s \cdot \frac{\pi^2 E}{12(1-\nu^2)} \left[\frac{h}{b}\right]^2$$

PRACTICE

Standard – EN – 1993 – 1 – 5: 2006

A.3 Shear buckling coefficients

(1) For plates with rigid transverse stiffeners and without long longitudinal stiffeners, the shear buckling coefficient k_τ can be obta

$$k_\tau = 5{,}34 + 4{,}00\,(h_w/a)^2 + k_{\tau s\ell} \quad \text{when } a/h_w \geq 1$$

$$k_\tau = 4{,}00 + 5{,}34\,(h_w/a)^2 + k_{\tau s\ell} \quad \text{when } a/h_w < 1$$

where $k_{\tau s\ell} = 9\left(\frac{h_w}{a}\right)^2 \sqrt[4]{\left(\frac{I_{s\ell}}{t^3 h_w}\right)^3}$ but not less than $\frac{2{,}1}{t}\sqrt[3]{\frac{I_{s\ell}}{h_w}}$

$\Longrightarrow$ **For infinitely long strip:**

$$\begin{cases} k_s = 5.34, & \text{simply supported} \\ k_s = 8.98, & \text{clamped support.} \end{cases}$$

Southwell & Skan (1924)

THEORY

For finite strips: aspect ratio $\alpha = a/b$.

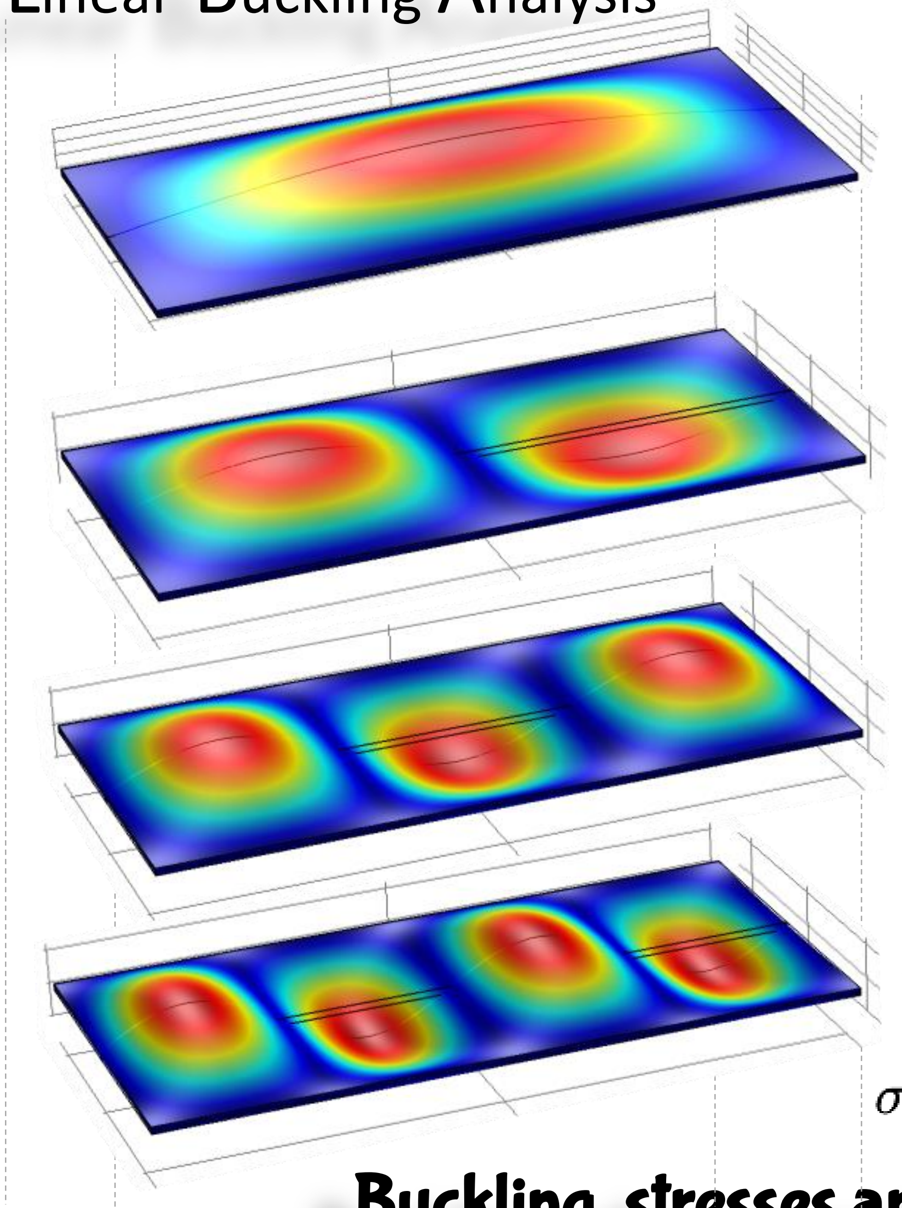
$$\begin{cases} k_s = 4.00 + 5.34/\alpha^2, & \text{for } \alpha \leq 1; \text{ four edges simply supported,} \\ k_s = 5.34 + 4.00/\alpha^2, & \text{for } \alpha \geq 1; \text{ four edges clamped.} \end{cases}$$

However, from where comes this PRACTICAL design formula? $\Longleftarrow \Longleftarrow$

[224] Galambos, T. V. (Ed.). 1998. *Guide to stability design criteria for metal structures.* New York: John & Sons.

Linear Buckling Analysis

3D-FE model



λ	σ_{cr}
1	..... 70 MPa
1.1	..... 77 MPa
1.5	..... 105 MPa
2.1	..... 147 MPa

$$\sigma_{cr,1} \equiv \sigma_E = 70 \text{ MPa}$$

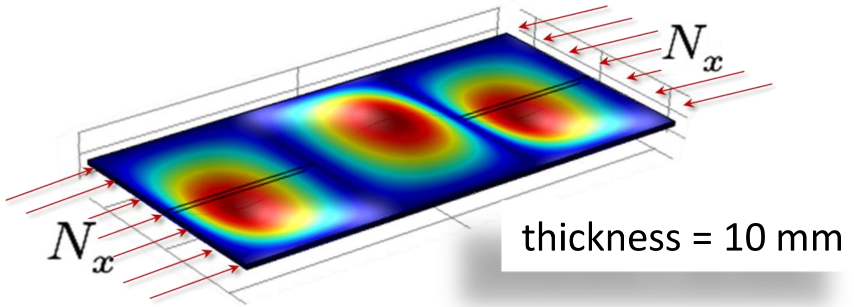
Buckling stresses and modes

$$N_x = \lambda N_{cr}$$

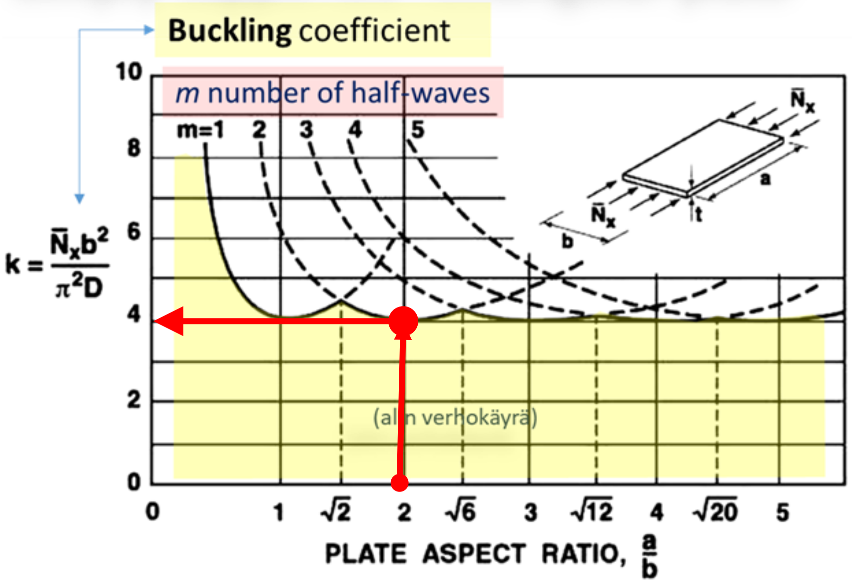
$$E = 70 \text{ GPa}, \quad \nu = 0.33$$

$$a = 1\text{m}, \; b = a/2, \quad t = 10 \text{ mm}$$

$$N_y = N_{xy} = 0$$

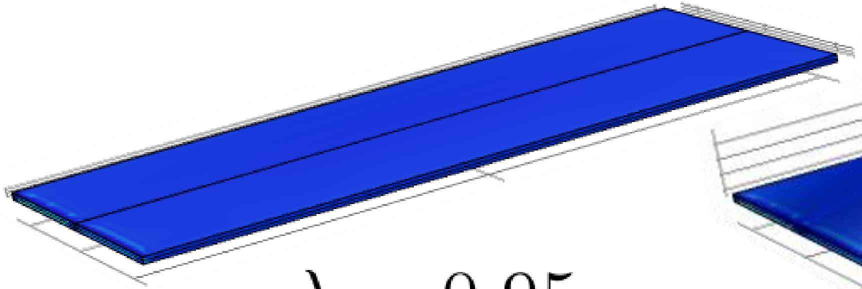


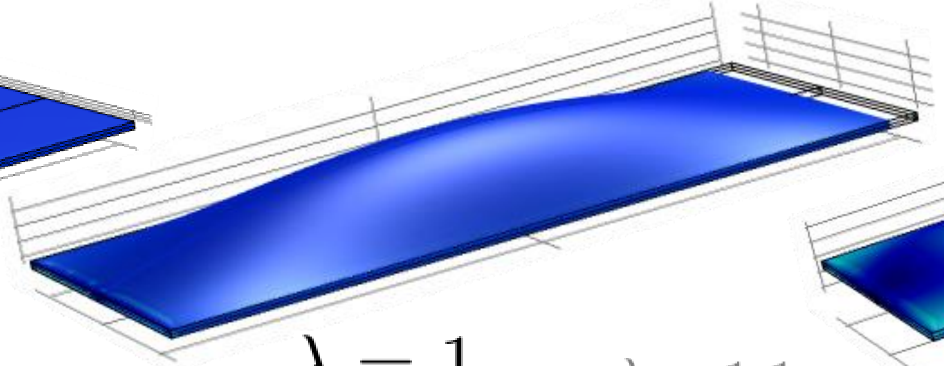
Simply supported rectangular plate

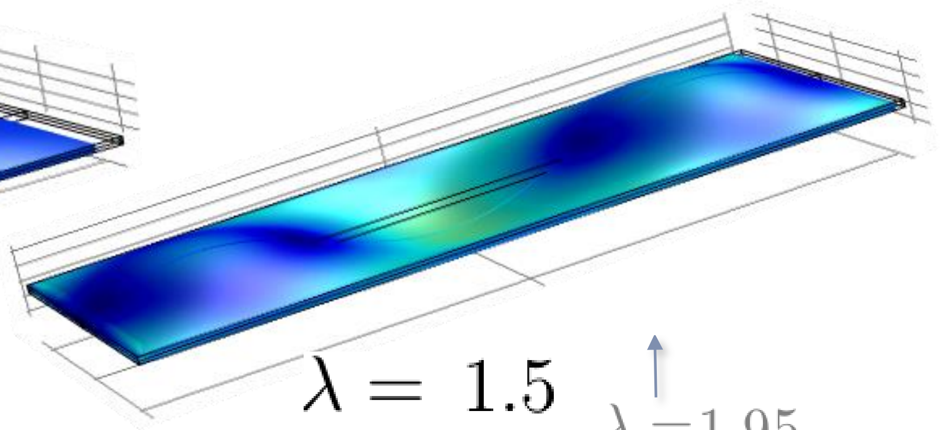


$\sigma_{cr,2D} = 4\pi^2 D/b^2 = 103.4$ MPa.

Post-Buckling Analysis







$\lambda = 0.95$

Remains in initial flat state

$\lambda = 1$

1st bifurcation

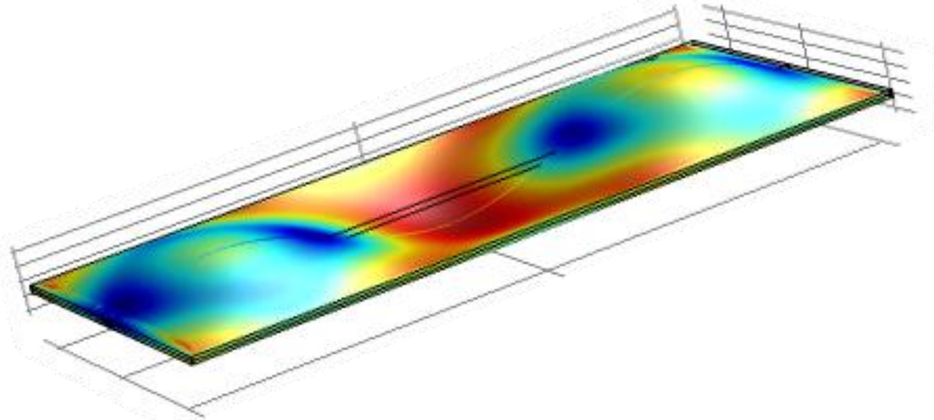
$\lambda = 1.1$

2nd bifurcation was not observed in simulation (unstable mode)

$\lambda = 1.5$

3rd bifurcation

$\lambda = 1.95$



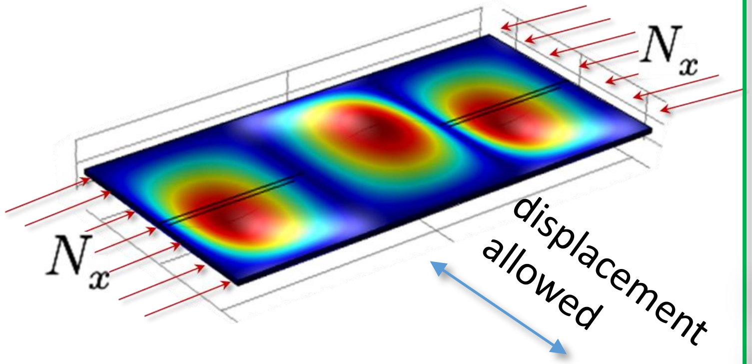
$N_x = \lambda N_{cr}$ $N_{cr} = 70 \text{ MPa}$

$E = 70 \text{ GPa}, \nu = 0.33$

$a = 1\text{m}, \ b = a/2, \quad t = 10 \text{ mm}$

$N_y = N_{xy} = 0$

Simply supported rectangular plate

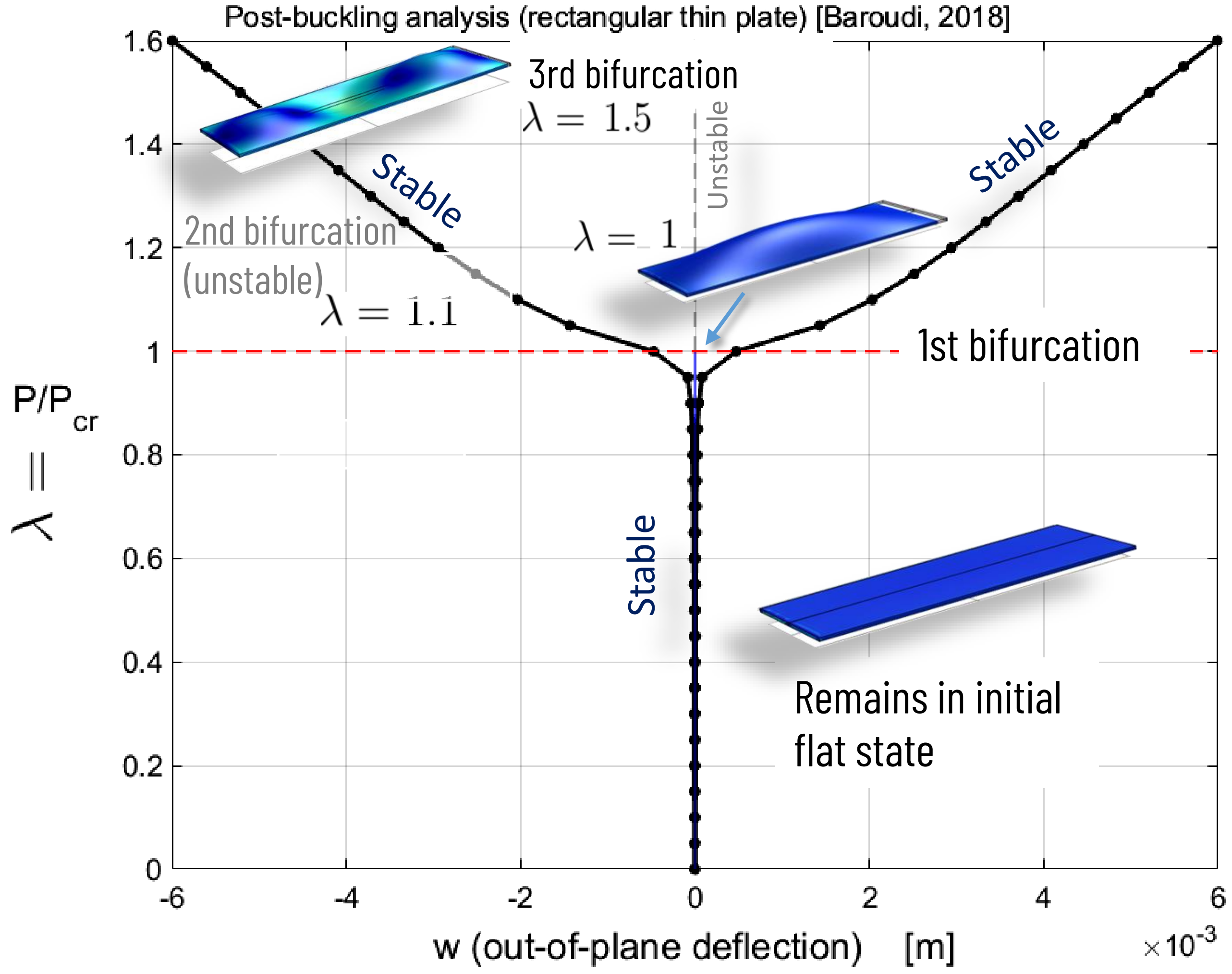


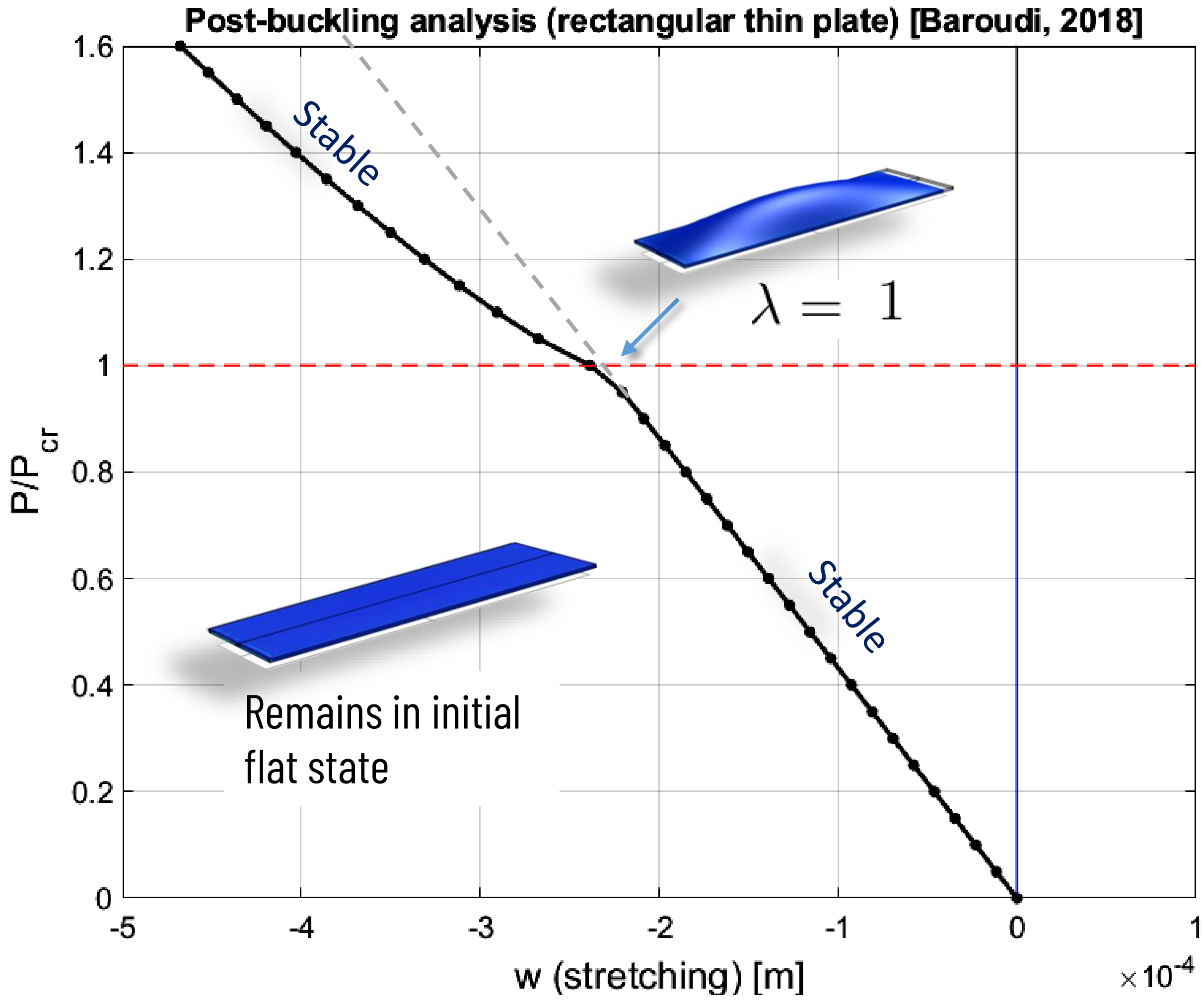
Displacements are multiplied by a scale factor which varies to make the deflections visible, especially, the first buckling

Post-Buckling Analysis

2nd bifurcation (unstable)
was not observed in simulation

Voima-ohjattuna





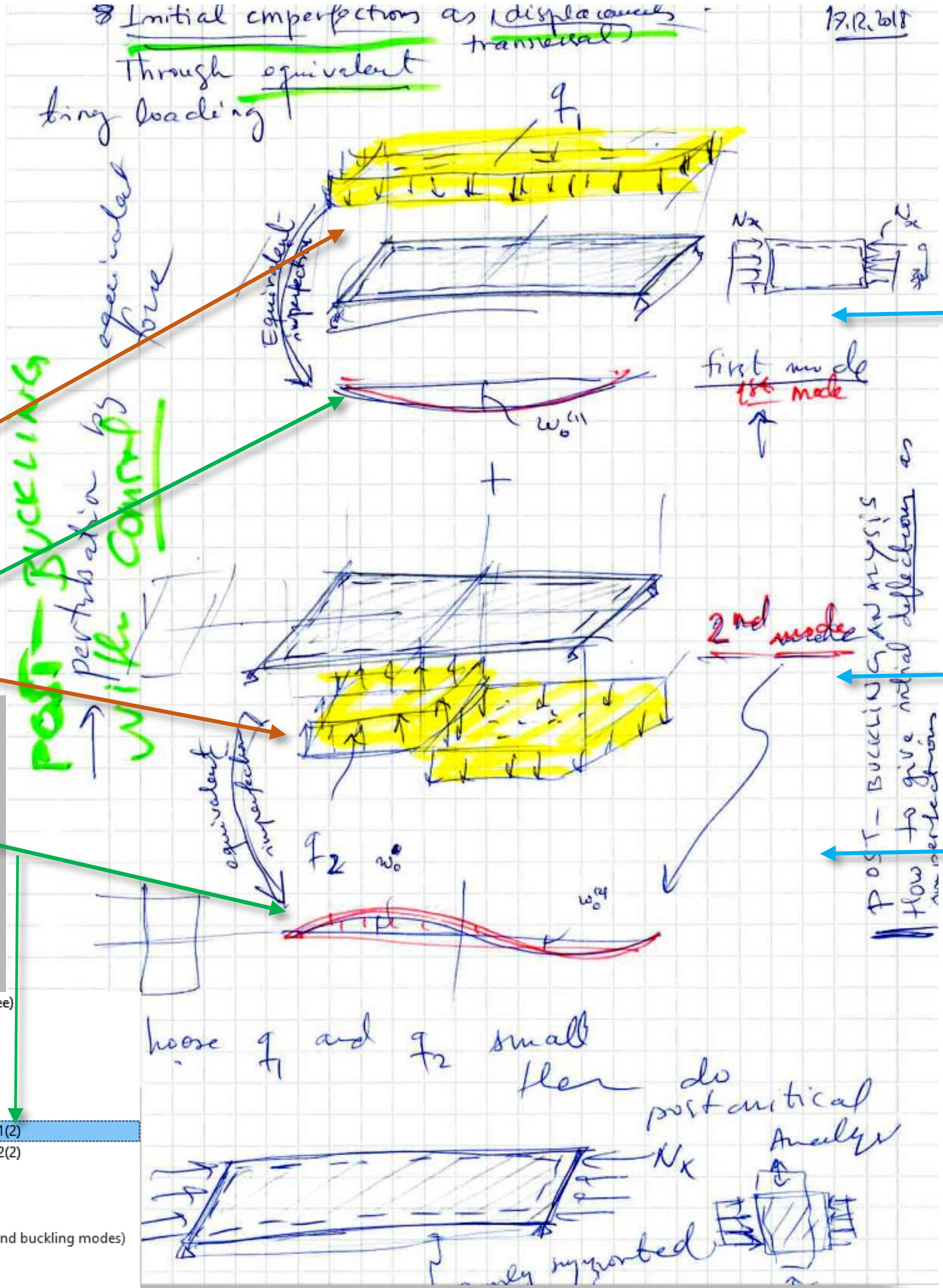
Computational Post-Buckling Analysis

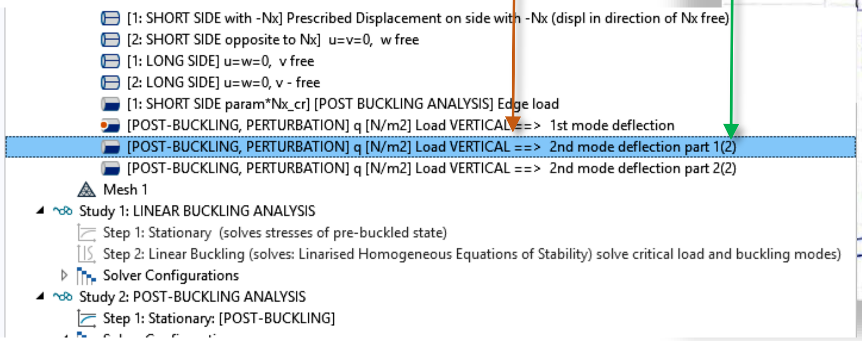
How to do?

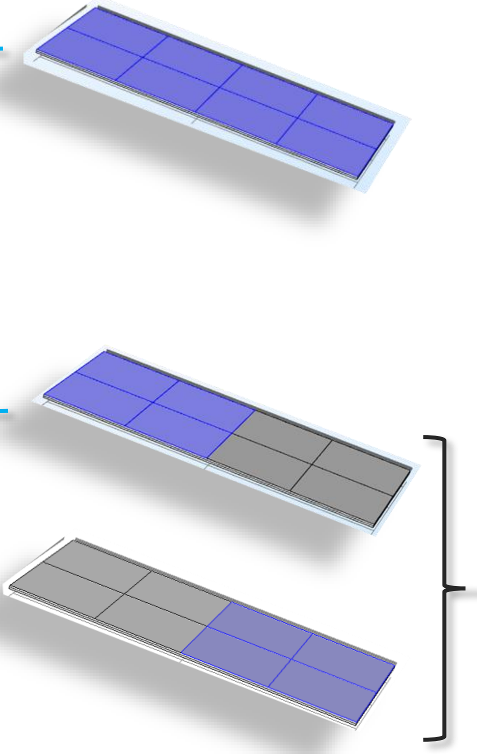
- How to give **easily** (lazily) initial tiny displacement shape as a perturbation? Usually, a combination of Eigen-modes is used. A random combination can be also used.

- One idea is to use equivalent tiny combination of loads resulting in equivalent deformations

Generating a tiny perturbation of the primary equilibrium configuration





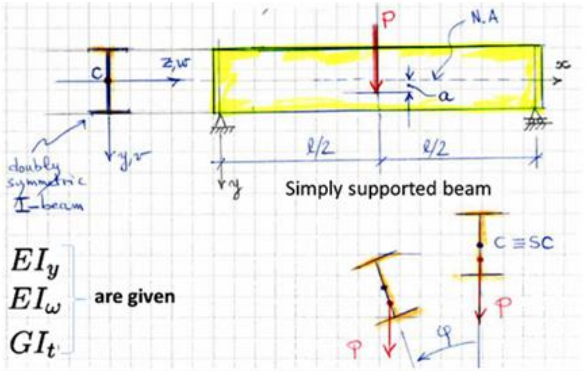


Voima-ohjattuna

Appendix & Miscellaneous

In a bit disorder now … will be updated

Lateral-torsional buckling by Rayleigh-Ritz




```
ϕ[x_] := B Sin[π x/L]
w[x_] := A Sin[π x/L]
Mz[x_] := P/2 x
```

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \int_0^\ell EI_\omega \phi''^2 \mathrm{d}x + \int_0^\ell (M_z^0\phi)' w' \mathrm{d}x + 1/2 Pa\phi(\ell)^2$$

```
Energy = 1/2 ∫_0^L (e Iy (w''[x])^2 + g It (ϕ'[x])^2 + e Iw (ϕ''[x])^2) dx + 2 ∫_0^(L/2) (D[Mz[x] ϕ[x], x]) w'[x] dx + 1/2 P a B^2
```

$\frac{1}{2} a B^2 P + \frac{1}{16} A B P (4 + \ldots \frac{\ldots^2 e\,Iy\,\pi^4 + B^2 (g\,It\,L^2 \pi^2 + e\,Iw\,\pi^4)}{4 L^3}$

```
Collect[D[Energy, A] // FullSimplify, {A, B}]
Collect[D[Energy, B] // FullSimplify, {A, B}]
```

Exact analytical solution:

$$\frac{P_E \ell}{4M_{ref}} \approx 1.35\left[\sqrt{1 + [0.54 P_{E,y} a/M_{ref}]^2} + 0.54 P_{E,y} a/M_{ref}\right]$$

$$M_{ref} = \sqrt{P_{E,y}[GI_t + \pi^2 EI_\omega/\ell^2]}$$

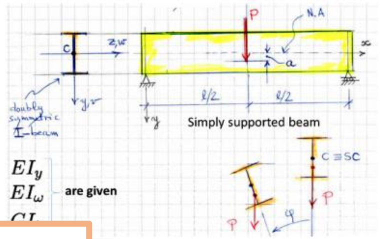
$$P_{E,y} = \pi^2 EI_y/\ell^2$$

$\frac{1}{16} A P (-4 + \pi^2 \ldots \ldots e\,Iw\,\pi^4)$

$$\begin{pmatrix} \frac{e\,Iy\,\pi^4}{2 L^3} & \ldots \\ \frac{1}{16} P (-4 + \pi^2) & \left(a P + \frac{g\,It\,L^2 \ldots}{2 L^3}\right. \end{pmatrix};$$

Criticality means that the **determinant vanishes.** The roots of the quadratic polynomial in *P* gives the critical loads. The smallest one is the buckling load.

Lateral-torsional buckling by Rayleigh-Ritz




```
ϕ[x_] := B Sin[π x/L]
w[x_] := A Sin[π x/L]
Mz[x_] := P/2 x
```

$$\Delta\Pi = \frac{1}{2}\int_0^{\ell} EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^{\ell} GI_t \phi'^2 \mathrm{d}x + \frac{1}{2}\int_0^{\ell} EI_\omega \phi''^2 \mathrm{d}x + \int_0^{\ell} (M_z^0 \phi)' w' \mathrm{d}x + 1/2 Pa\phi(\ell)^2$$

```
Energy = 1/2 ∫_0^L (e Iy (w''[x])^2 + g It (ϕ'[x])^2 + e Iw (ϕ''[x])^2) dx + 2 ∫_0^(L/2) (D[Mz[x] ϕ[x], x]) w'[x] dx + 1/2 P a B^2
```

$$\frac{1}{2} a B^2 P + \frac{1}{16} A B P (4 + \pi^2) + \frac{A^2 e\, Iy\, \pi^4 + B^2 (g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4)}{4 L^3}$$

```
Collect[D[Energy, A] // FullSimplify, {A, B}]
Collect[D[Energy, B] // FullSimplify, {A, B}]
```

$$\frac{A\, e\, Iy\, \pi^4}{2 L^3} + \frac{1}{16} B P (-4 + \pi^2)$$

$$\frac{1}{16} A P (-4 + \pi^2) + B \left(a P + \frac{g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4}{2 L^3}\right)$$

Exact analytical solution:

$$\frac{P_E \ell}{4 M_{ref}} \approx 1.35 \left[\sqrt{1 + [0.54 P_{E,y} a / M_{ref}]^2} + 0.54 P_{E,y} a / M_{ref}\right]$$

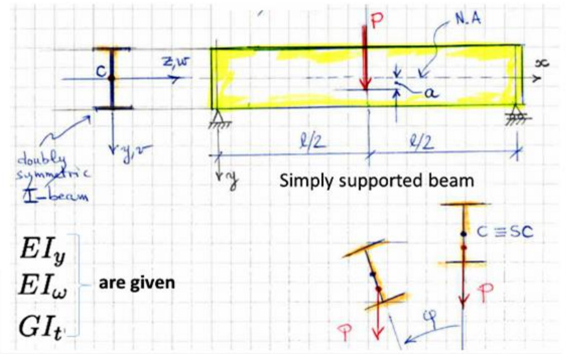
$$M_{ref} = \sqrt{P_{E,y}[GI_t + \pi^2 EI_\omega/\ell^2]}$$

$$P_{E,y} = \pi^2 EI_y/\ell^2$$

$$\begin{pmatrix} \frac{e\, Iy\, \pi^4}{2 L^3} & \frac{1}{16} P (-4 + \pi^2) \\ \frac{1}{16} P (-4 + \pi^2) & \left(a P + \frac{g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4}{2 L^3}\right) \end{pmatrix};$$

Criticality means that the **determinant vanishes**. The roots of the quadratic polynomial in *P* gives the critical loads. The smallest one is the buckling load.

Lateral-torsional buckling by Rayleigh-Ritz




```
ϕ[x_] := B Sin[π x/L]
w[x_] := A Sin[π x/L]
Mz[x_] := P/2 x
```

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \int_0^\ell EI_\omega \phi''^2 \mathrm{d}x + \int_0^\ell (M_z^0\phi)' w' \mathrm{d}x + 1/2 Pa\phi(\ell)^2$$

```
Energy = 1/2 ∫_0^L (e Iy (w''[x])^2 + g It (ϕ'[x])^2 + e Iw (ϕ''[x])^2) dx + 2 ∫_0^(L/2) (D[Mz[x] ϕ[x], x]) w'[x] dx + 1/2 P a B^2
```

$$\frac{1}{2} a B^2 P + \frac{1}{16} A B P \left(4+\pi^2\right) + \frac{A^2 e\, Iy\, \pi^4 + B^2 \left(g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4\right)}{4 L^3}$$

```
Collect[D[Energy, A] // FullSimplify, {A, B}]
Collect[D[Energy, B] // FullSimplify, {A, B}]
```

$$\frac{A\, e\, Iy\, \pi^4}{2 L^3} + \frac{1}{16} B P \left(-4+\pi^2\right)$$

$$\frac{1}{16} A P \left(-4+\pi^2\right) + B\left(a P + \frac{g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4}{2 L^3}\right)$$

Exact analytical solution:

$$\frac{P_E \ell}{4M_{ref}} \approx 1.35\left[\sqrt{1+[0.54 P_{E,y} a/M_{ref}]^2} + 0.54 P_{E,y} a/M_{ref}\right]$$

$$M_{ref} = \sqrt{P_{E,y}[GI_t + \pi^2 EI_\omega/\ell^2]}$$

$$P_{E,y} = \pi^2 EI_y/\ell^2$$

$$\begin{pmatrix} \frac{e\, Iy\, \pi^4}{2 L^3} & \frac{1}{16} P \left(-4+\pi^2\right) \\ \frac{1}{16} P \left(-4+\pi^2\right) & \left(a P + \frac{g\, It\, L^2 \pi^2 + e\, Iw\, \pi^4}{2 L^3}\right) \end{pmatrix};$$

Criticality means that the **determinant vanishes.** The roots of the quadratic polynomial in *P* gives the critical loads. The smallest one is the buckling load.

$$\begin{pmatrix} \frac{e\,Iy\,\pi^4}{2\,L^3} & \frac{1}{16} P\left(-4+\pi^2\right) \\ \frac{1}{16} P\left(-4+\pi^2\right) & \left(a\,P + \frac{g\,It\,L^2\,\pi^2 + e\,Iw\,\pi^4}{2\,L^3}\right) \end{pmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad a)$$

$$\frac{\pi^2}{2\,L} \begin{pmatrix} \frac{e\,Iy\,\pi^2}{L^2} & \frac{1}{16}\,\frac{2\,L}{\pi^2}\,P\left(-4+\pi^2\right) \\ \frac{1}{16}\,\frac{2\,L}{\pi^2}\,P\left(-4+\pi^2\right) & \left(a\,P\,\frac{2\,L}{\pi^2} + g\,It\,\frac{e\,Iw\,\pi^2}{L^2}\right) \end{pmatrix}; \qquad b)$$

$$AA := \frac{\pi^2}{2\,L} \begin{pmatrix} Pey & P\,L\,\frac{\left(-4+\pi^2\right)}{8\,\pi^2} \\ P\,L\,\frac{\left(-4+\pi^2\right)}{8\,\pi^2} & \left(a\,P\,\frac{2\,L}{\pi^2} + \frac{M^2}{Pey}\right) \end{pmatrix}; \qquad c)$$

$$M_{ref} = \sqrt{P_{E,y}[GI_t + \pi^2 EI_\omega/\ell^2]}$$

$$P_{E,y} = \pi^2 EI_y/\ell^2$$

9.2 LATERAL STABILITY OF THE NODES OF PLANE TRUSSES

It is well known that the nodes of the compressed chords of trusses have to be supported against lateral displacement. However, it is less well known that in some cases also the nodes of the *chords in tension* need such a support. In the following we will present the criterion of KIRSTE [1950], which allow us to decide whether such a support is necessary or not.

We assume that the nodes of the truss have spherical hinges. Let us give a virtual displacement v to one of the nodes, denoted by C *(Fig. 9–10)*. Supposing that all neighbouring nodes are rigidly supported against lateral displacement, the restoring force V acting on the node C is given by the expression

$$V = v \sum_i \frac{N_i}{l_i}, \tag{9–43}$$

where N_i is the bar force (positive if tension) acting in the ith bar joining the node C, and l_i is the length of the ith bar.

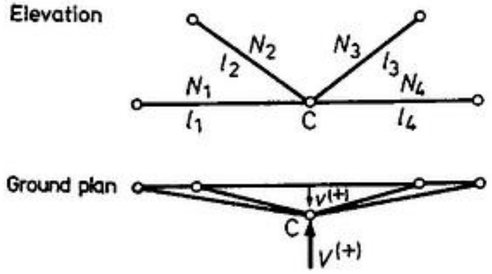


Fig. 9–10 A node of a truss

The original position of the node is stable if $\sum_i \frac{N_i}{l_i}$ has a positive sign, since in this case V becomes a restoring force. If this sum is equal to zero, then the position of the node is indifferent, and if the sum has a negative sign, then the node is unstable since V pushes it further in the direction of the displacement.

In [KIRSTE, 1950] we also find the method to deal with the case if the neighbouring nodes are not rigidly supported laterally.

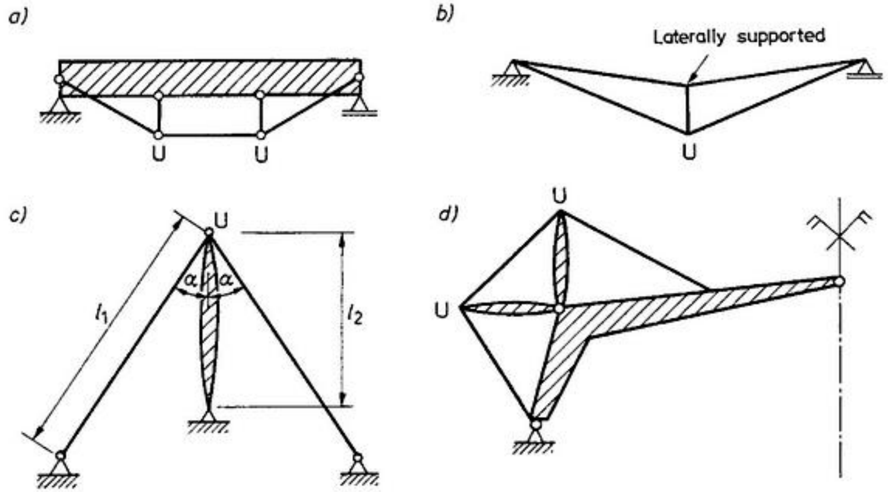
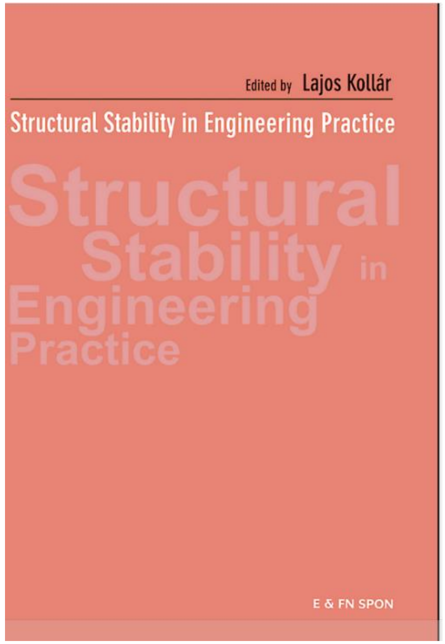


Fig. 9–11 Structures with unstable nodes



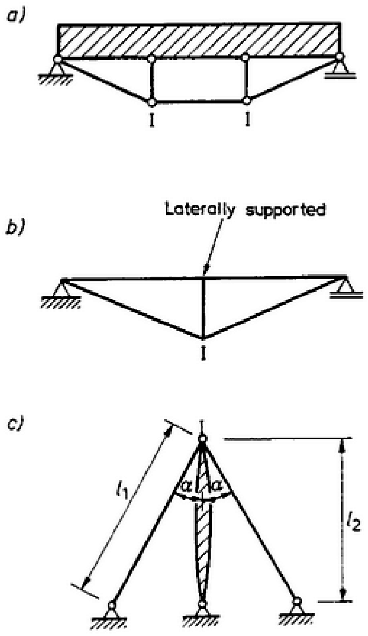


Fig. 9–12 Structures with indifferent nodes

The nodes of the actual trusses have no spherical hinges, nevertheless the criterion of *Kirste* is a good basis to decide whether the nodes need a lateral support or not. To illustrate the practical importance of this instability phenomenon, we show some examples. In *Fig. 9–11* structural arrangements with unstable nodes (U), in *Fig. 9–12* with indifferent ones (I), and in *Fig. 9–13* with stable ones (S) are depicted. Where we marked two nodes with U, there the two nodes displace simultaneously in lateral direction.

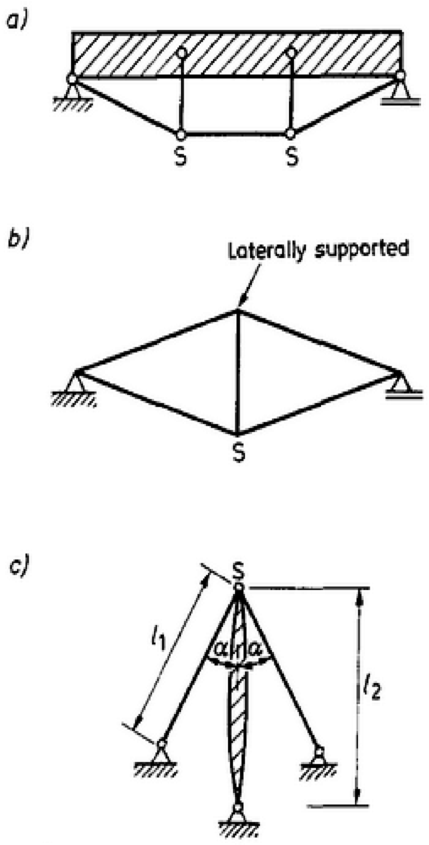


Fig. 9–13 Structures with stable nodes

To show the application of the method let us consider the structure of *Fig. 9–12c*, whose variants with longer and shorter prestressing cables are depicted in *Figs 9–11c* and *9–13c* respectively. After prestressing the cables, the vertical equilibrium of the upper node yields $N_2=-2N_1\cos\alpha$, where N_1 and N_2 are the forces arising in the inclined cables and in the vertical post, respectively. The sum appearing in Eq. (9–43) becomes:

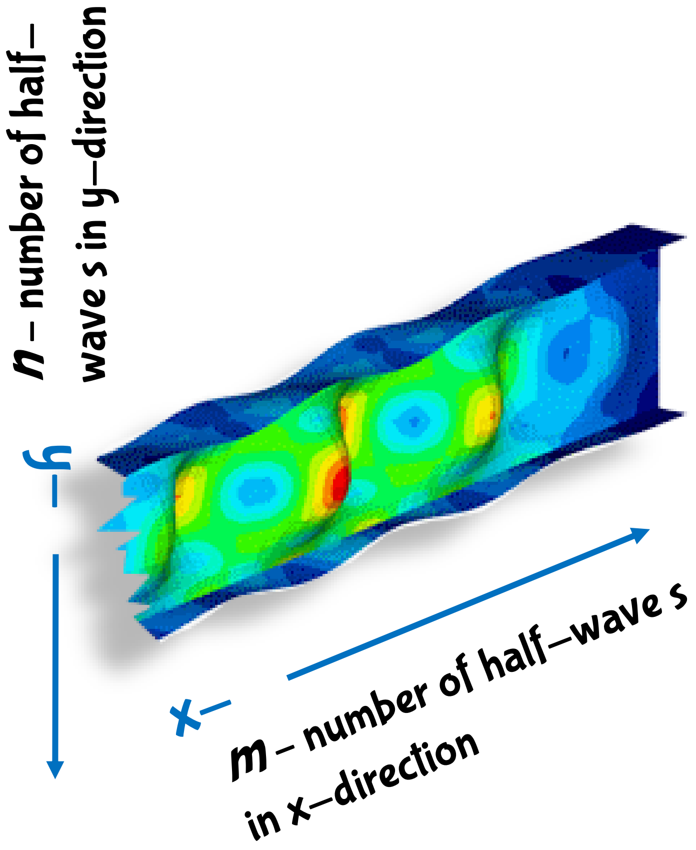
$$\sum_{i=1}^{2}\frac{N_i}{l_i}=2\frac{N_1}{l_1}+\frac{N_2}{l_2}=2\frac{N_1}{l_1}-2\frac{N_1}{l_2/\cos\alpha}.$$

This sum is positive if $l_1<l_2/\cos\alpha$, i.e. in the case of *Fig. 9–13c*, negative if $l_1>l_2/\cos\alpha$, i.e. in the case of *Fig. 9–11c;* and equal to zero in the case of *Fig. 9–12c.*

In practice it is not always enough to know that the node is stable, but we may also ask: how stable is it? That is, if the stable state of equilibrium is close to the indifferent (neutral) state, then the node may be considered almost as 'unstable' as if it were indeed in an unstable state.

TOMKA [1997] presented a very simple method to establish the 'measure of stability' of a node of a truss or cable structure. He applies two equal but opposite auxiliary forces S on the compressed bar(s) joining the node *(Fig. 9–14)*, and determines their value necessary to bring the node into an indifferent state of equilibrium. If the necessary forces S cause compression, the structure is stable, and the ratio of S to the actual compressive force acting in the bar yields the measure of stability.

Buckling of a simply supported rectangular plate - in-plane bending and compression



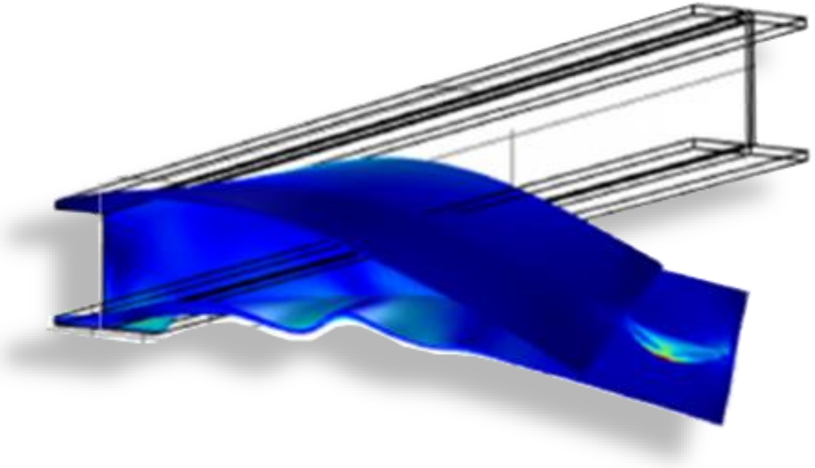


Plate buckling