Elastic Stability of Structures

CIV-E4100 - Stability of Structures D, 01.03.2021-18.04.2021

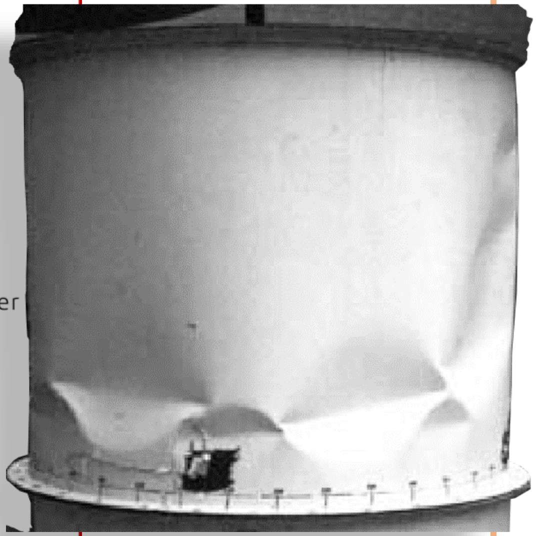
Week #5-6 – Lecture series

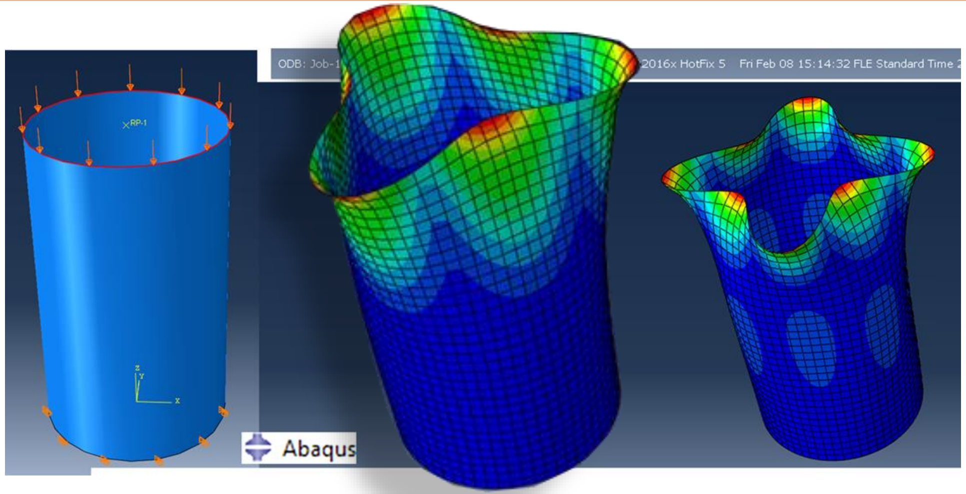
Stability of cylindrical shells

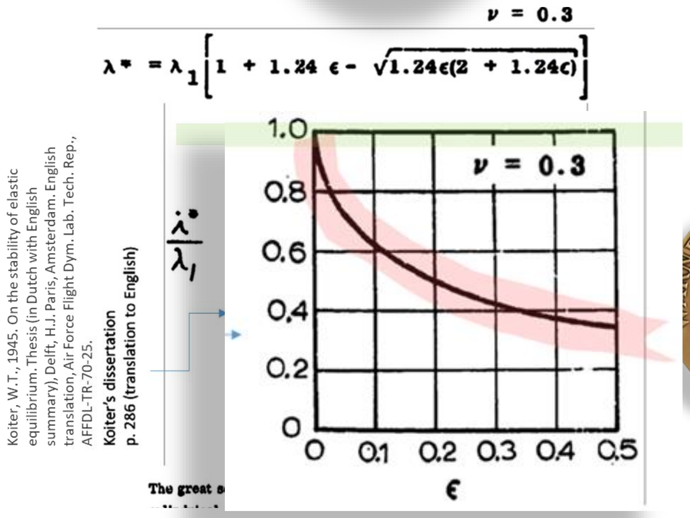
- What makes shells imperfection-sensitive structures? Why Plates are not?
  - ➢ Recall: types of bifurcational instabilities
- **Buckling of axisymmetric cylindrical shells**
- Equilibrium equation of axisymmetric cylindrical shells
  - ➢ Large deflection Donnell-type theory
- Energy criteria for stability loss of thin cylindrical shell
- Deriving stability loss equation
- The linear stability loss equations
- Axisymmetric buckling of circular cylindrical shells under uniform axial compression
- Buckling solution using Donnell's equations for axially compressed thin cylinder
- Computational Linear buckling analysis
  - ➢ Finite Element Example Buckling of thin-walled cylindrical shells
- Post-buckling behavior of thin shells
  - ➢ Effect of imperfection on post-buckling behavior
  - ➢ Effects of initial geometric imperfections on stability of thin shells

DO NOT MISS this course:

- ❑ **CIV–E4080: New course on material modelling – constitutive modellings**
- ❑ **CIV-E4080 - Material Modelling in Civil Engineering L,**







Lecture slides for internal use only

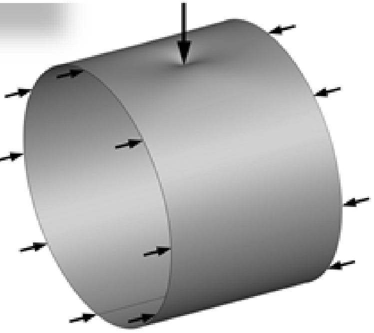
D. Baroudi, Dr. & Eng. (DI)

Donnell large-deflection (stability) equation

$$D\nabla^8 w + \frac{Eh}{R^2}\frac{\partial^4 w}{\partial x^4} + 2Sh\nabla^4\left(\frac{\partial^2 w}{\partial x \partial s}\right) = 0$$

Torsion of thin cylinders

Donnel: Report # 479 (1933).



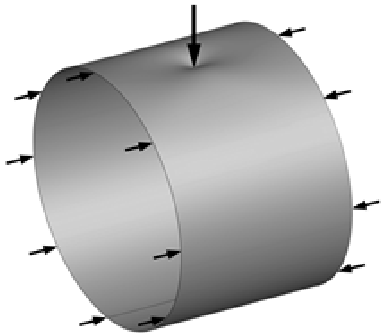
Version 6.4.2021

Content

STARTS

- What makes shells imperfection-sensitive structures? Why Plates are not?
  - Recall: types of bifurcational instabilities

- Buckling of axisymmetric cylindrical shells
- Equilibrium equation of axisymmetric cylindrical shells
  - Large deflection Donnell-type theory
- Energy criteria for stability loss of thin cylindrical shell
- Deriving stability loss equation
- The linear stability loss equations



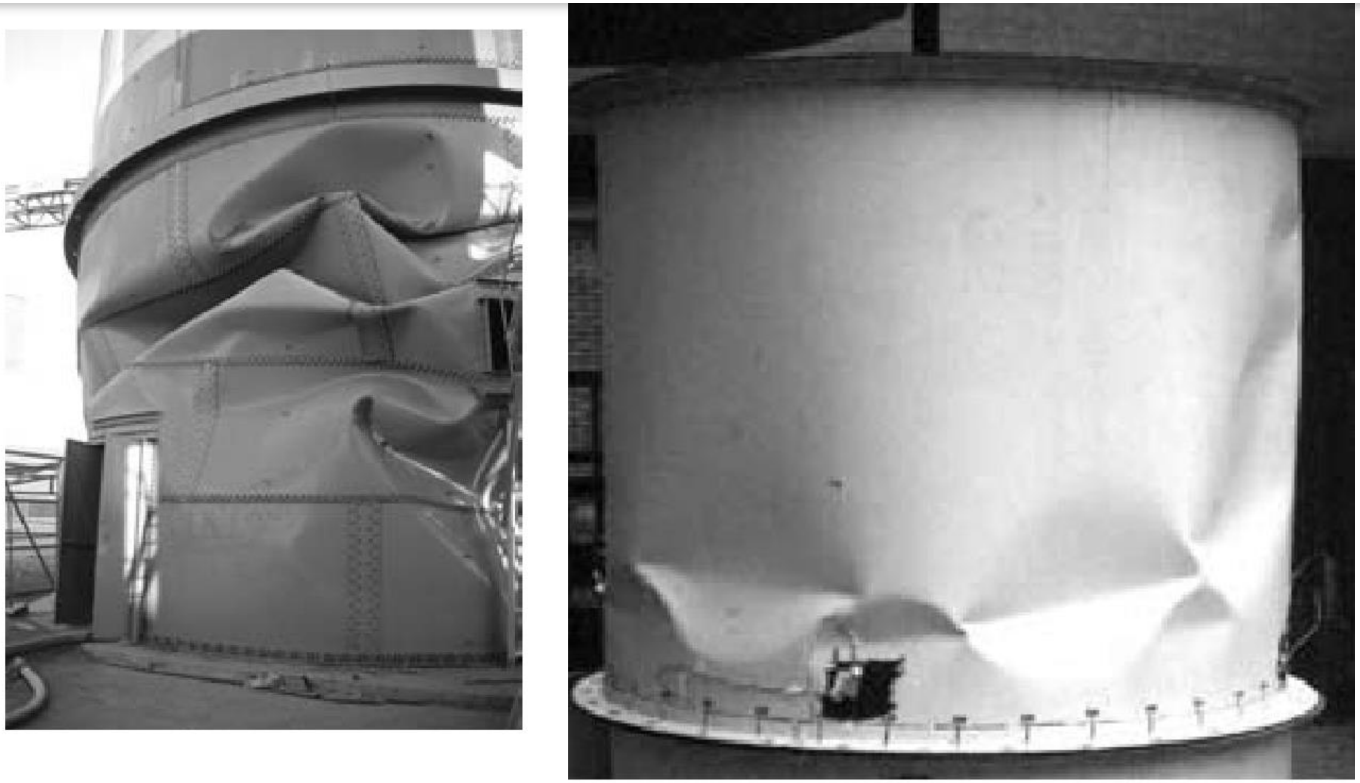
- Axisymmetric buckling of circular cylindrical shells under uniform axial compression
- Buckling solution using Donnell's equations for axially compressed thin cylinder
- Computational Linear buckling analysis
  - Finite Element Example Buckling of thin-walled cylindrical shells
- Post-buckling behavior of thin shells
  - Effect of imperfection on post-buckling behavior
  - Effects of initial geometric imperfections on stability of thin shells
- FE-based (non-linear) F.E.M. analysis of imperfection sensitivity

International Journal on Bifurcation & Chaos in Applied Science Engineering

DO NOT MISS this course:

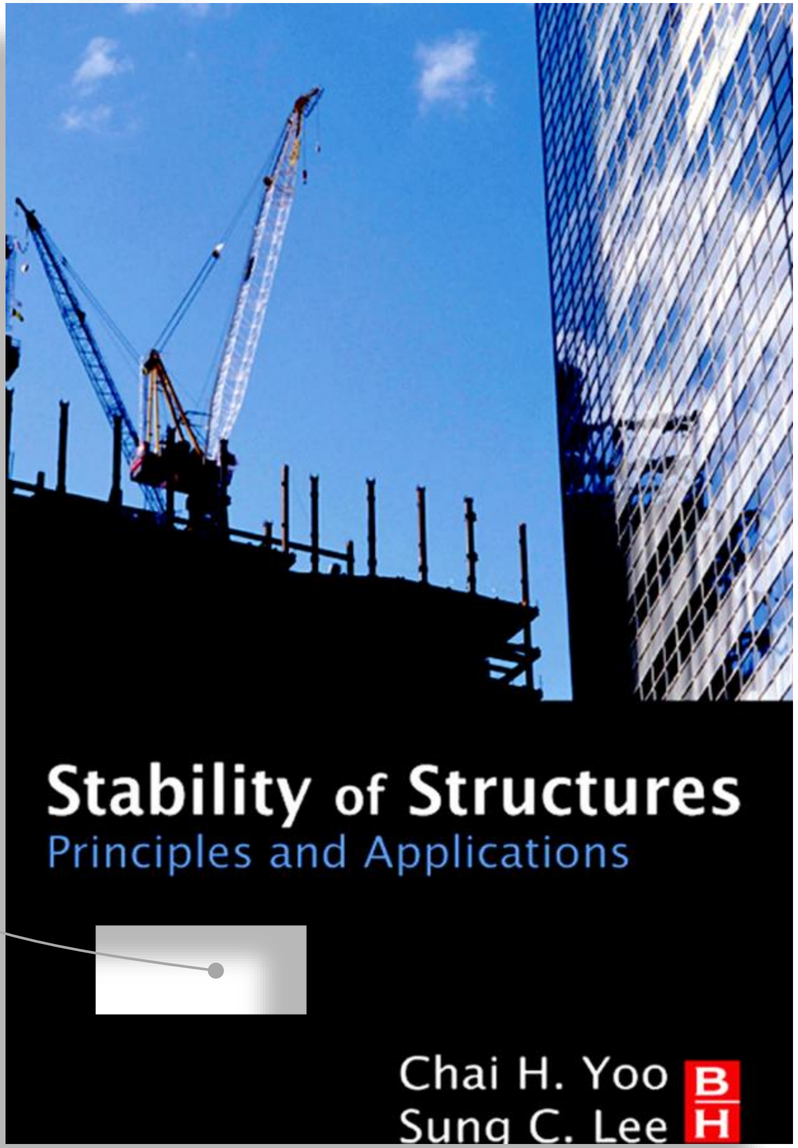
- ❑ CIV–E4080: New course on material modelling – constitutive modellings
- ❑ CIV-E4080 - Material Modelling in Civil Engineering L, 15.04.2019 to 27.05.2019

(a)

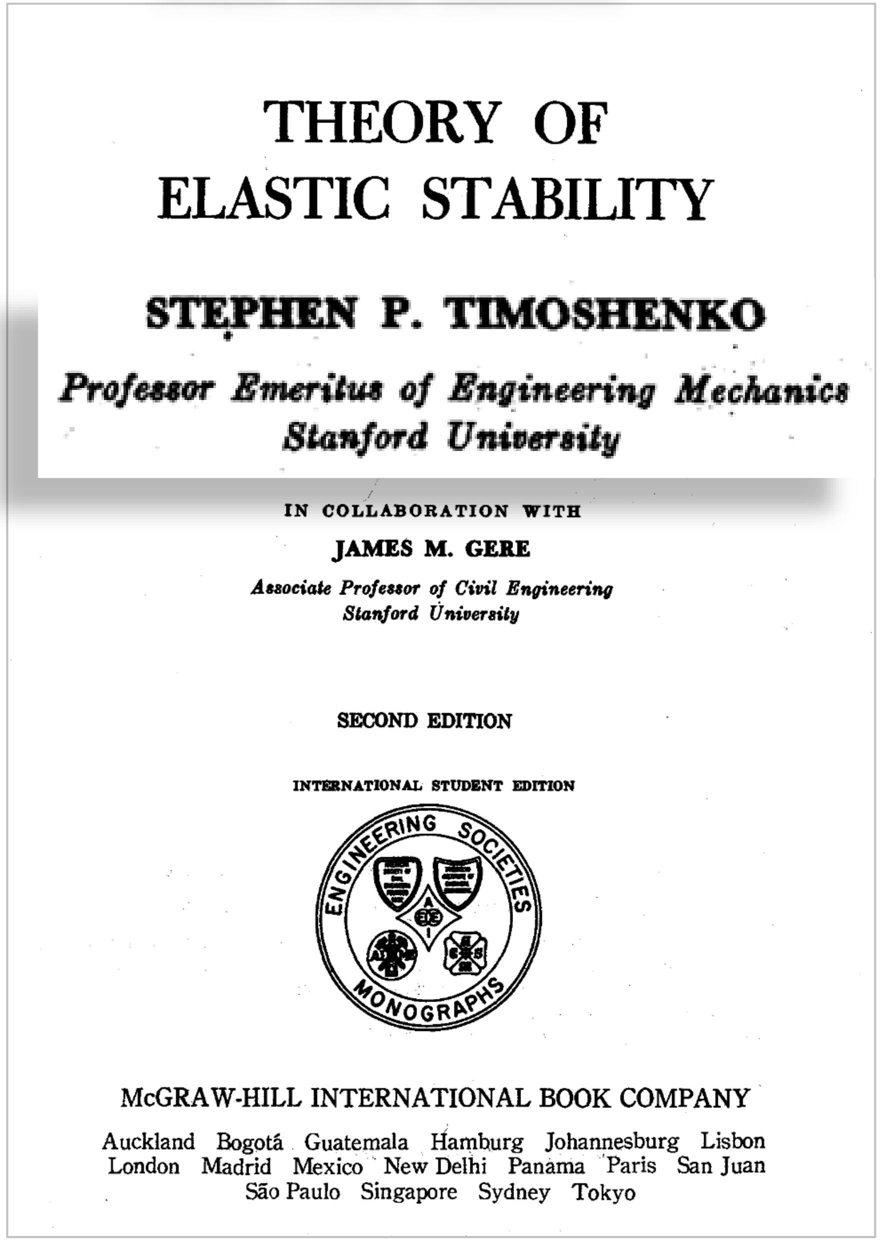


re 2.4 Typical appearance of axial compression buckles: (a) failure in service; (b) test
) in laboratory (Knödel and Schulz 1992).

This course textbook
e-book



Must read classics


THEORY OF ELASTIC STABILITY

STEPHEN P. TIMOSHENKO

Professor Emeritus of Engineering Mechanics
Stanford University

IN COLLABORATION WITH

JAMES M. GERE

Associate Professor of Civil Engineering
Stanford University

SECOND EDITION

INTERNATIONAL STUDENT EDITION

ENGINEERING SOCIETIES MONOGRAPHS

McGRAW-HILL INTERNATIONAL BOOK COMPANY

Auckland Bogotá Guatemala Hamburg Johannesburg Lisbon London Madrid Mexico New Delhi Panama Paris San Juan São Paulo Singapore Sydney Tokyo

This week

Chapter 9. Buckling of Thin Cylindrical Elements

Thin shell example - Ariane

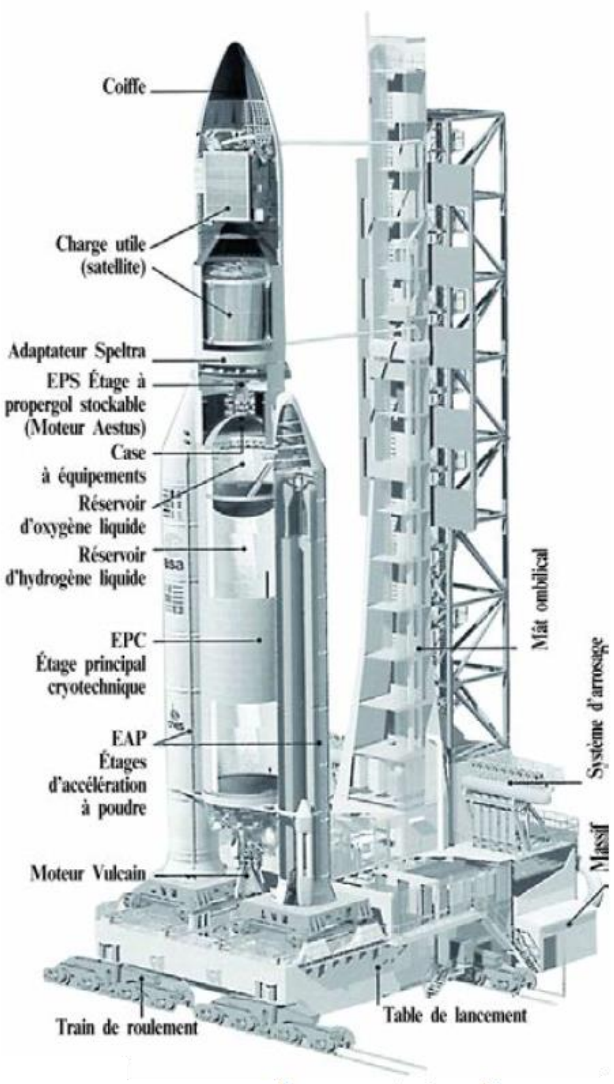


Photo du réservoir cryogénique

Jérôme Didier. Etude du comportement au flambage des coques cylindriques multicouches métal/matériau mousse sous chargements combinés pression interne/cisaillement/flexion. Mécanique [physics.med-ph]. INSA de Lyon, 2014. Français. NNT : 2014ISAL0069 . tel-01221817

Stability of cylindrical shells

Buckling of thin shells in aeronautics

- many launchers of space structures such cryogenic containers made of thin aluminium shells with extreemly light thick insulating material (thick foam). This is a sandwich-type of (multilayered) thin cylindrical shells.
- the ratio R/t can be more than 650
- **various load combinations**: internal pressure, wind load (bending & shear), own weight and weight of the liquid oxygene and hydrogen (axial load) in the static regime. For dynamic regimes, we should add acceleration forces

http://**theses**.insa-lyon [**J. Didier**], [2014], INSA de Lyon

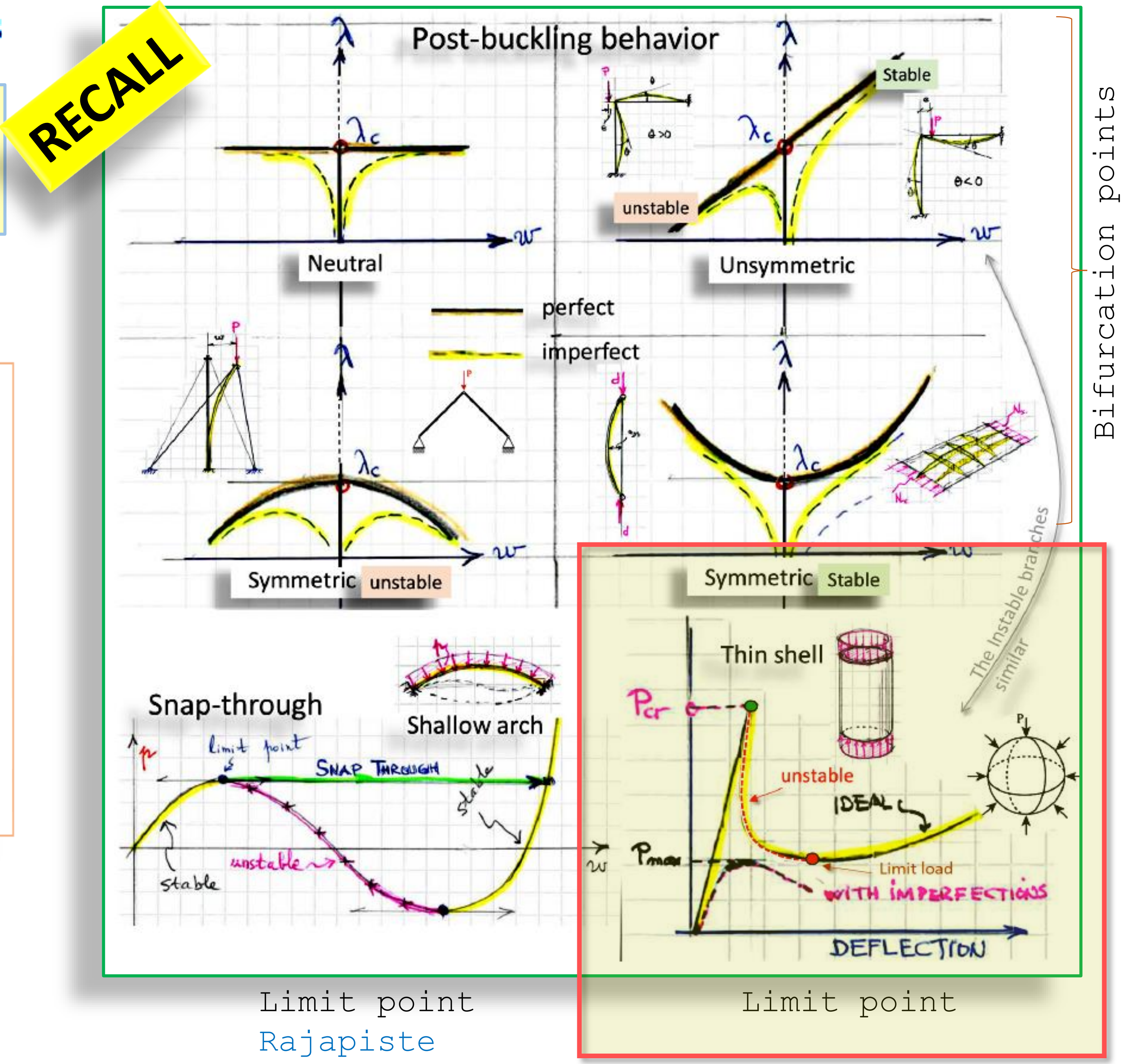
Types of bifurcational instabilities

- The nature of post-buckling behavior determines to a large extend safety and the robustness of the structural design

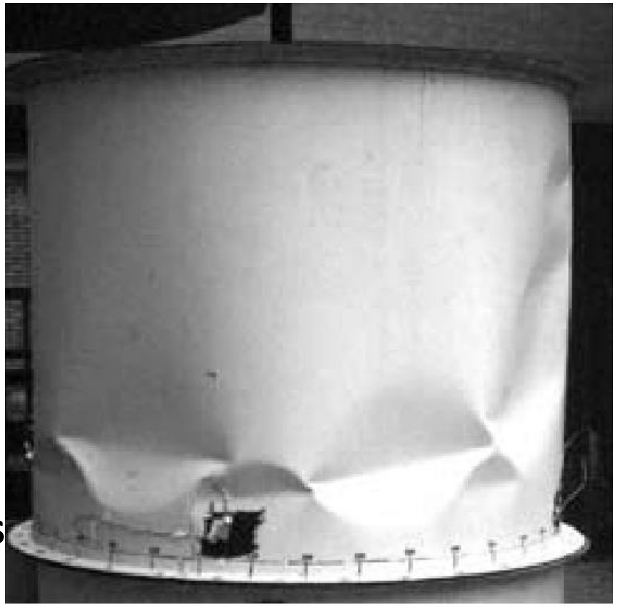
Basic types bifurcations Haarautuminen

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*
- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*
- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

- **Snap-through**
  - ✓ Such dynamic behavior is pathological not desired behavior and is locally like an asymmetric branching on equilibrium path

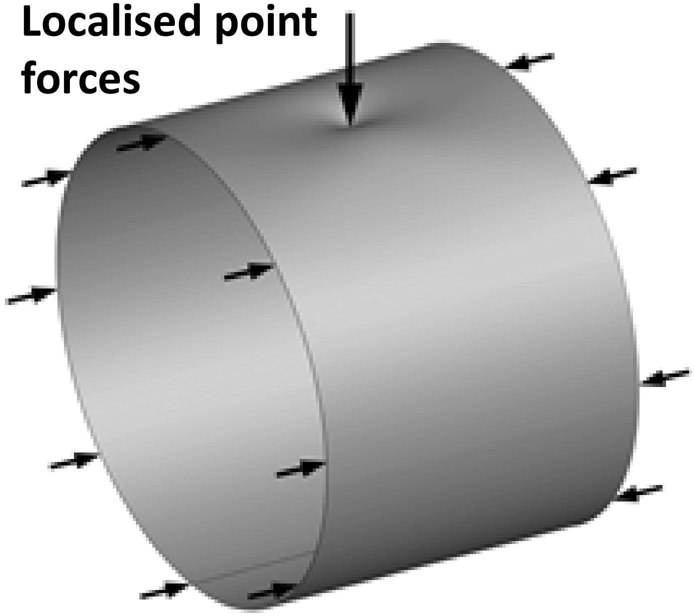


The next important observation is good to keep in mind: In general, and this observation is based on experimental evidences for decades, buckling of thin shells is *a localised phenomenon* due to their high imperfection sensitivity. Such imperfections can be localised loading, geometry imperfections of the shell, boundary conditions, local change in rigidity, in curvature, in supporting and load-transfer areas, etc. All this is one of the reasons for which experimental analysis and design of such structures is not avoidable despite availability of computational technology. All these imperfections should be characterized together with heir effects for the specific design of pre-design.

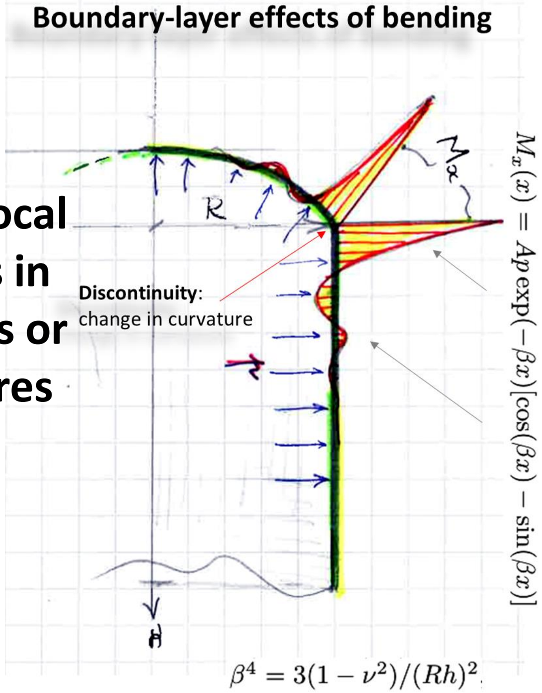


stiffer supports
(additional local bending)

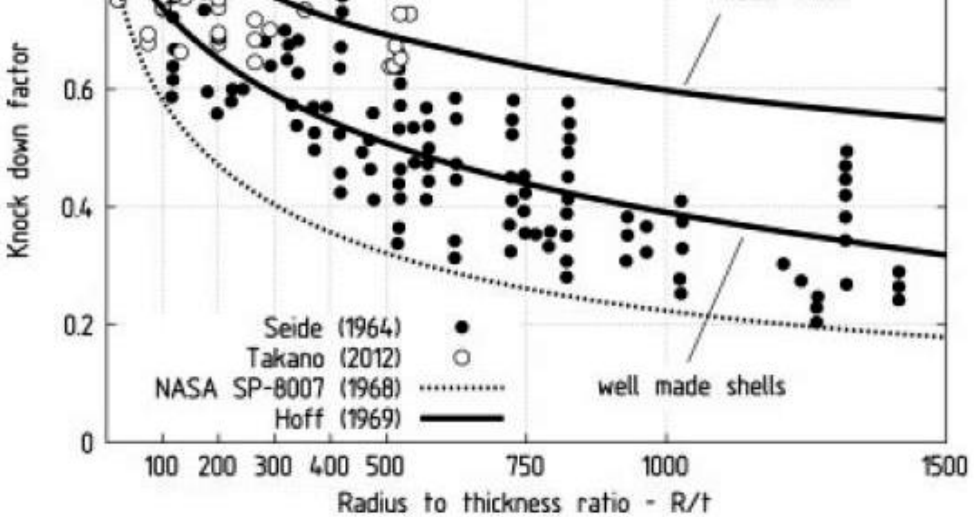
Localised point forces

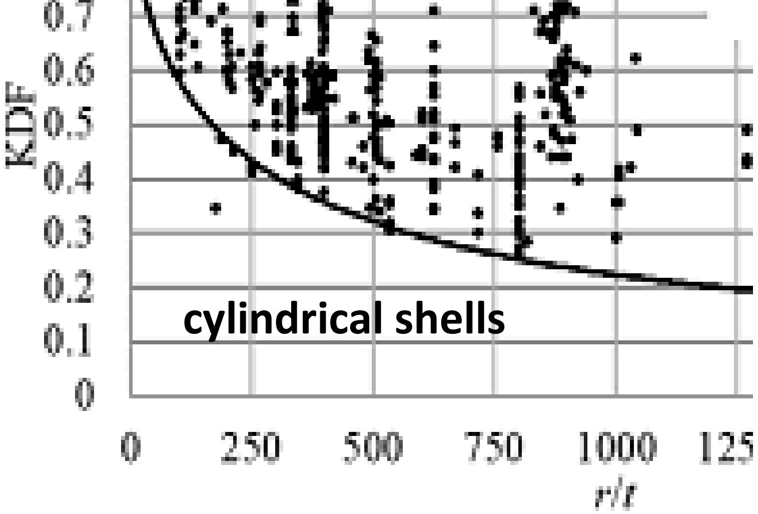


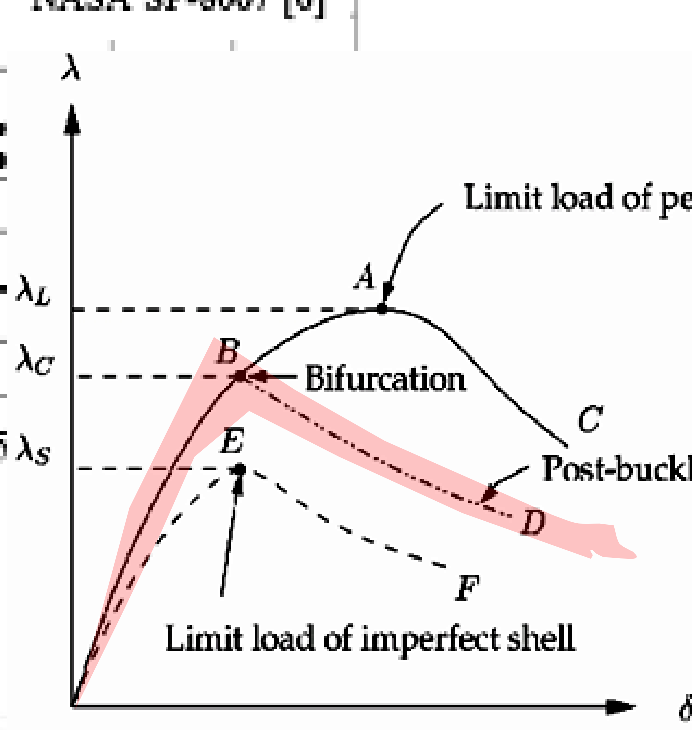
Brutal local changes in rigidities or curvatures

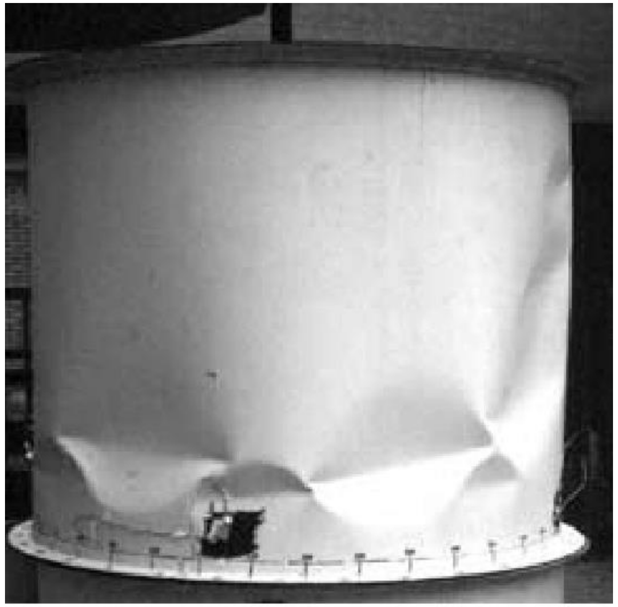


Knock-down fac



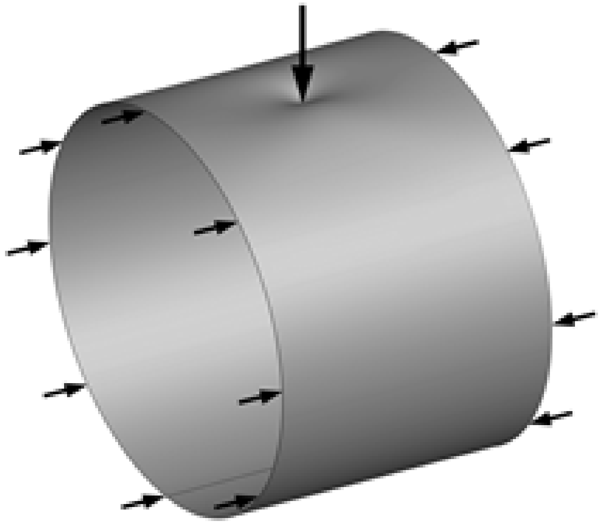




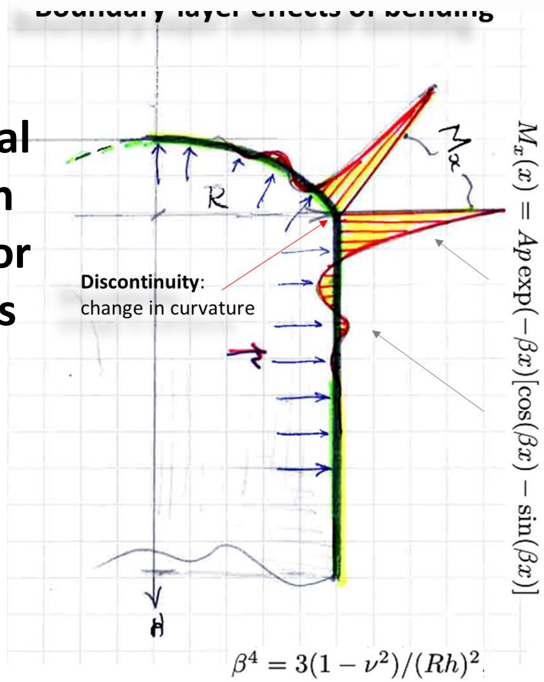


Stiffer supports
(additional local bending)

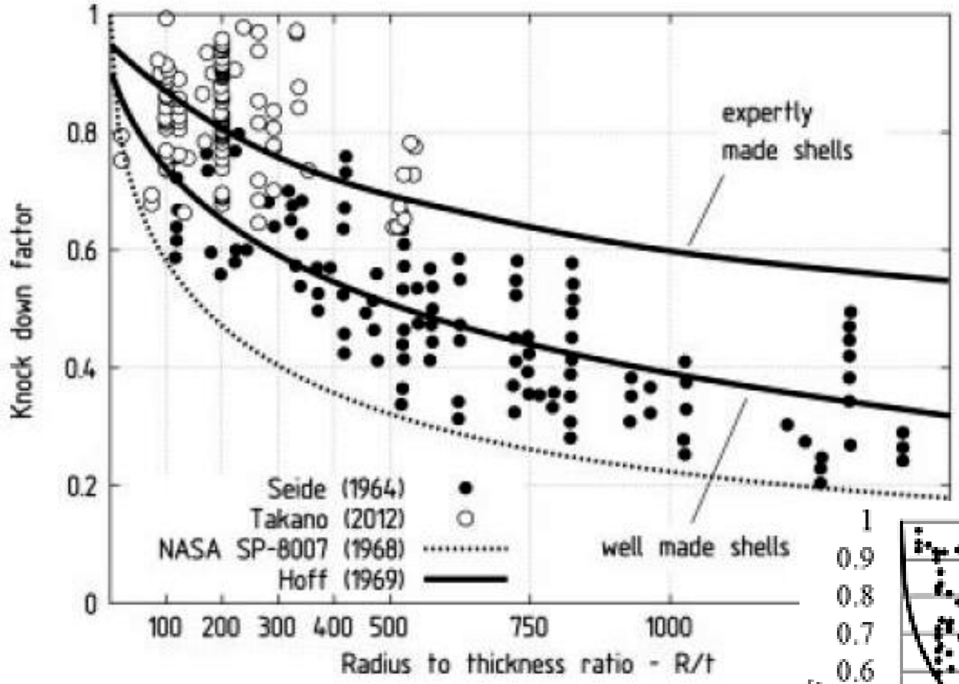
Localised point forces



Brutal local changes in rigidities or curvatures

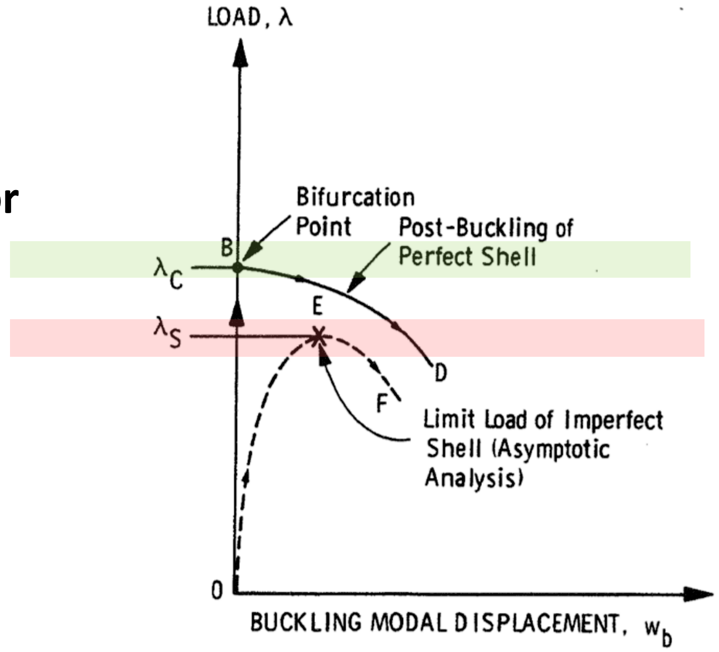


A key experimental fact about buckling of thin shells

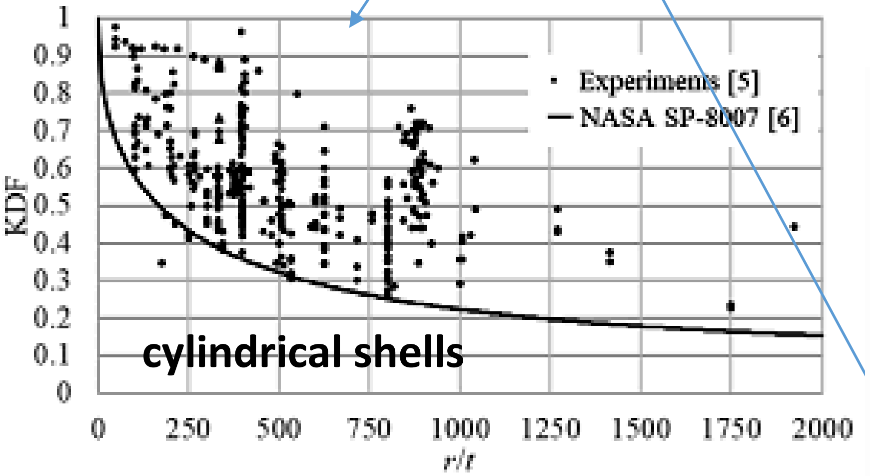


Limit load = Perfect * reduction factor

Reduction factor = knock down factor



Deflection of-mid-plane



[Latin American Journal of Solids and Structures](#)

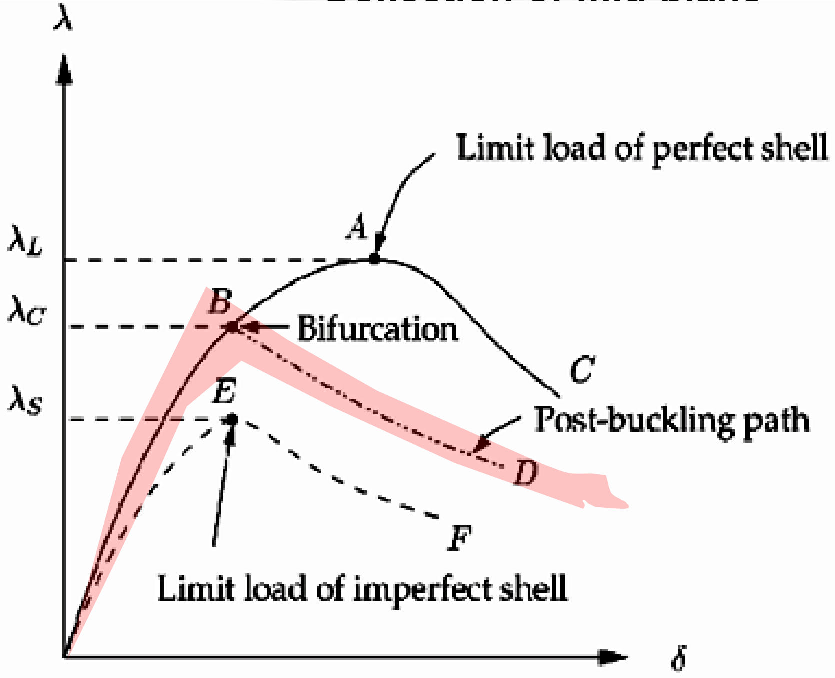
Print version ISSN 1679-7817 *On-line version* ISSN 1679-7825

Lat. Am. j. solids struct. vol.17 no.8 Rio de Janeiro 2020 Epub Dec 11, 2020

https://doi.org/10.1590/1679-78256197

ORIGINAL ARTICLE

An Overview of Buckling and Imperfection of Cone-Cylinder Transition under Various Loading Condition



The next important observation is good to keep in mind: In general, and this observation is based on experimental evidences for decades, buckling of thin shells is *a localised phenomenon* due to their high imperfection sensitivity. Such imper-

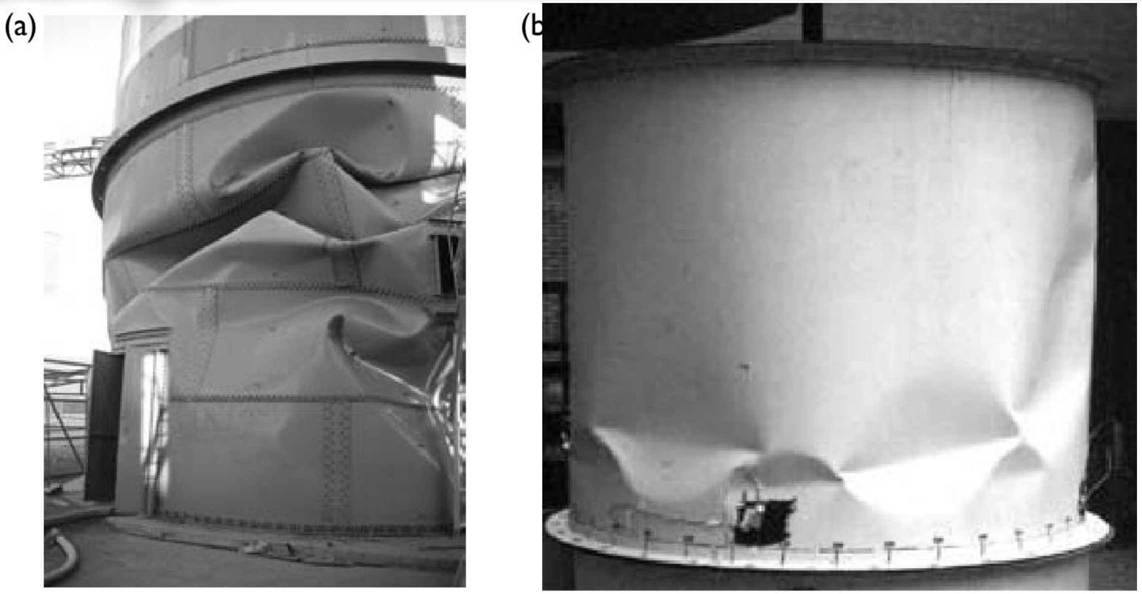
Eurocode 3 - Design of steel structures - Part 1-6: Strength and Stability of Shell Structures

Eurocode 3 - Calcul des structures en acier - Partie 1-6 : Résistance et stabilité des structures en coque

Eurocode 3 - Bemessung und Konstruktion von Stahlbauten - Teil 1-6: Festigkeit und Stabilität von Schalen

This amendment A1 modifies the European Standard EN 1993-1-6:2007; it was approved by CEN on 17 January 2017.

CEN members are bound to comply with the CEN/CENELEC Internal Regulations which stipulate the conditions for inclusion of this amendment into the relevant national standard without any alteration. Up-to-date lists and bibliographical references concerning such national standards may be obtained on application to the CEN-CENELEC Management Centre or to any CEN member.



...re 2.4 Typical appearance of axial compression buckles: (a) failure in service; (b) test in laboratory (Knödel and Schulz 1992).

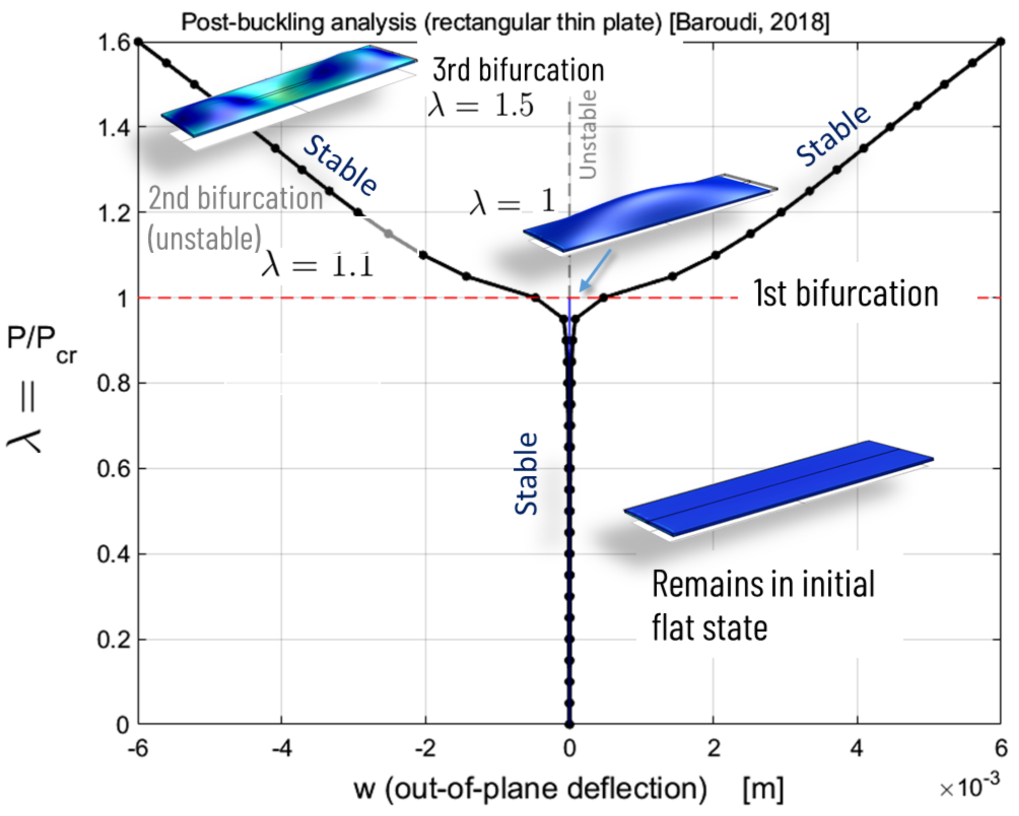
Plates (& columns) are not imperfection-sensitive structures

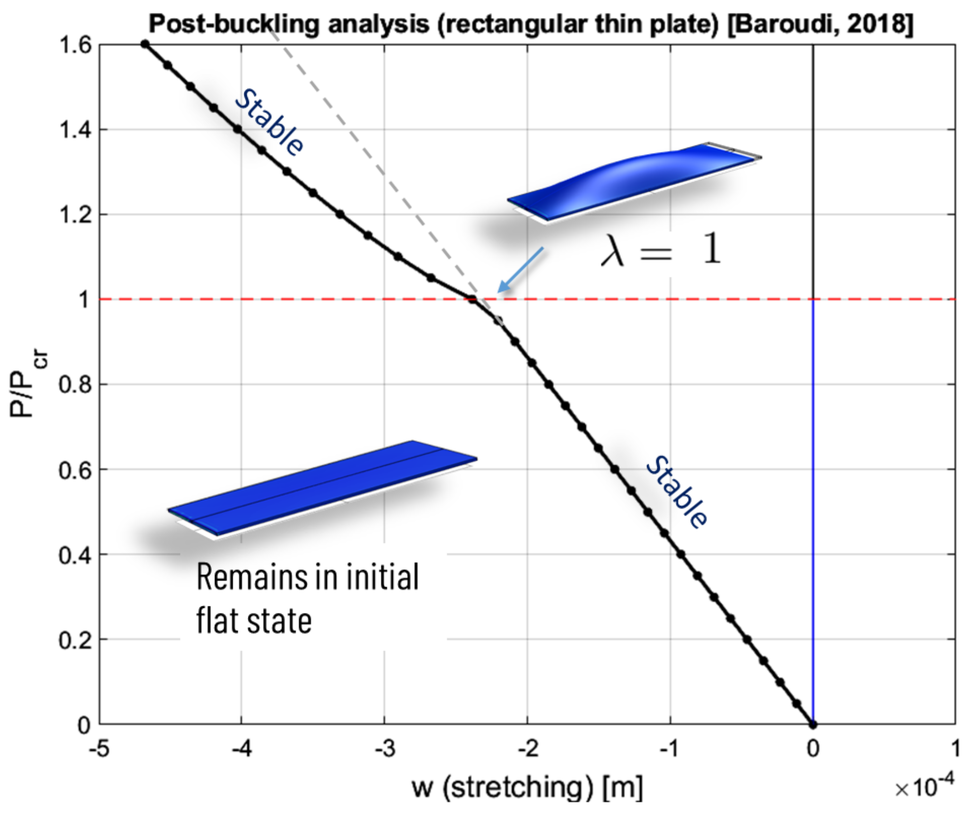
Recall last lecture series

Post-Buckling Analysis

2nd bifurcation (unstable) was not observed in simulation

Voima-ohjattuna



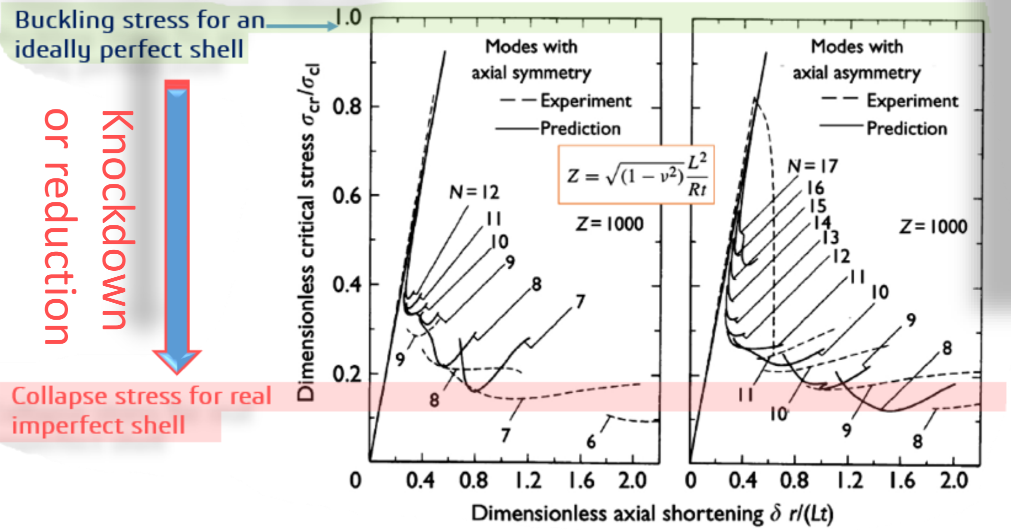


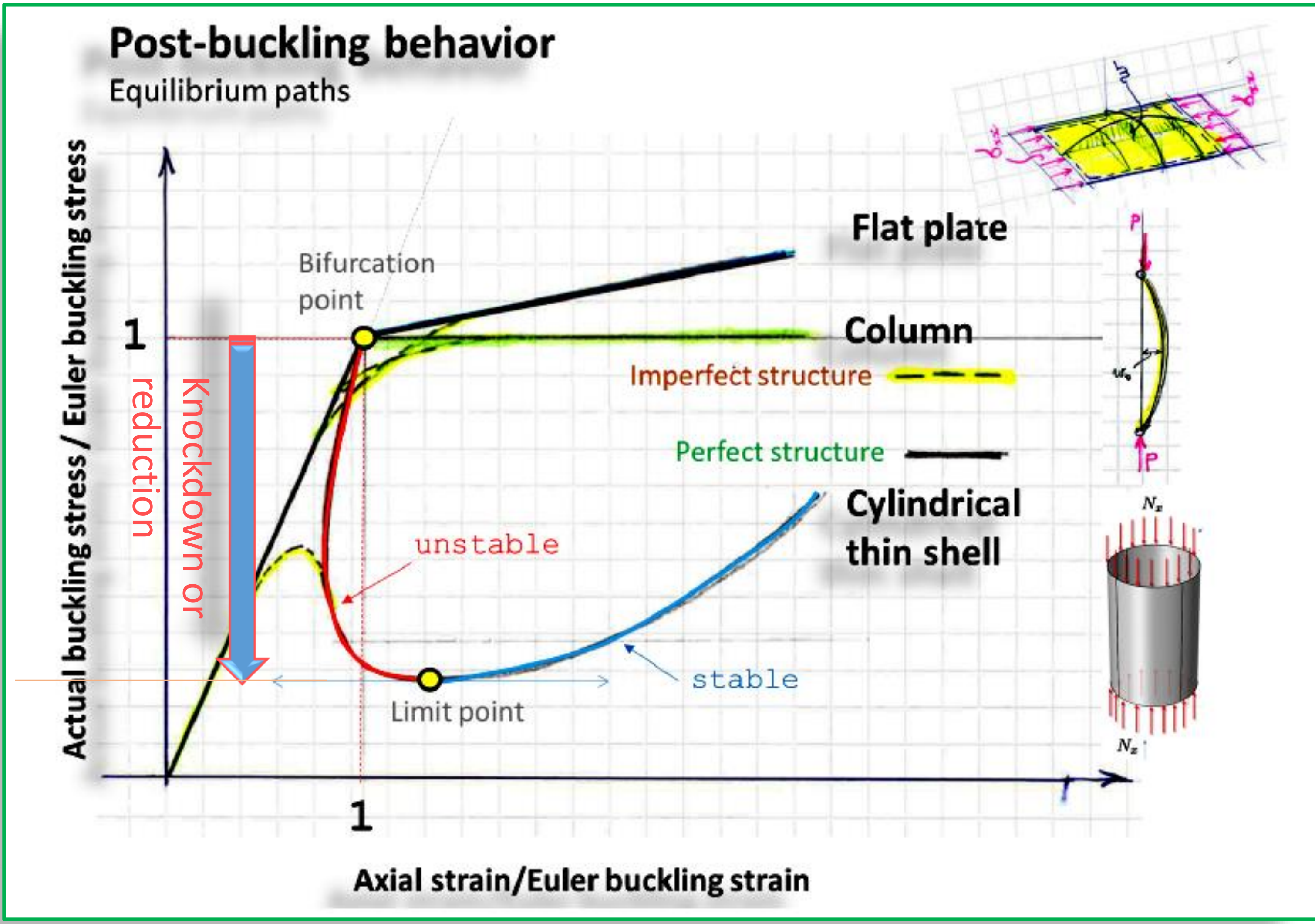
Shells are imperfection-sensitive structures

What to take with you

Experimental evidence

Thin Cylindrical Shells

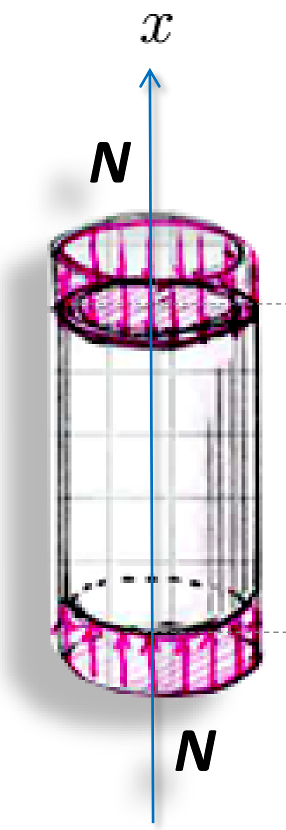


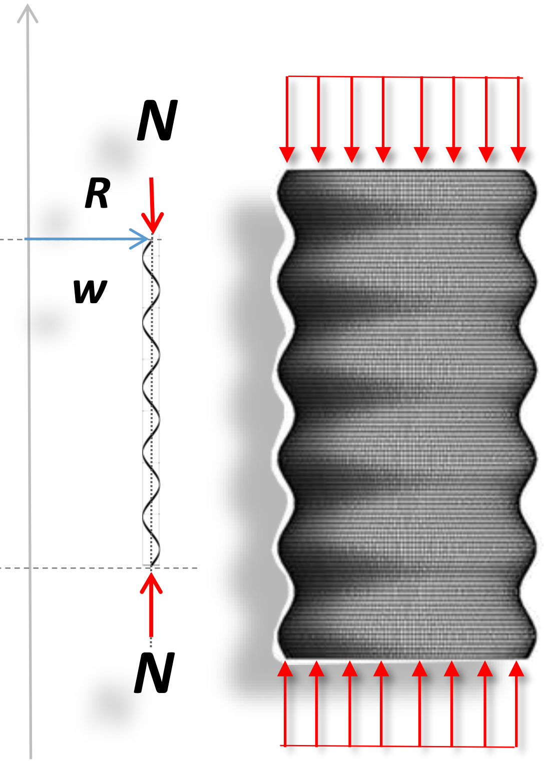


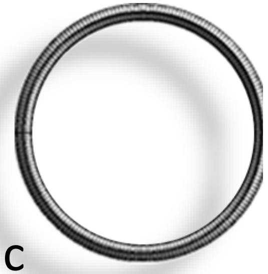
Schematic of fundamental characteristics of post-buckling behaviour for plates, columns and thin shells.

Buckling of axisymmetric cylindrical shells

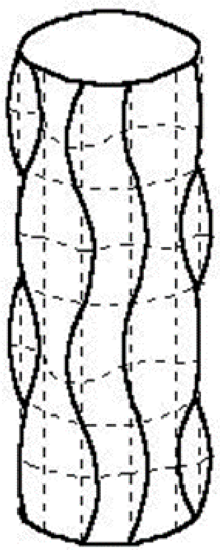
Buckling of axisymmetric cylindrical shells

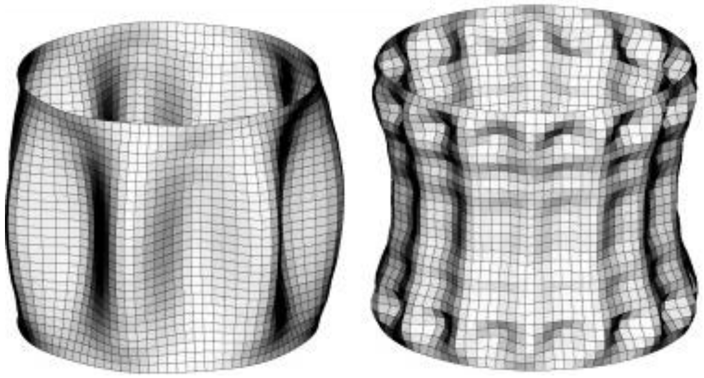


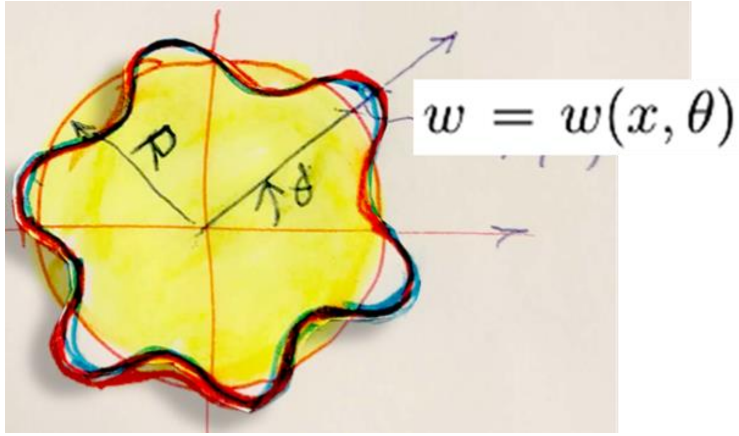




a) axisymmetric buckling mode

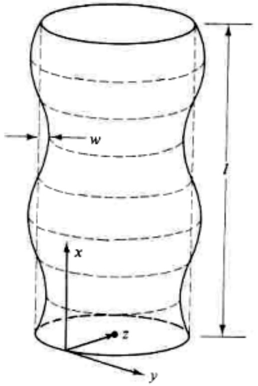






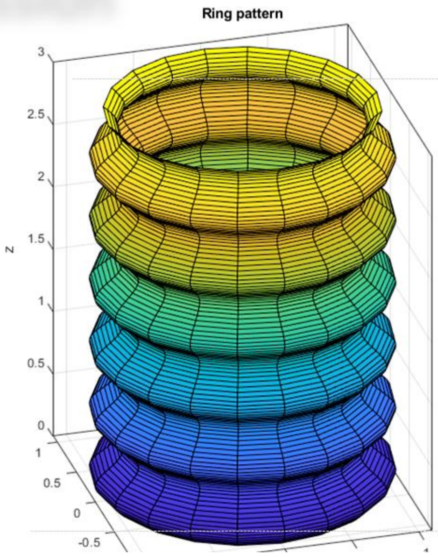
b) asymmetric buckling mode

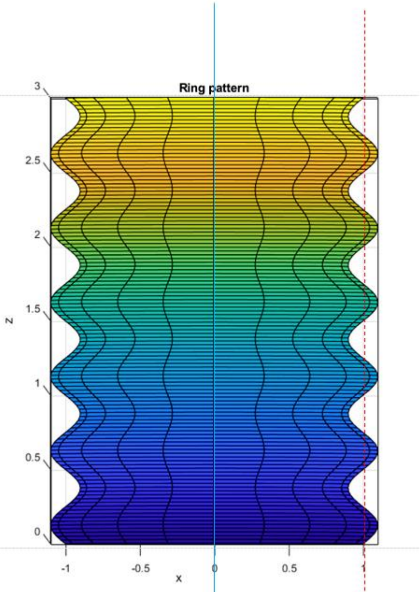
Buckling of axisymmetric cylindrical shells under uniform compression



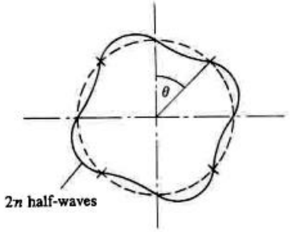
Ring patterns:

$$w(x) = w_0 \sin \frac{m\pi x}{\ell}$$



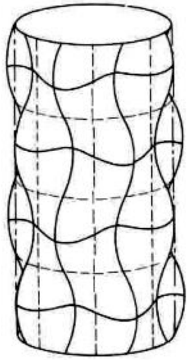


a)

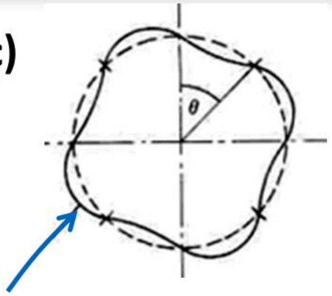


Chessboard patterns:

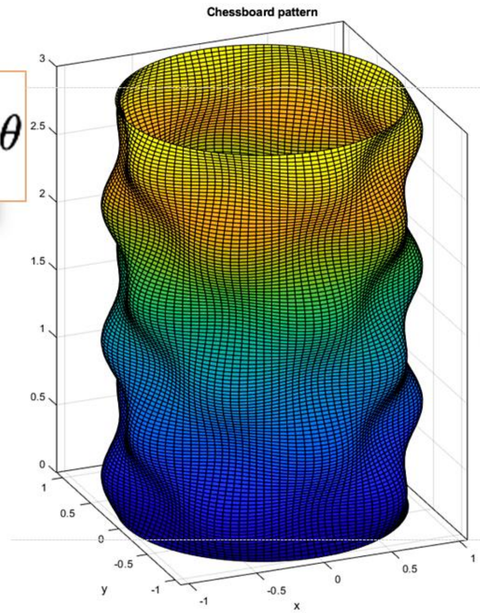
$$w(x) = w_0 \sin \frac{m\pi x}{\ell} \sin n\theta$$

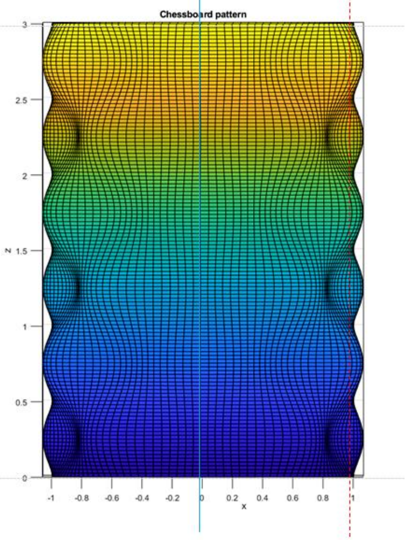


c)



2π-half waves





b)

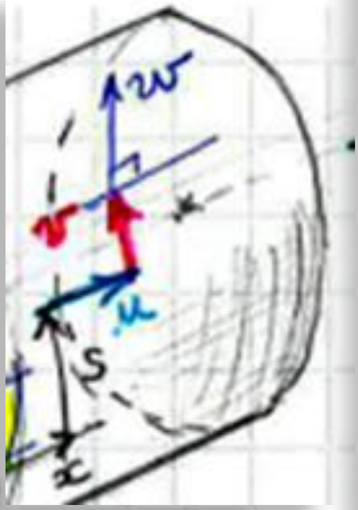
```
% asymmetric chessboard bucklin of cylinder-shells
% Plotting only.
% Aythor: D. Baroudi, 2019
% ------------------------------------
% Z     - cylinder axis Z = [0, L]
% THETA - 0:2*pi
%% --------------------------------------------------------
R0 = 1;     % Radius
L = 3*R0;  % length
m = 6;     % nbre 1/2-waves in z-direction
n = 5;     % nbre 1/2-waves in theta direction
w0 = R0/10
NP = 4 * 30 ;
% ----------------------------
theta = linspace(0,2*pi, NP);
z     = linspace(0,L,  NP);

% Generating the mesh ---
Z = meshgrid(z);
[R , THETA] = meshgrid(z,  theta);

% The radial displacement w(z, theta)  at (z, theta) ---
w_z_theta = w0 * sin( m * pi * Z / L) .* sin(n  * THETA);

% Ploting the cylinder surface ---
% 1) Chessboard pattern
% ------------------------------------------------------------------
figure
surf( (R0 + w_z_theta) .* cos(THETA), (R0 + w_z_theta) .* sin(THETA), Z);
axis square
grid on
box on
axis equal
title('Chessboard pattern')
xlabel('x')
ylabel('y')
zlabel('z')

% -------
% 2) Ring pattern
figure
m = 8;
[X, Y, Z] = cylinder(R0 + w0 * sin( m * pi * z / L));
surf(X,Y,Z*L)
axis square
grid on
box on
axis equal
title('Ring pattern')
xlabel('x')
ylabel('y')
zlabel('z')
% ---------------------------------------------------------------
```

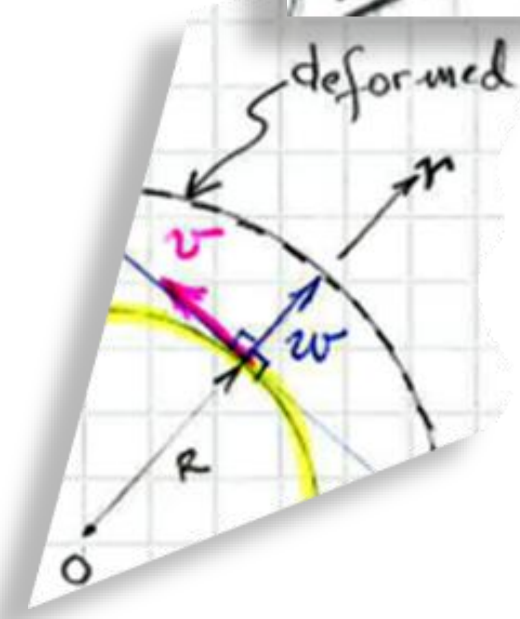
Buckling of axisymmetric cylindrical shells

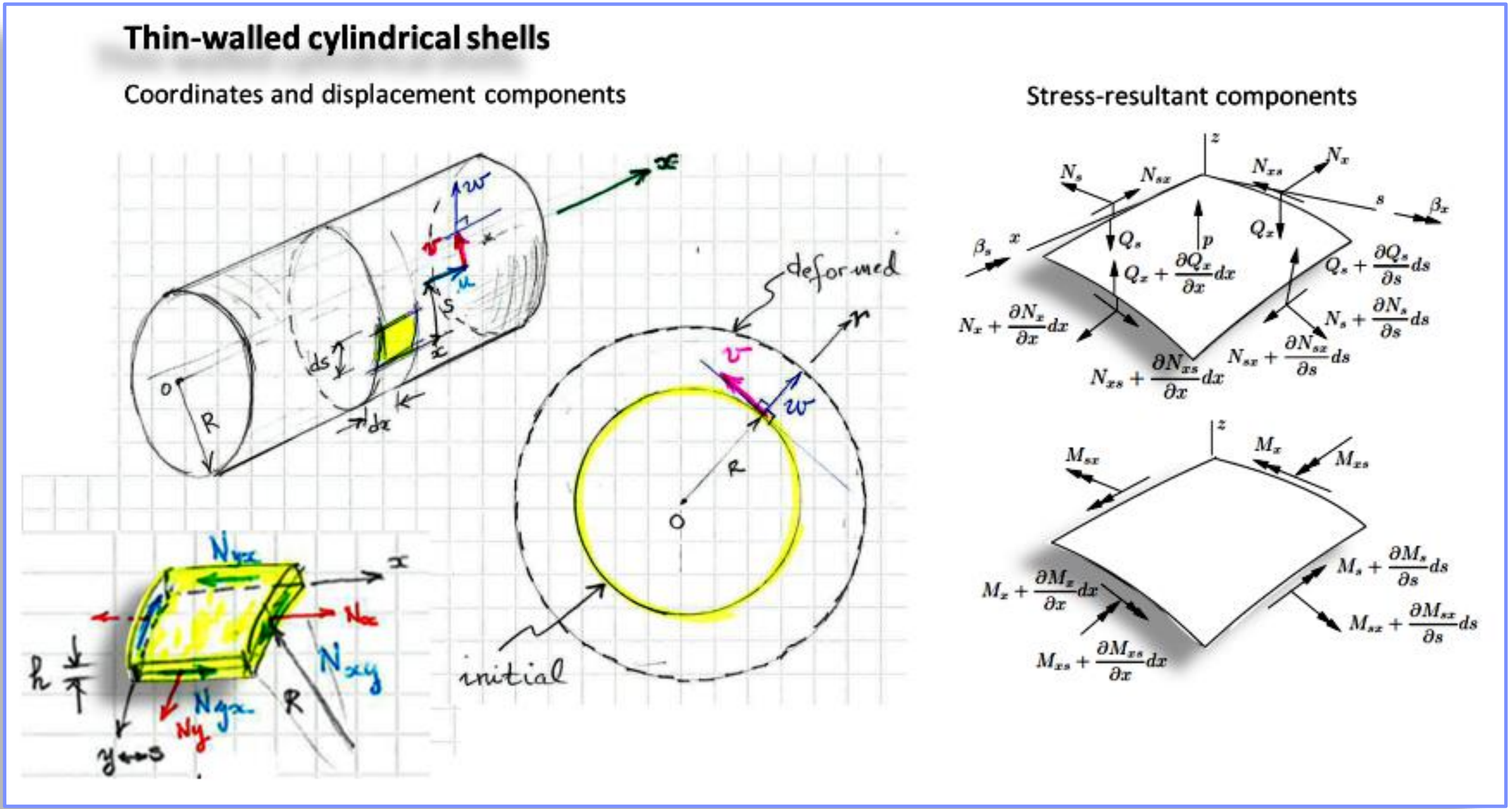
Deriving loss of stability equations

Buckling of axisymmetric cylindrical shells

Coordinate system, displacements and stress resultants







Thin-walled cylindrical shell, coordinate system, displacements u, v, w of the mid-plane and stress resultants components. In some writings the coordinate $y \equiv s$). The rotations in the left part of the figure are defined by $\beta_x = -w_{,x}$ and $\beta_s = -w_{,s}$.

Equilibrium equation of axisymmetric cylindrical shells

Large deflection Donnell-type theory

Physical problem: thin-walled tubular shell with both axial loading and transversal pressure *p*

Kinematics: membrane deformations and curvatures:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2, \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2},$$

$$\epsilon_s = \frac{\partial v}{\partial s} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2, \qquad \kappa_s = -\frac{\partial^2 w}{\partial s^2}$$

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial s}, \qquad \kappa_{xs} = -\frac{\partial^2 w}{\partial x \partial s}.$$

Isotropic elastic material:
Constitutive relations

membrane stress-resultants

$$N_x = C(\epsilon_x + \nu\epsilon_s),$$
$$N_s = C(\epsilon_s + \nu\epsilon_x),$$
$$N_{xs} = C(1-\nu)\epsilon_{xs},$$

membrane rigidity

$$C = \frac{Eh}{1-\nu^2}$$

bending stress-resultants

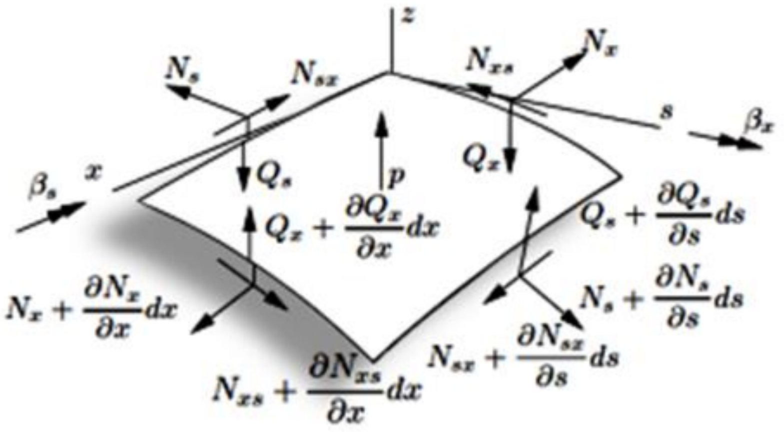
$$M_x = D(\kappa_x + \nu\kappa_s),$$
$$M_s = D(\kappa_s + \nu\kappa_x),$$
$$M_{xs} = D(1-\nu)\kappa_{xs},$$

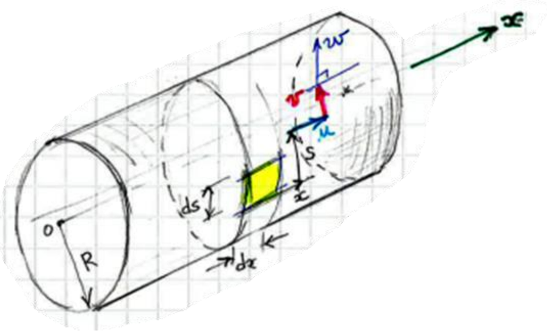
bending rigidity

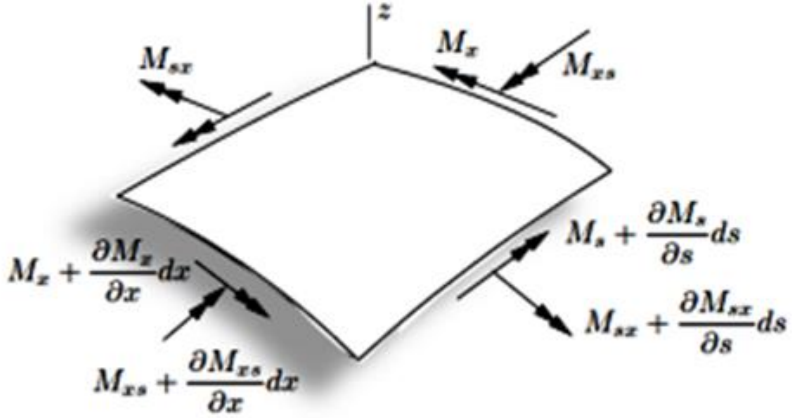
$$D = \frac{Eh^3}{12(1-\nu^2)}$$

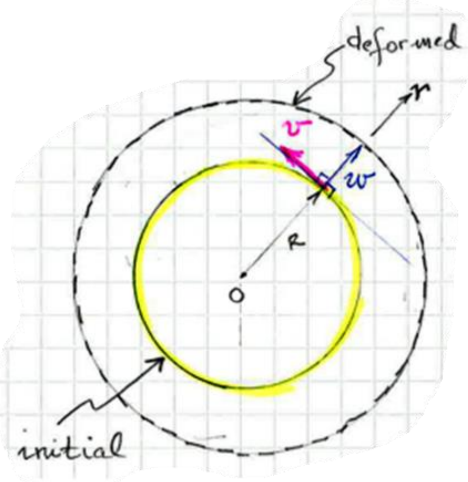
u, v and w being *total displacements*,

Stress-resultant components









There is analogy with the von-Kármán large-deflection plate theory

Equilibrium equation of axisymmetric cylindrical shells

Large deflection Donnell-type theory

three *coupled* partial differential equations

$$\begin{cases} N_{x,x} + N_{xs,s} = 0 \\ N_{xs,x} + N_{s,s} = 0 \\ D\left[w_{,xxxx} + 2w_{,xx}w_{,ss} + w_{,ssss}\right] - \left[N_x w_{,xx} + 2N_{xs}w_{,xs} + N_s\left(w_{,ss} - \frac{1}{R}\right)\right] = p, \end{cases}$$

$$N_{\alpha,\beta} = N_{\alpha,\beta}(u, v, w),$$

Large-deflection Donnell-type equilibrium equations for the analysis of cylindrical shells

Physical problem: thin-walled tubular shell with both axial loading and transversal pressure *p*

- given the external pressure *p* (external loads) we *can solve uniquely* all the displacement components and internal force from the non-linear coupled equilibrium equations for elasticity and given the kinematic relations + boundary conditions
- This set of coupled non-linear equations represents large-deflection equilibrium equations which are also valid for the post-buckled configuration, naturally.
- known as **Large-deflection Donnell-type equations** (some time the name of **von-Karman** is associated (see also von-Karman large deflection plate theory)

N.B. Now, **membrane stress-resultants depend on the deflection *w* and the displacement components *u* and *v* as well [coupling].**

Isotropic elastic material: Constitutive relations

membrane stress-resultants

$$N_x = C(\epsilon_x + \nu\epsilon_s),$$
$$N_s = C(\epsilon_s + \nu\epsilon_x),$$
$$N_{xs} = C(1-\nu)\epsilon_{xs},$$

membrane rigidity

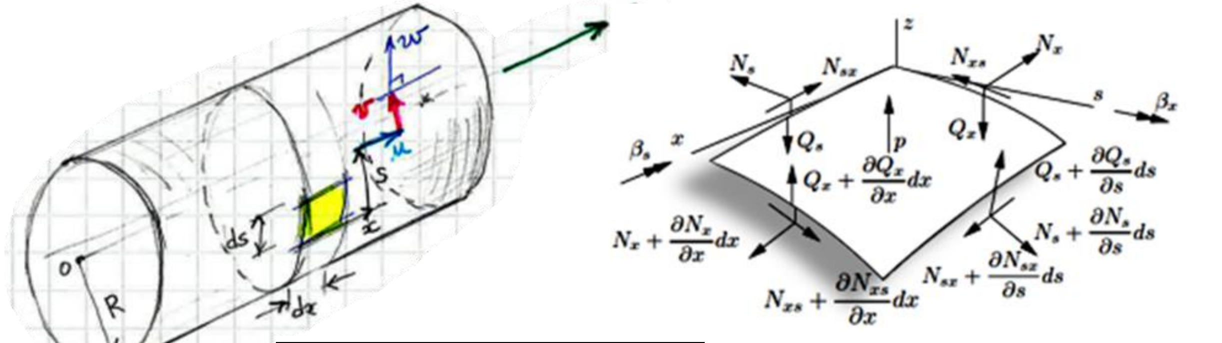
$$C = \frac{Eh}{1-\nu^2}$$

bending stress-resultants

$$M_x = D(\kappa_x + \nu\kappa_s),$$
$$M_s = D(\kappa_s + \nu\kappa_x),$$
$$M_{xs} = D(1-\nu)\kappa_{xs},$$

bending rigidity

$$D = \frac{Eh^3}{12(1-\nu^2)}$$



There is analogy with the von-Kármán large-deflection plate theory

Donnell, L.H., *Beams, Plates and Shells*. 1976, New York: McGraw-Hill, Inc.

The Linear stability equations:

The Linear stability equations: To derive linearised stability loss equations (Eigen-value problem) we consider an infinitely tiny perturbation[261] of the equilibrium state leading to a transition between the pre-buckled state to buckled equilibrium state. We introduce such a perturbation to the non-linear couples equilibrium equation system (Equation 1.687) for the purpose of linearise buckling equations in order to determine the lowest critical stress (buckling stress) as from solving the Eigen-value problem. Let for the moment keep in mind that N_x^0, N_s^0 and N_{xs}^0 are *statically admissible*[262] Then the following perturbation expansion is be used

$$w^* = \underbrace{w^{(0)}}_{=0} + \underbrace{\Delta w}_{\equiv w}, \quad u^* = u^{(0)} + \underbrace{\Delta u}_{\equiv v}, \quad v^* = v^{(0)} + \Delta v$$

$$N_x = N_x^0 + \Delta N_x, \quad N_s^0 + \Delta N_s, \quad N_{xs} = N_{xs}^0 + \Delta N_{xs}$$

Naturally, the perturbed state fulfil the all the three equilibrium equations 1.687). Inserting this perturbed state to the equilibrium equations and accounting for elasticity and kinematics (plus some simplifications; shallow shell, ...) one obtains the linearised stability equations (For details and the linearised stability equations, refer to our textbook Section 9.4 [Chia Yoo]).

Isotropic elastic material: Constitutive relations

membrane stress-resultants

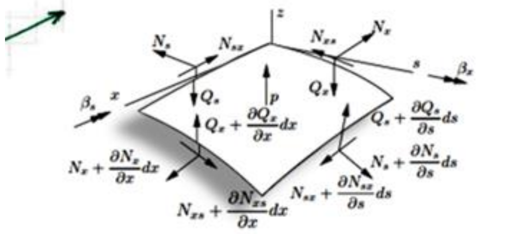
$$N_x = C(\epsilon_x + \nu\epsilon_s), \quad N_s = C(\epsilon_s + \nu\epsilon_x), \quad N_{xs} = C(1-\nu)\epsilon_{xs},$$

membrane rigidity $C = \frac{Eh}{1-\nu^2}$

bending stress-resultants

$$M_x = D(\kappa_x + \nu\kappa_s), \quad M_s = D(\kappa_s + \nu\kappa_x), \quad M_{xs} = D(1-\nu)\kappa_{xs},$$

bending rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$



N.B. Now, **membrane stress-resultants** depend on the deflection ***w*** and the displacement components ***u*** and ***v*** as well [**coupling**].

Large-deflection Donnell-type equilibrium equations for the analysis of cylindrical shells

three *coupled* partial differential equations

$$\begin{cases} N_{x,x} + N_{xs,s} = 0 \\ N_{xs,x} + N_{s,s} = 0 \\ D\left[w_{,xxxx} + 2w_{,xx}w_{,ss} + w_{,ssss}\right] - \left[N_x w_{,xx} + 2N_{xs}w_{,xs} + N_s\left(w_{,ss} - \frac{1}{R}\right)\right] = p, \end{cases}$$

$$N_{\alpha,\beta} = N_{\alpha,\beta}(u, v, w),$$

refer to our textbook Section 9.4 [Chia Yoo]

Energy criteria for stability loss of thin-walled cylindrical shell

Bryan form of the increment of the total potential energy

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\Delta U:\ \text{membrane + bending}} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part in } \Delta W(\sigma^0)}$$

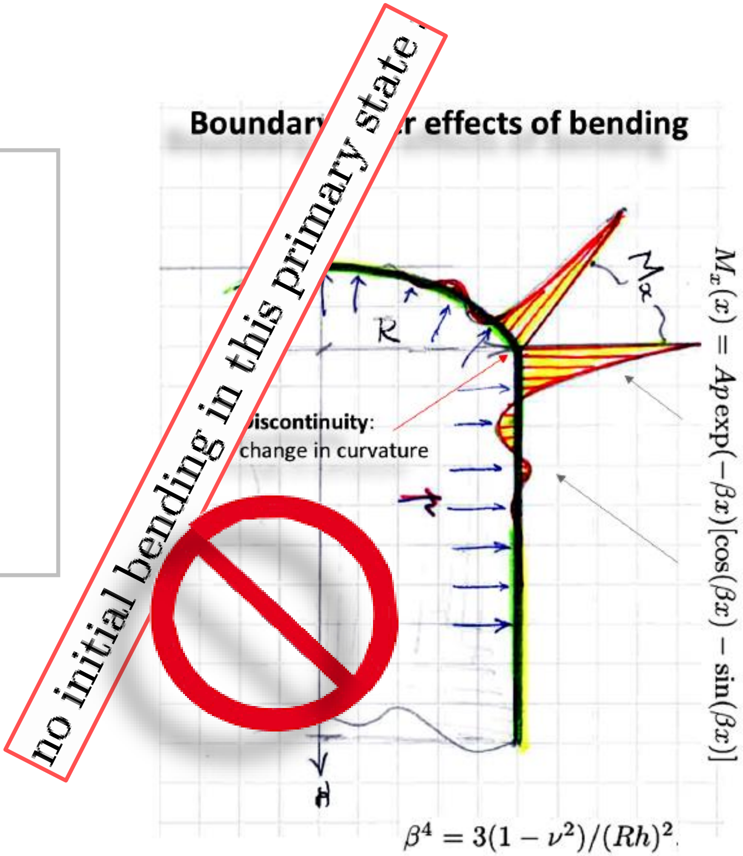
$$= \Delta U_{memb} + \Delta U_{bend} + \Delta W(\sigma^0).$$

pre-buckling pure membrane state

no initial bending in this primary state

N_x^0, N_s^0 and N_{xs}^0 are statically admissible.

$$u^* = u^0 + u, \qquad v^* = v^0 + v \qquad w^* = w^0 + w.$$



The components linear ($\mathbf{L}$) part of the strain increment ϵ_1 are

$$e_x = \frac{\partial u}{\partial x}, \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2},$$

$$e_s = \frac{\partial v}{\partial s} + \frac{w}{R}, \qquad \kappa_s = -\frac{\partial^2 w}{\partial s^2},$$

$$\mathbf{L}\gamma_{xs} = 2e_{xs} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}, \qquad \kappa_{xs} = -\frac{\partial^2 w}{\partial x \partial s},$$

quadratic part (second-order)

$$\epsilon_x^* \approx \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2,$$

$$\epsilon_s^* \approx \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2,$$

$$\gamma_{xs}^* \approx \frac{\partial^2 w}{\partial x \partial s}.$$

$$\Delta U_{bend} = \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,ss}^2 + 2\nu w_{,xx}w_{,ss} + 2(1-\nu)w_{,xs}^2\right]\mathrm{d}A$$

Energy criteria for stability loss of thin-walled cylindrical shell

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\Delta U:\ \text{membrane + bending}} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part in } \Delta W(\sigma^0)}$$

$$\Delta U_{bend} = \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,ss}^2 + 2\nu w_{,xx}w_{,ss} + 2(1-\nu)w_{,xs}^2\right]\mathrm{d}A$$

$$\begin{aligned}\Delta U_{memb} =& \frac{1}{2}C\int_A \left[e_x^2 + e_s^2 + 2\nu e_x e_s + 2(1-\nu)e_{xs}^2\right]\mathrm{d}A\\ =& \frac{1}{2}C\int_A \Big[u_{,x}^2 + v_{,s}^2 + 2\nu u_{,x}v_{,s} + \frac{1-\nu}{2}u_{,s}^2 + \frac{1-\nu}{2}v_{,s}^2 + \\ & + (1-\nu)u_{,s}v_{,x} + \frac{w^2}{R^2} + 2v_{,s}\frac{w}{R} + 2\nu u_{,x}\frac{w}{R}\Big]\mathrm{d}A.\end{aligned}$$

$$\Delta W(\sigma^0, \epsilon_2) = \int_A \left[N_x^0 \frac{1}{2}w_{,x}^2 + N_s^0 \frac{1}{2}w_{,s}^2 + N_{xs}^0 w_{,x}w_{,s}\right]\mathrm{d}A.$$

$$\delta(\Delta\Pi) = 0, \quad \forall \delta u, \delta v, \delta w$$

$$\Downarrow$$

Stability equations

$$\begin{aligned} C\left(u_{,xx} + \frac{1-\nu}{2}u_{,ss} + \frac{1+\nu}{2}v_{,xs} + \frac{\nu}{R}w_{,s}\right) &= 0\\ C\left(v_{,ss} + \frac{1-\nu}{2}v_{,xx} + \frac{1+\nu}{2}u_{,xs} + \frac{\nu}{R}w_{,s}\right) &= 0\\ D\nabla^4 w + \frac{C}{R}\left(\frac{w}{R} + v_{,s} + \nu u_{,x}\right) - \left(N_x^0 w_{,xx} + 2N_{xs}^0 w_{,x}w_{,s} + N_s^0 w_{,ss}\right) &= 0\end{aligned}$$

Energy criteria for stability loss of thin-walled cylindrical shell

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V \epsilon_1{}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\Delta U:\ \text{membrane + bending}} + \underbrace{\int_V \epsilon_2{}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part in } \Delta W(\sigma^0)}$$

$$\Delta U_{bend} = \frac{1}{2}D\int_A \left[w_{,xx}^2 + w_{,ss}^2 + 2\nu w_{,xx}w_{,ss} + 2(1-\nu)w_{,xs}^2\right] \mathrm{d}A$$

$$\Delta U_{memb} = \frac{1}{2}C\int_A \left[e_x^2 + e_s^2 + 2\nu e_x e_s + 2(1-\nu)e_{xs}^2\right] \mathrm{d}A$$
$$= \frac{1}{2}C\int_A \left[u_{,x}^2 + v_{,s}^2 + 2\nu u_{,x}v_{,s} + \frac{1-\nu}{2}u_{,s}^2 + \frac{1-\nu}{2}v_{,s}^2 + (1-\nu)u_{,s}v_{,x} + \frac{w^2}{R^2} + 2v_{,s}\frac{w}{R} + 2\nu u_{,x}\frac{w}{R}\right] \mathrm{d}A.$$

$$\Delta W(\sigma^0, \epsilon_2) = \int_A \left[N_x^0 \frac{1}{2}w_{,x}^2 + N_s^0 \frac{1}{2}w_{,s}^2 + N_{xs}^0 w_{,x}w_{,s}\right] \mathrm{d}A.$$

$$\delta(\Delta\Pi) = 0, \quad \forall \delta u, \delta v, \delta w$$

$$N_{x,x} + N_{xs,s} = 0$$
$$N_{xs,x} + N_{s,s} = 0$$

+ Constitutive relations

Loss of Stability equations:

$$C\left(u_{,xx} + \frac{1-\nu}{2}u_{,ss} + \frac{1+\nu}{2}v_{,xs} + \frac{\nu}{R}w_{,s}\right) = 0$$

$$C\left(v_{,ss} + \frac{1-\nu}{2}v_{,xx} + \frac{1+\nu}{2}u_{,xs} + \frac{\nu}{R}w_{,s}\right) = 0$$

$$D\nabla^4 w + \frac{C}{R}\left(\frac{w}{R} + v_{,s} + \nu u_{,x}\right) - \left(N_x^0 w_{,xx} + 2N_{xs}^0 w_{,x}w_{,s} + N_s^0 w_{,ss}\right) = 0$$

- The unknown **displacements** ***u***, ***v*** and ***w*** are now **coupled**
- This is a set of **three coupled** partial differential equations for the ***u***, ***v*** and ***w***
- Eliminating ***u***, ***v*** from the third equation of equilibrium using the reaming two equations leads to the well-known **Donnell-type** large-deflection equation (ref. Donnell report):

Cf. To buckling equations of rings and arches in the Emeritus prof. J. Paavola pdf-material

The pressure is usually constant and its gradients vanishes.

$$D\nabla^8 w + \frac{Eh}{R^2}w_{xxxx} - \nabla^4\left(N_x^0 w_{,xx} + 2N_{xs}^0 w_{,x}w_{,s} + N_s^0 w_{,ss}\right) = 0 \quad = \nabla^4 p$$

Similar equation for torsion buckling of thin-walled tube by Donnell, see **ref**:

L.H. Donnell. Stability of thin-walled tubes under (uniform) torsion (*NACA Technical Report 479*). (1933)

shallow shell approximation, $v/R << w_{,s}$)

Some classical solutions

$$\ldots_{cr} \equiv \sigma_{\min} = \ldots \frac{\ldots}{\sqrt{3(1-\nu^2)}} \frac{h}{R}.$$

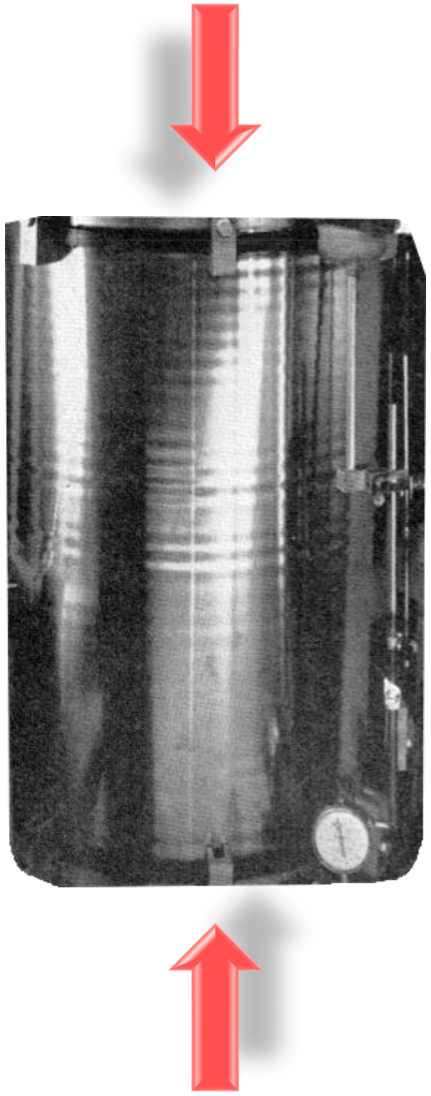
The questions:

- Derive `the formula` (Equation 1)
- What does the coefficient γ in formula (equation 2) represent? […] and where this formula is used in structural design. (Give a short concise answ[…])

Classical results

Axisymmetric buckling
of
circular cylindrical shells
under
uniform axial compression

Buckles
wrinkles



Axisymmetric buckling of circular cylindrical shells under uniform axial compression

We assume that the length of the shell is enough for the boundary conditions to not perturb such buckling patterns to form.

- Isotropic thin **cylindrical** shell of radius *R* under uniform axial compression **buckling**

- In general, the out-of-plan-mid-plane displacement is

$$w = w(x, \theta)$$

- To derive the formula for of Euler buckling stress we investigate separately ring patterns and chessboard modes separately

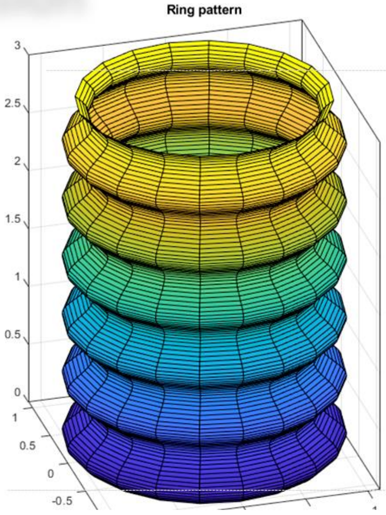
- The Euler buckling stress will be the smallest of the two

For the geometrically ideally perfect shell

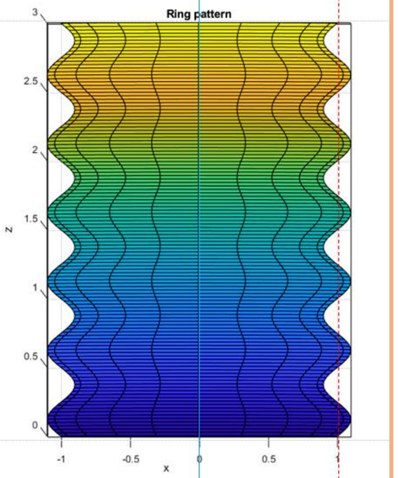
Buckling of axisymmetric cylindrical shells under uniform compression

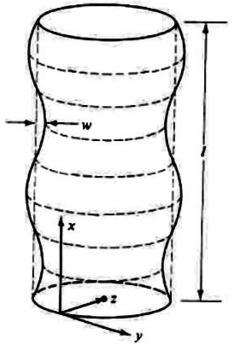
Ring patterns:

$$w(x) = w_0 \sin \frac{m\pi x}{\ell}$$



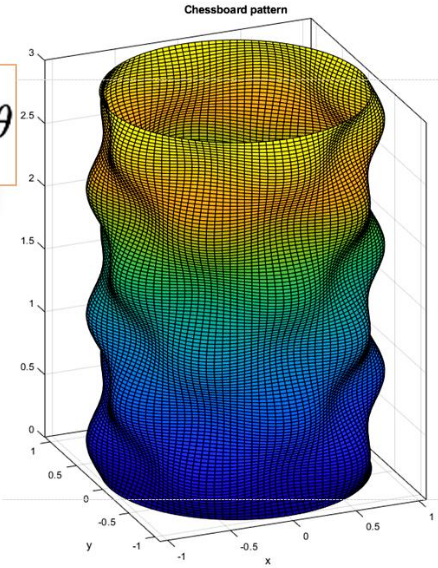
a)



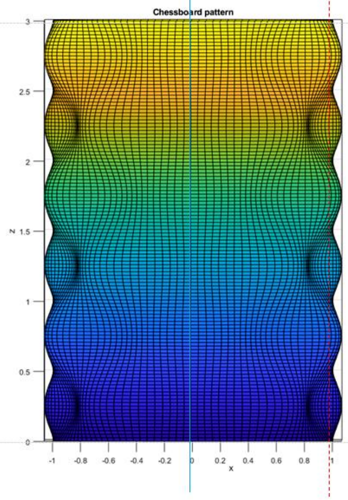


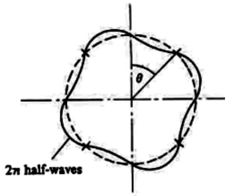
Chessboard patterns:

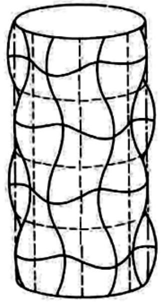
$$w(x) = w_0 \sin \frac{m\pi x}{\ell} \sin n\theta$$



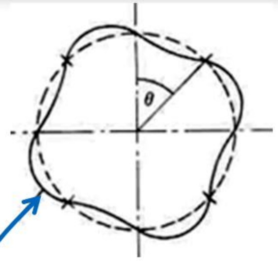
b)







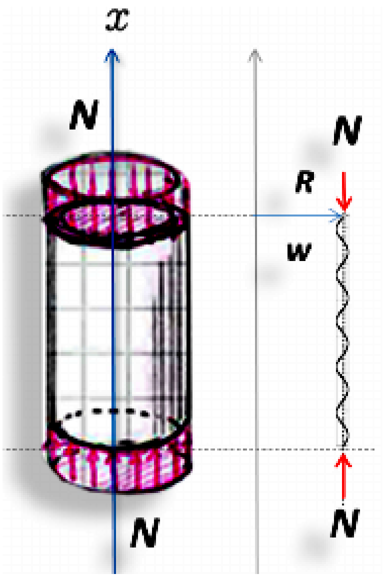
c)



2π-half waves

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

1(3)



Cylindrical shell under uniform compression. Axisymmetric buckling mode.

- Isotropic thin **cylindrical** shell of radius *R* under uniform axial compression **buckling** in an **axisymmetric mode** (ring mode)

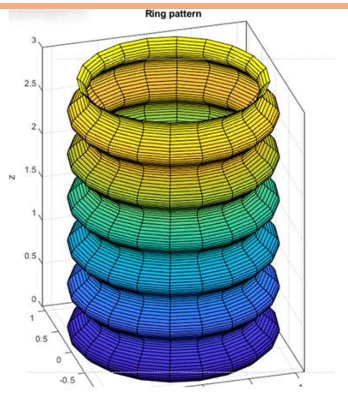
$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$

Cf. the textbook, there they solve the critical stress from the **Donnell-equations**. The result for this case is the same as when solving these **simplified Equations**

$$D\nabla^8 w + \frac{Eh}{R^2} w_{xxxx} - \nabla^4 N_x^0 w_{,xx} = 0$$

Chess-board mode:

$$w(x,s) = A \sin\frac{m\pi x}{\ell} \sin\frac{n\pi s}{\ell}$$



Ring mode

cylindrical shell under uniform compression

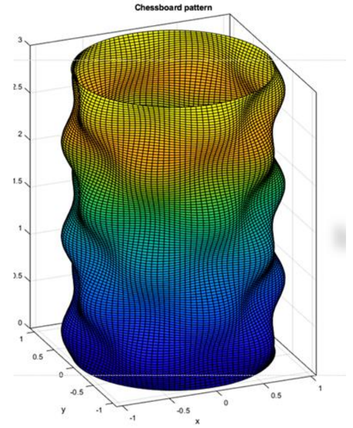
$$D\frac{d^4 w}{dx^4} + N\frac{d^2 w}{dx^2} + \frac{Eh}{R^2} w = 0,$$

Timoshenko

Similarity

$$EI\frac{d^4 w}{dx^4} + N\frac{d^2 w}{dx^2} + kw = 0,$$

axially compressed column-beam bounded to an elastic foundation



Chess-board mode

Ring patterns:

$$w(x) = w_0 \sin\frac{m\pi x}{\ell}$$

$$D = Eh^3/12(1-\nu^2),$$

$$D \to EI, \quad \frac{Eh}{R^2} \to k$$

This means that **both solutions** are also mathematically **similar**

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

- consider an isotropic thin **cylindrical** shell of radius R under uniform axial compression ($N > 0$)
- consider **buckling** in an **axisymmetric mode** (ring mode)

($N > 0$, compression)

$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$

Timoshenko

$$D\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{Eh}{R^2}w = 0,$$

$$D = Eh^3/12(1-\nu^2),$$

Trial solution in the form
(kin. admissible)

$$w(x) = w_0 \sin\frac{m\pi x}{\ell}$$

(Ring patterns)

$$\Downarrow$$

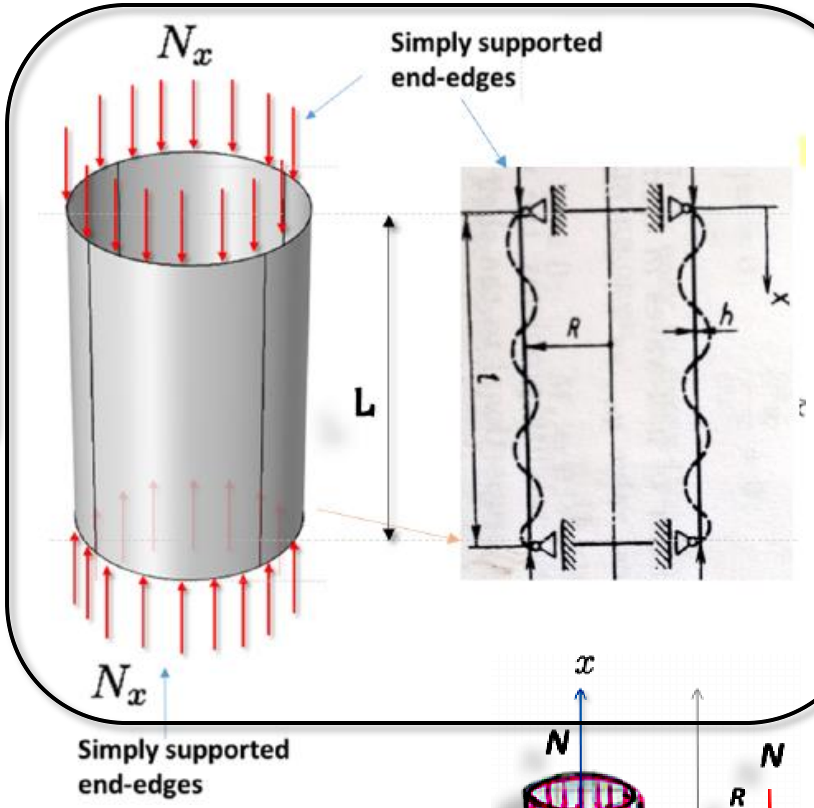
substitution of the solution into the ODE gives the critical load

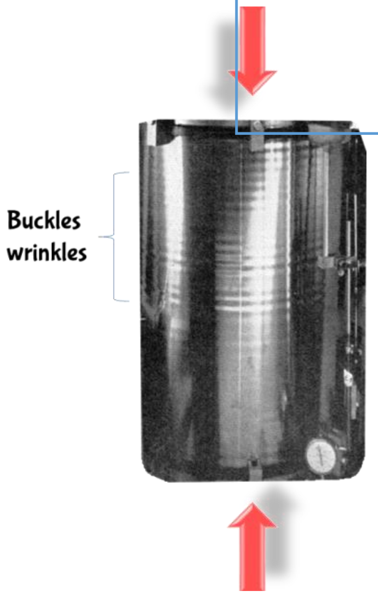
$$N_{cr} = D\left[\alpha_m^2 + \frac{Eh}{\alpha_m^2 R^2 D}\right] \qquad \alpha_m = \frac{m\pi}{\ell}$$

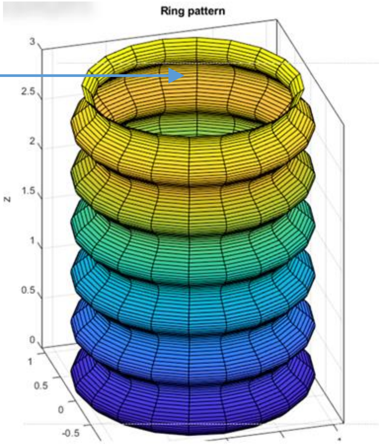
$$\Downarrow$$

= 0.6 (steel)

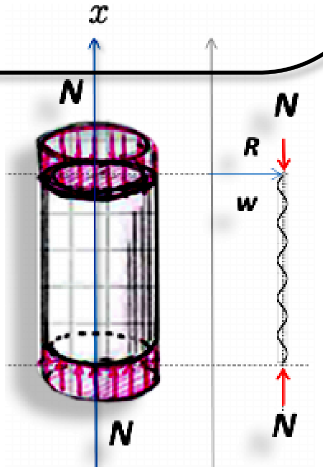
$$N_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R} \rightarrow \sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}$$







Ring mode



Cylindrical shell under uniform compression. Axisymmetric buckling mode.

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

- consider an isotropic ideally perfect thin **cylindrical** shell of radius *R* under uniform axial compression ($N > 0$)
- consider **buckling** in an **axisymmetric mode** (ring mode)

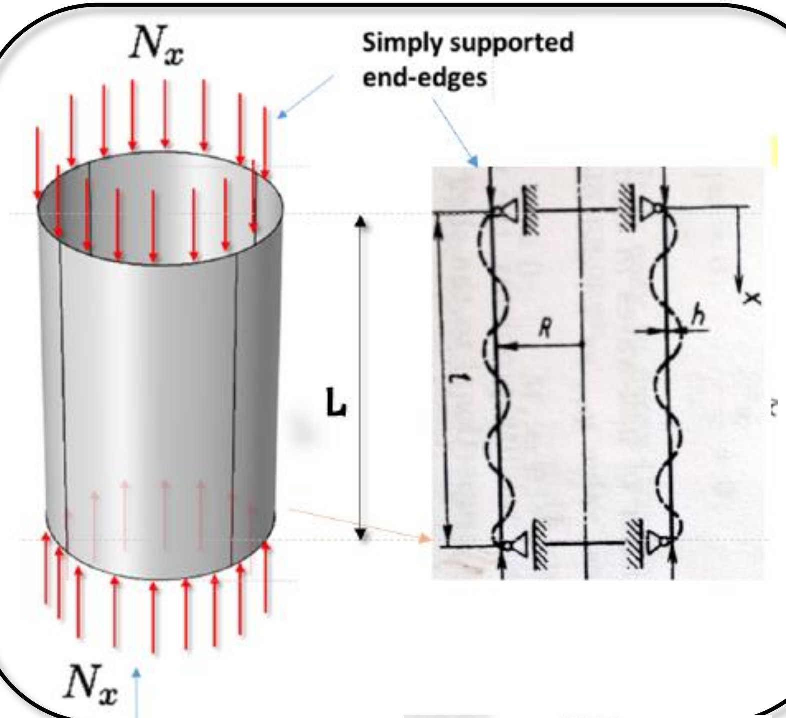
($N > 0$, compression)

$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$

$$D = Eh^3/12(1-\nu^2),$$

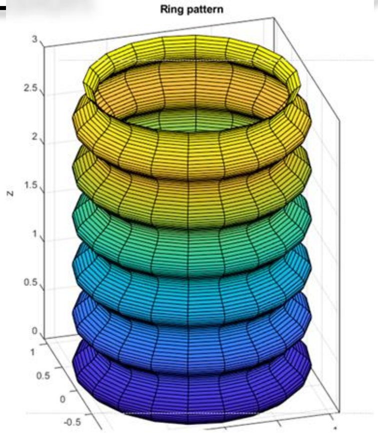
$$\Downarrow$$

$$N_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R} \rightarrow \sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}$$



Obtained using Ring patterns mode as trial

$$w(x) = w_0 \sin\frac{m\pi x}{\ell}$$



Ring mode

- The critical stress does not depend on the length for relatively long cylinders ($L > 2R$)
- Using the chess-board mode as a trial, one obtains the same result as above (exercise)

$$w(x) = w_0 \cdot \sin\frac{m\pi x}{\ell} \cdot \sin n\theta,$$

where $2n$ being the number of half-waves

$$\sigma_{cr} = 0.6 \cdot \frac{Eh}{R} \quad \text{for } \nu = 0.3.$$

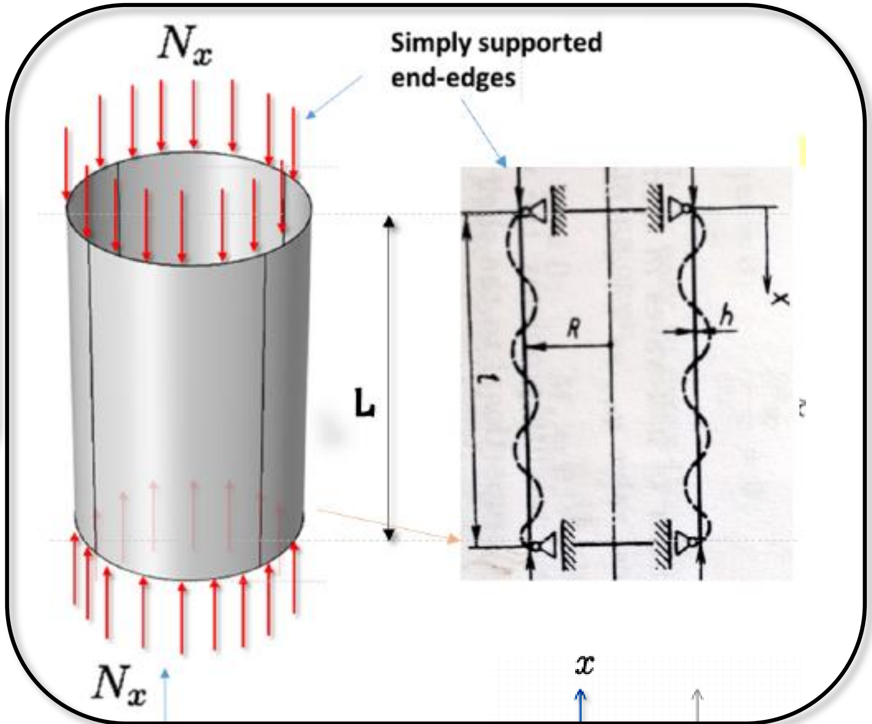
This is a famous **classical** result

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

- consider an isotropic thin **cylindrical** shell of radius *R* under uniform axial compression ($N > 0$)
- consider **buckling** in an **axisymmetric mode** (ring mode)

($N > 0$, compression)

$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$



Simply supported end-edges

$$D\frac{d^4w}{dx^4} + N\frac{d^2w}{dx^2} + \frac{Eh}{R^2}w = 0,$$

Timoshenko

$$D = Eh^3/12(1-\nu^2),$$

Trial solution in the form

(kin. admissible)

$$w(x) = w_0 \sin\frac{m\pi x}{\ell}$$

(Ring patterns)

$$\Downarrow$$

substitution of the solution into the ODE gives the critical load

$$N_{cr} = D\left[\alpha_m^2 + \frac{Eh}{\alpha_m^2 R^2 D}\right] \quad \alpha_m = \frac{m\pi}{\ell}$$

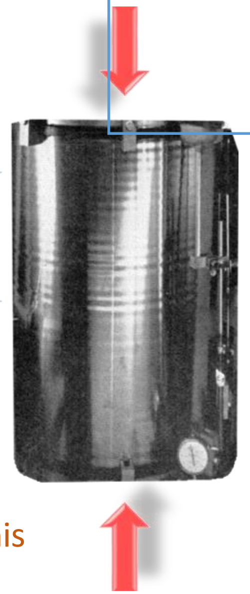
for $\nu = 0.3$.

$$\sigma_{\text{cr}} = 0.6 \cdot \frac{Eh}{R}$$

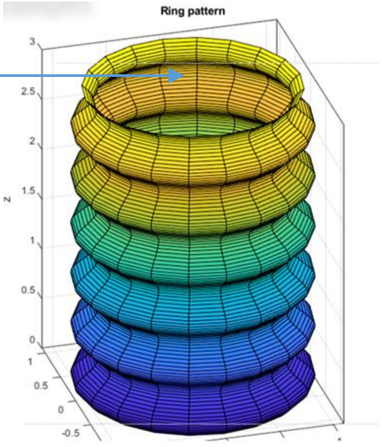
$$\Downarrow$$

$$N_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R} \rightarrow \sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}$$

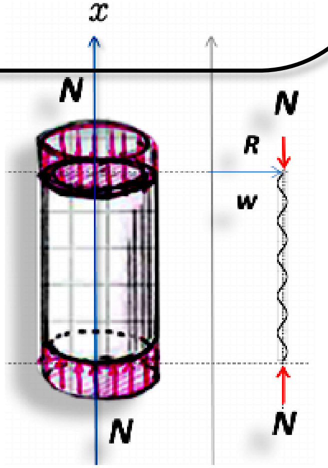
Buckles wrinkles



Note the ring mode of the wrinkles in this experiment



Ring mode



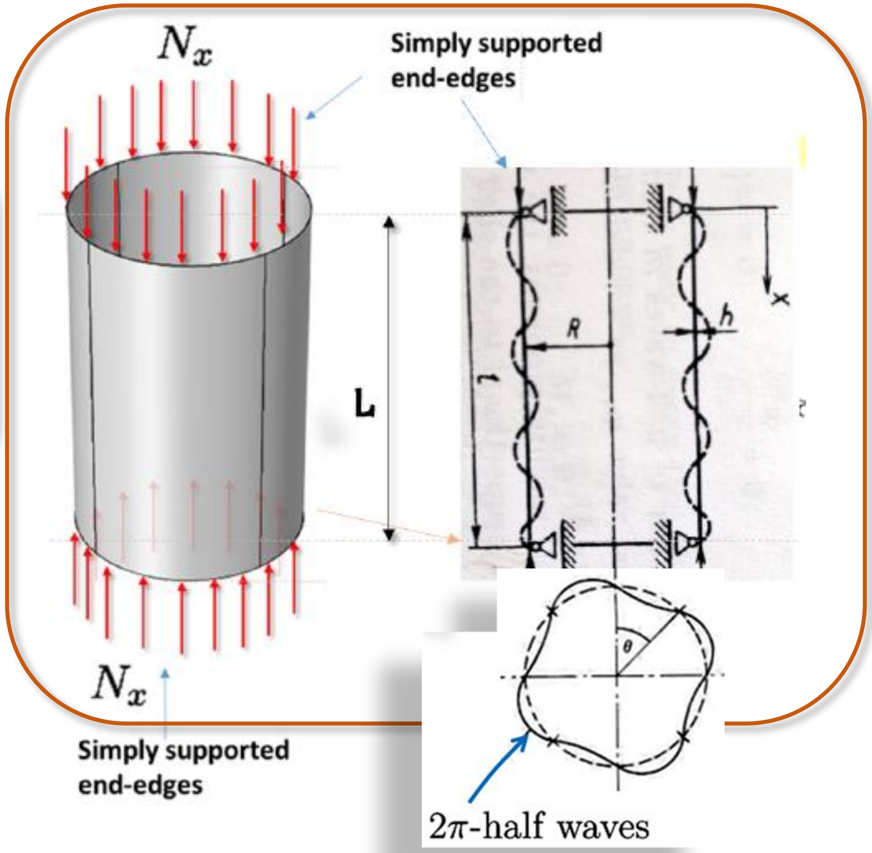
Cylindrical shell under uniform compression. Axisymmetric buckling mode.

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

- consider an isotropic thin **cylindrical** shell of radius *R* under uniform axial compression ($N > 0$)
- consider **buckling** in an **axisymmetric mode** (ring mode)

($N > 0$, compression)

$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$



$$D\frac{d^4 w}{dx^4} + N\frac{d^2 w}{dx^2} + \frac{Eh}{R^2}w = 0,$$

Timoshenko

$$D = Eh^3/12(1-\nu^2),$$

Trial solution in the form
(kin. admissible)

$$w(x) = w_0 \cdot \sin\frac{m\pi x}{\ell} \cdot \sin n\theta,$$

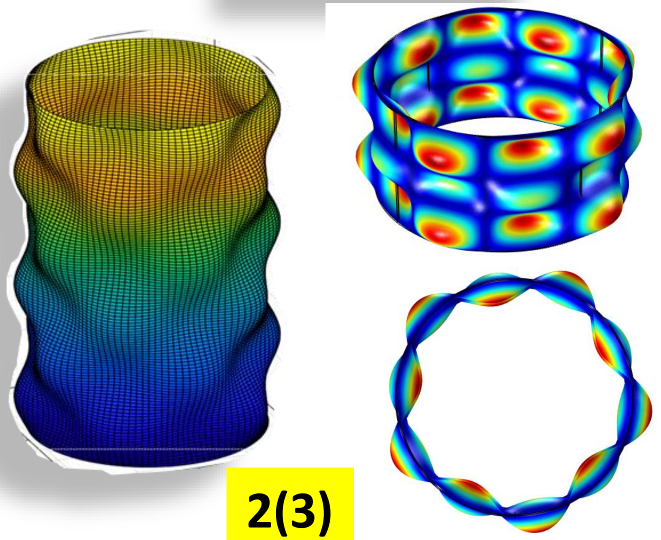
chessboard patterns

$\Downarrow$

substitution of the solution into the ODE gives the critical load

$$\sigma_{cr} = \frac{Eh^2}{12(1-\nu^2)} \underbrace{\left[\frac{\left(n^2 + (m\pi R/\ell)^2\right)^2}{R^2 (m\pi R/\ell)^2} + \frac{E(m\pi R/\ell)^2}{\left(n^2 + (m\pi R/\ell)^2\right)^2}\right]}_{\min},$$

Cchessboard patterns



Through minimisation, the buckling stress (lower stress) occurs for

$$\frac{n^2\ell^2 + (m\pi R)^2}{m\pi R\ell} = \left[\frac{Eh}{R^2 D}\right]^{1/4}, \Longrightarrow$$

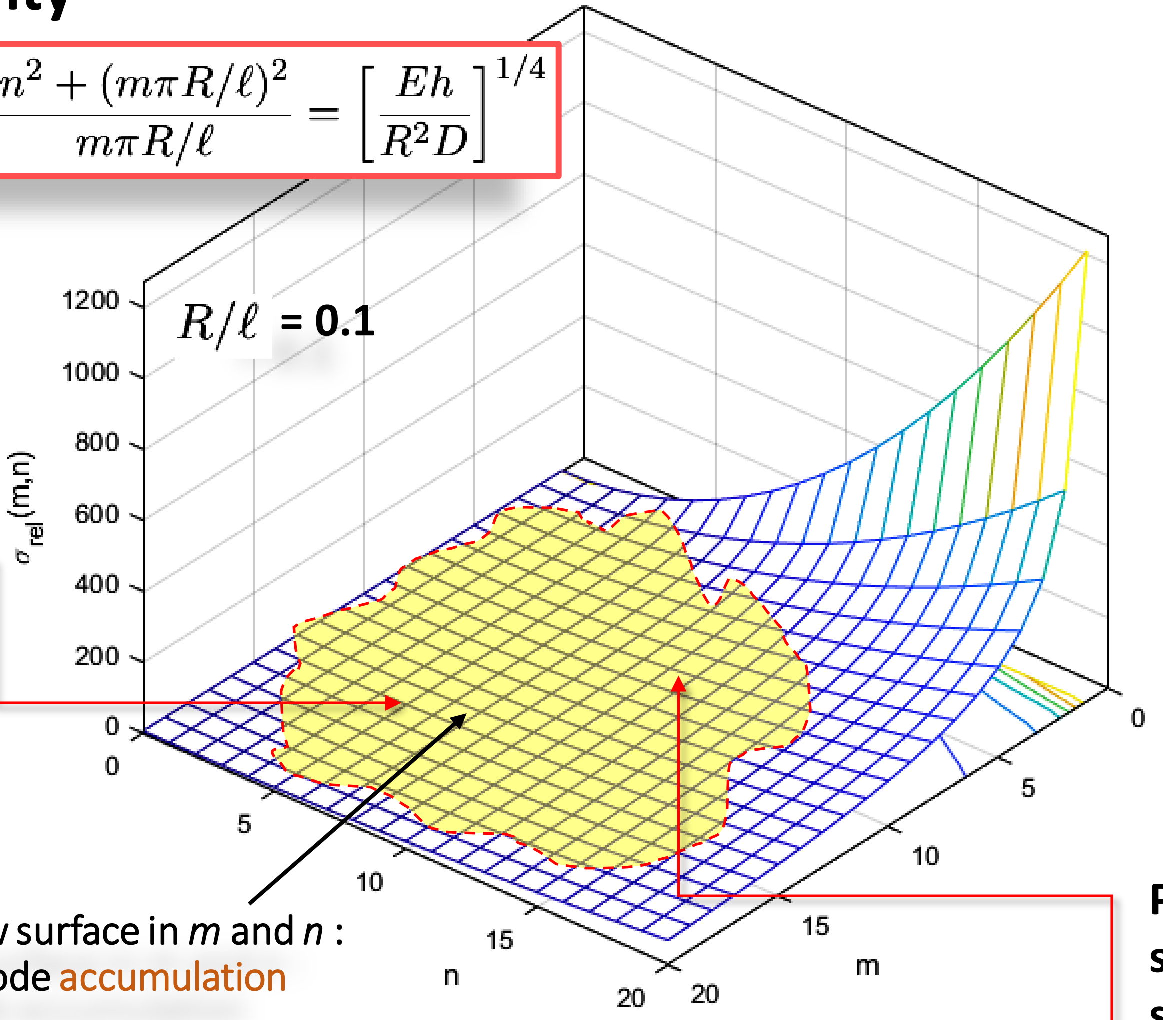
$$\sigma_{cr} \equiv \sigma_{\min} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}.$$

$= 0.6$ (steel)

Very shallow surface in *m* and *n*: Note the mode accumulation

Mode accumulation - sensitivity

$$\frac{n^2 + (m\pi R/\ell)^2}{m\pi R/\ell} = \left[\frac{Eh}{R^2 D}\right]^{1/4}$$



Note that <u>many buckling modes are close to each</u> other (yellow region)

$\Downarrow$

This explains partly Imperfection-sensitivity

Very shallow surface in *m* and *n* :
Note the mode accumulation

Practically the same buckling stress of many modes

Through minimisation, the buckling stress (lower stress) occurs for

$$\frac{n^2 + (m\pi R/\ell)^2}{m\pi R/\ell} = \left[\frac{Eh}{R^2 D}\right]^{1/4} \Longrightarrow \sigma_{cr} \equiv \sigma_{\min} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}.$$

elastic-plastic buckling

Good to know: Interestingly, when critical buckling stress is higher than yield stress, the failure occurs through an elastic-plastic buckling. It is observed, based on experiments, that the formula for critical stress (Eq. 1.1303) still holds after being updated and gives good results in accordance with tests. The correction is to replace the elastic E the elastic modulus by the effective modulus $E^* = \sqrt{E_s E_t}$, where E_s and E_t being respectively, the secant- and the tangent modulus (see the reference after the formula below). The critical buckling stress becomes simply

$$\sigma_{cr} \equiv \sigma_{\min} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}.$$

$$\sigma_{cr} = \frac{\sqrt{E_s E_t}}{\sqrt{3(1-\nu^2)}}\frac{h}{R}.$$

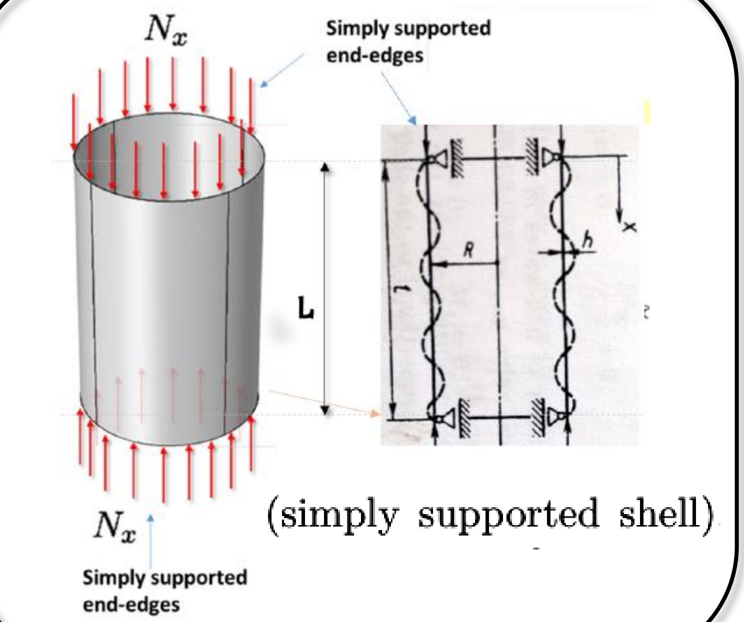
The reference: Gerard, G., *Compressive and Torsional Buckling of Thin-Wall Cylinders in Yield Region*. NACA TN 3726, Washington: National Advisory Committee for Aeronautics, 1956, 40p.

Solution using Donnell's linearised buckling equations for axially loaded cylinder

$$D\nabla^8 w + \frac{Eh}{R^2} w_{,xxxx} - \nabla^4 \left(\underbrace{N_x^0}_{-\frac{P}{2\pi R}} w_{,xx} + \underbrace{2N_{xs}^0 w_{,x} w_{,s} + N_s^0 w_{,ss}}_{=0} \right) = \underbrace{\nabla^4 p}_{=0}$$

$$\implies D\nabla^8 w + \frac{Eh}{R^2} w_{,xxxx} + \underbrace{\frac{P}{2\pi R}}_{\bar{\sigma}_x} \nabla^4 w_{,xx} = 0.$$

Eigen-value problem represents an eight-order differential equation



(simply supported shell)

Kinimetaically admissible trial:

$$w(x,s) = A\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi s}{\pi R}\right) = A\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{\beta\pi s}{\ell}\right)$$

$$\Downarrow \quad \beta = n\ell/(\pi R)$$

$$D\left(\frac{\pi}{\ell}\right)^8 (m^2+\beta^2) + \frac{Eh}{R^2} m^4 \left(\frac{\pi}{\ell}\right)^4 - \bar{\sigma}_x h \left(\frac{\pi}{\ell}\right)^6 m^2 (m^2+\beta^2)^2 = 0.$$

Batdorf parameter $Z = \frac{\ell^2}{Rh}\sqrt{(1-\nu^2)}$

The relative length is reflected well in this parameter

buckling stress parameter $k_x = \frac{\bar{\sigma}_x h \ell^2}{D\pi^2},$

minimum of k_x is attained when $\frac{[m^2+\beta^2(n)]^2}{m^2} = \sqrt{\frac{12Z^2}{\pi^4}}$

the critical stress parameter $\Downarrow$

$$k_x = \frac{(m^2+\beta^2(n))^2}{m^2} + \frac{12Z^2 m^2}{\pi^4 (m^2+\beta^2(n))^2}$$

$\Longrightarrow$ The buckling stress being the smallest k_x,

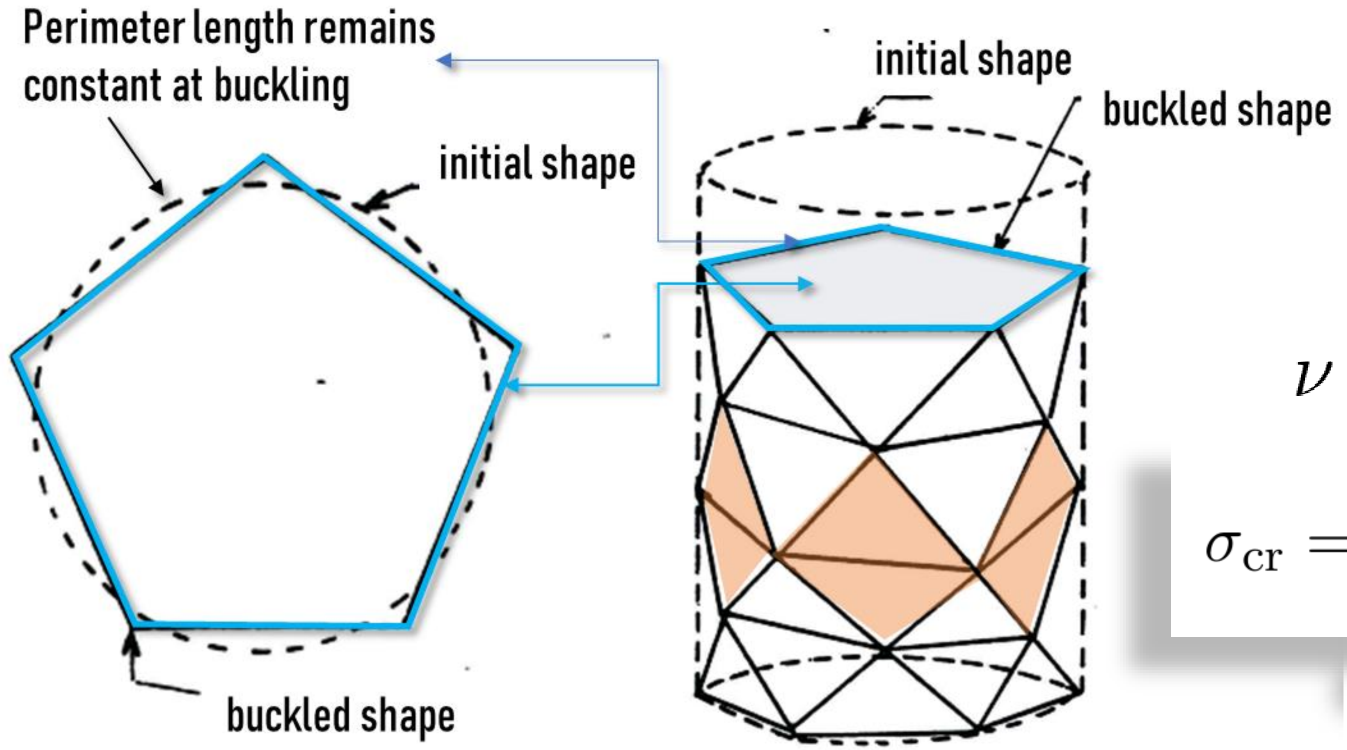
$$k_{x,min} = \frac{4\sqrt{3}}{\pi^2} Z \implies \sigma_{cr,min} \equiv \sigma_E = \frac{1}{\sqrt{3(1-\nu^2)}} \frac{Eh}{R},$$

For short cylinders ($Z < 2.85$), $k_x = 1 + \frac{12Z^2}{\pi^4}.$

This is a famous classical result

Diamond-shaped buckling of cylindrical shells under uniform axial compression

$$w(x,y) = \frac{1}{2} w_0 \cdot \cos^2\left(\frac{n\pi x}{2\ell} - \frac{my}{2R}\right) \cdot \cos^2\left(\frac{n\pi x}{2\ell} + \frac{my}{2R}\right)$$



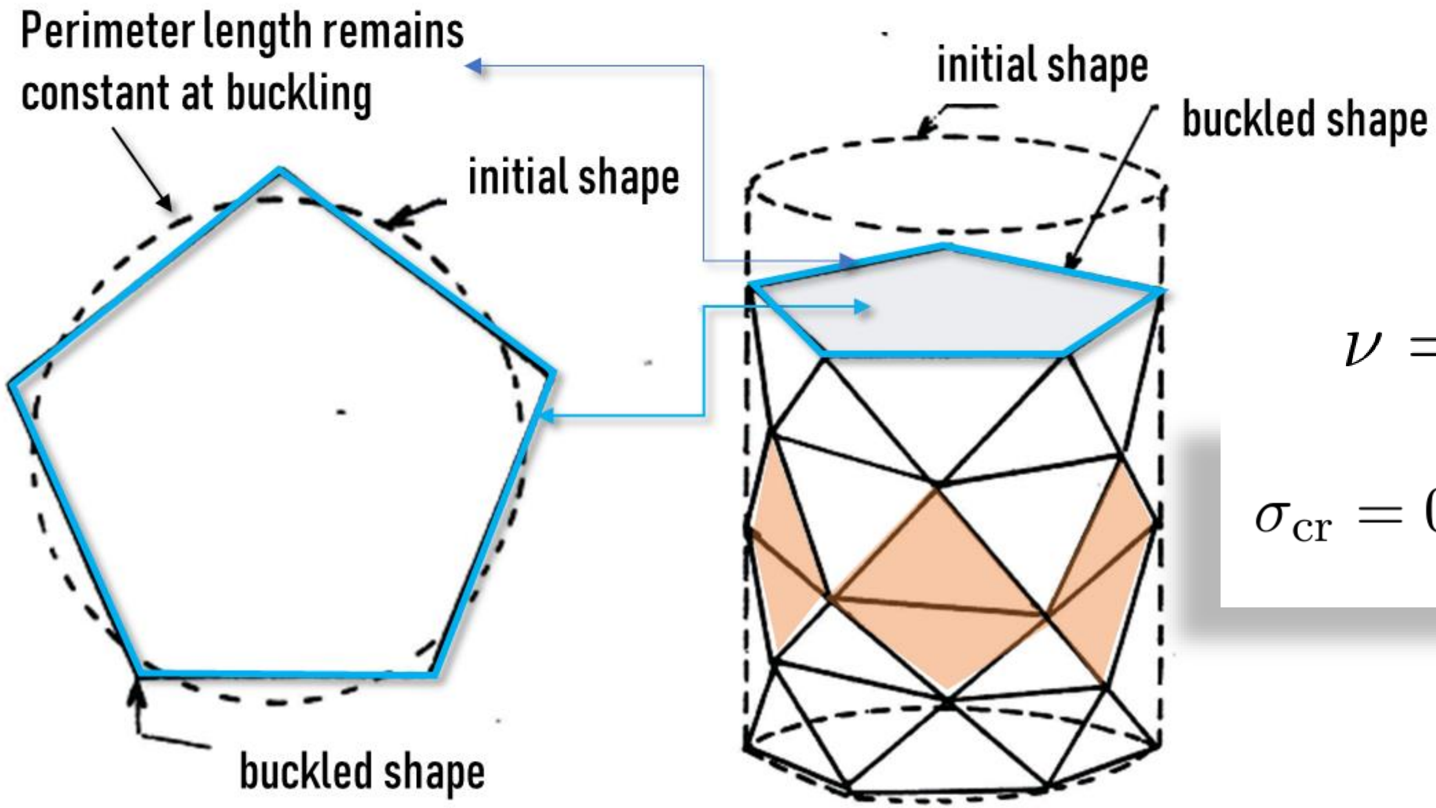
$$\nu = 0.3$$

$$\sigma_{\mathrm{cr}} = 0.19 \cdot \frac{Eh}{R}$$

E.R. Lancaster et al. / *International Journal of Mechanical Sciences* **42** (2000) 843-865

Diamond-shaped buckling of cylindrical shells under uniform axial compression

$$w(x,y) = \frac{1}{2} w_0 \cdot \cos^2\left(\frac{n\pi x}{2\ell} - \frac{my}{2R}\right) \cdot \cos^2\left(\frac{n\pi x}{2\ell} + \frac{my}{2R}\right)$$



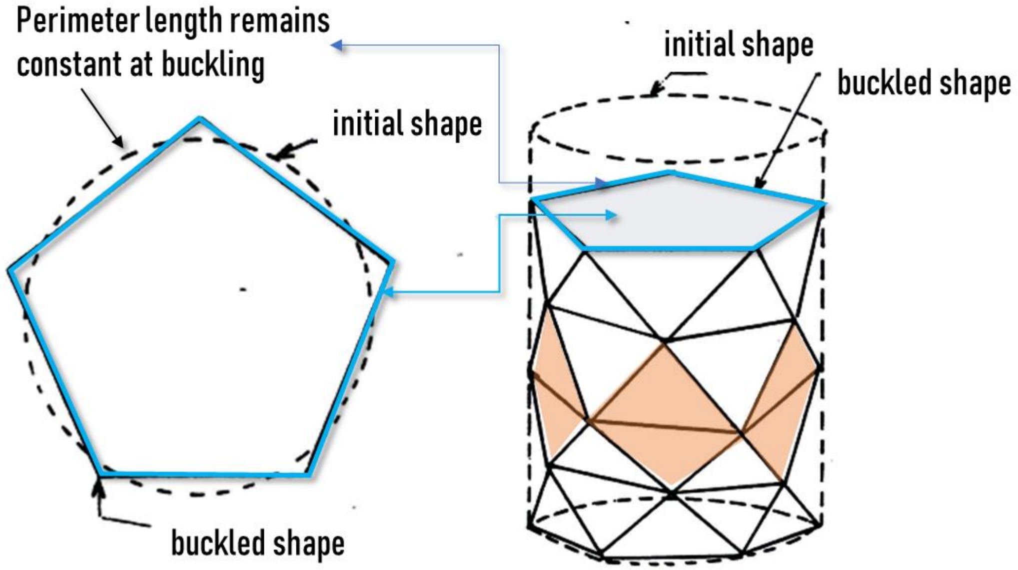
$$\nu = 0.3$$

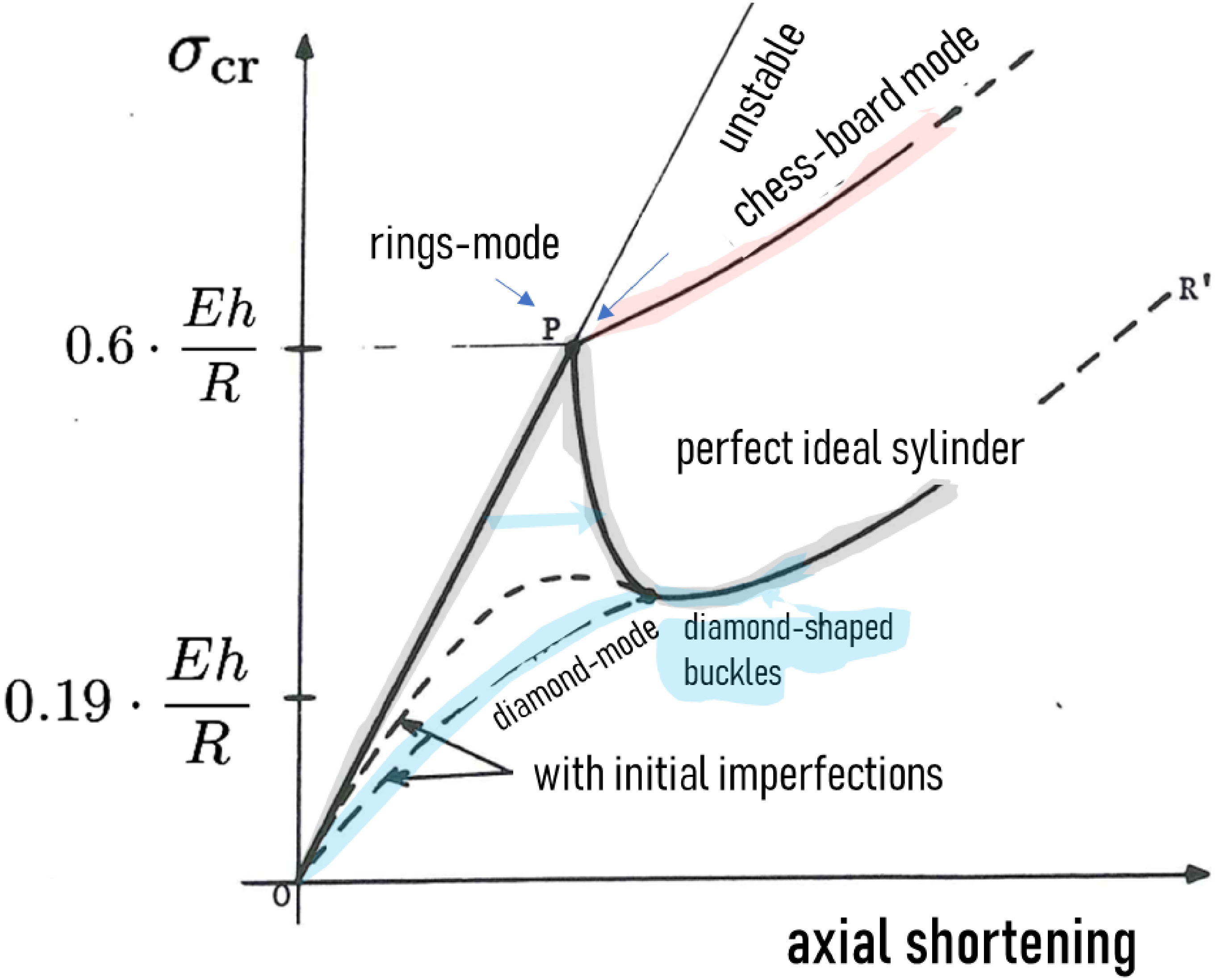
$$\sigma_{cr} = 0.19 \cdot \frac{Eh}{R}$$

Ref: Doctoral thesis - FR

Diamond-shaped buckling of cylindrical shells under uniform axial compression

$$w(x,y) = \frac{1}{2} w_0 \cdot \cos^2\left(\frac{n\pi x}{2\ell} - \frac{my}{2R}\right) \cdot \cos^2\left(\frac{n\pi x}{2\ell} + \frac{my}{2R}\right)$$



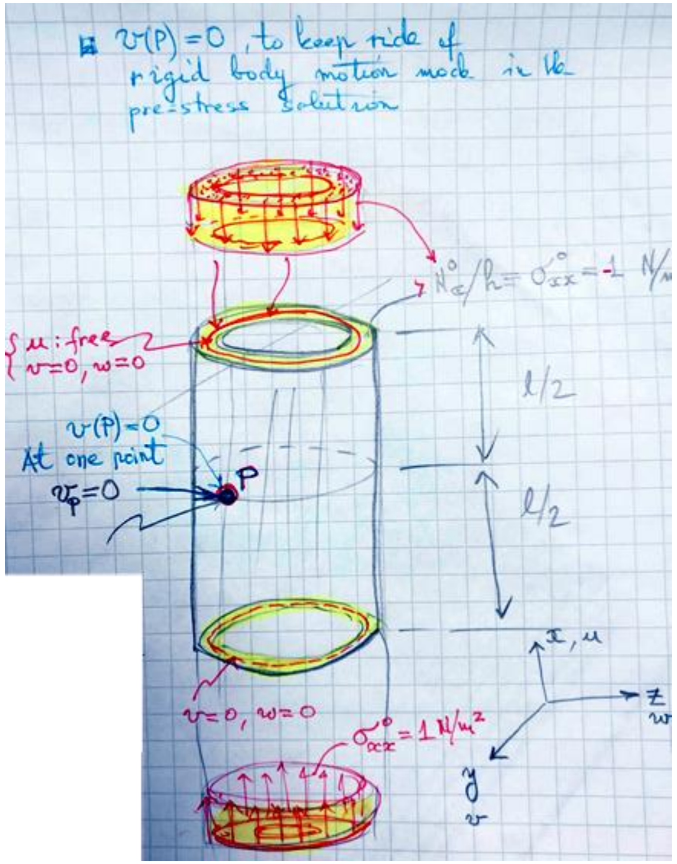


Finite Element Example

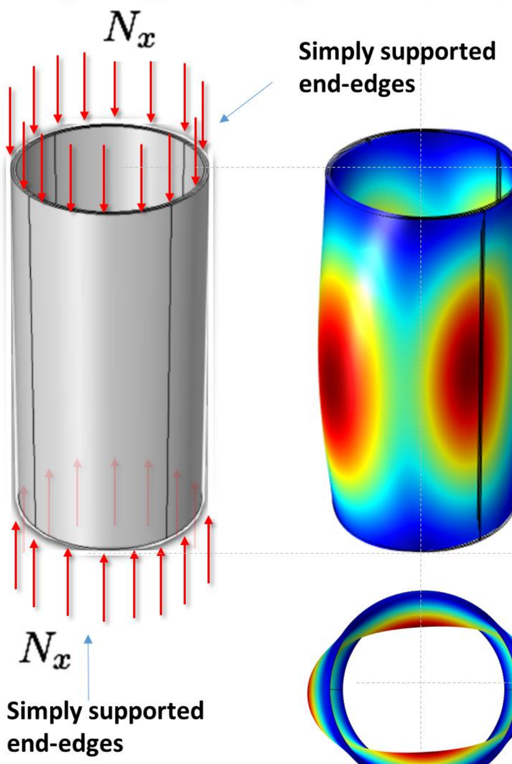
Buckling of thin-walled cylindrical shells

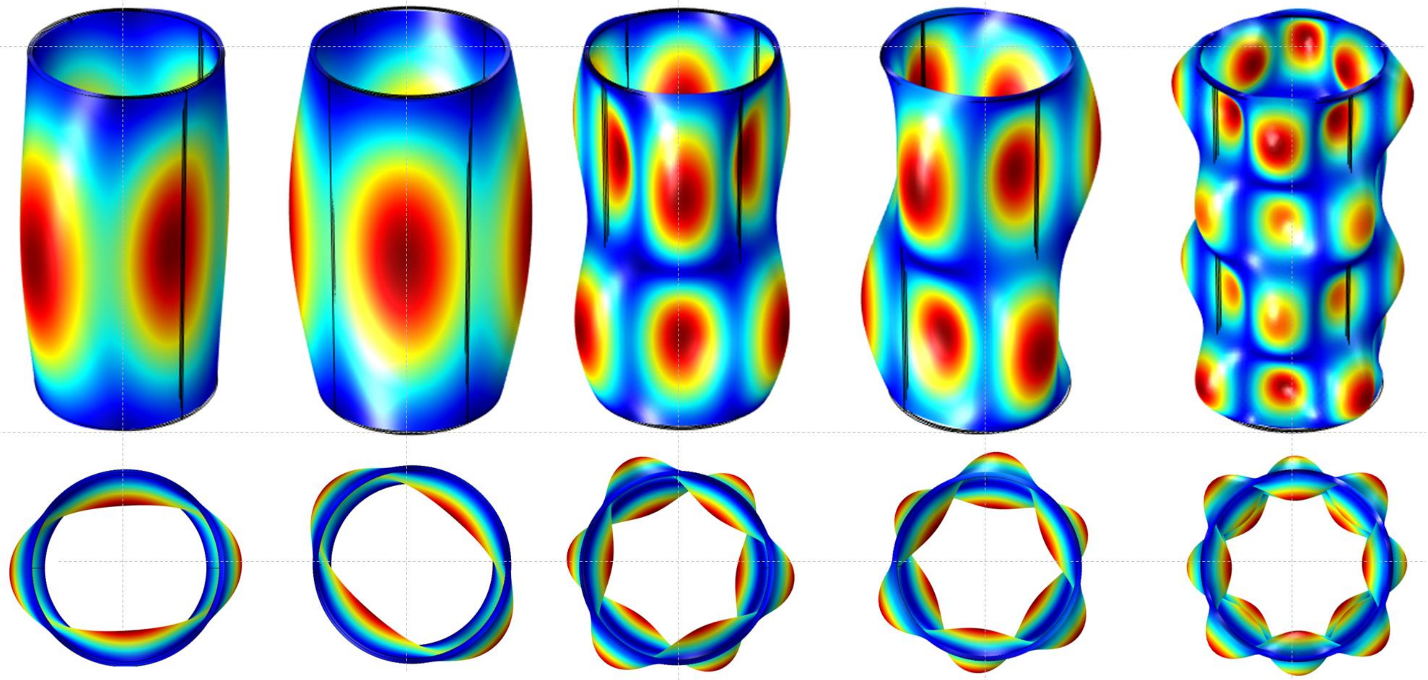
Computational example

A relatively longer shell



Axially compressed cylindrical shell





Buckling modes:	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$(N_{x,cr})_i / (N_{x,cr})_1 =$	1.0000	1.0001	1.0083	1.0083	1.0861

Parameters

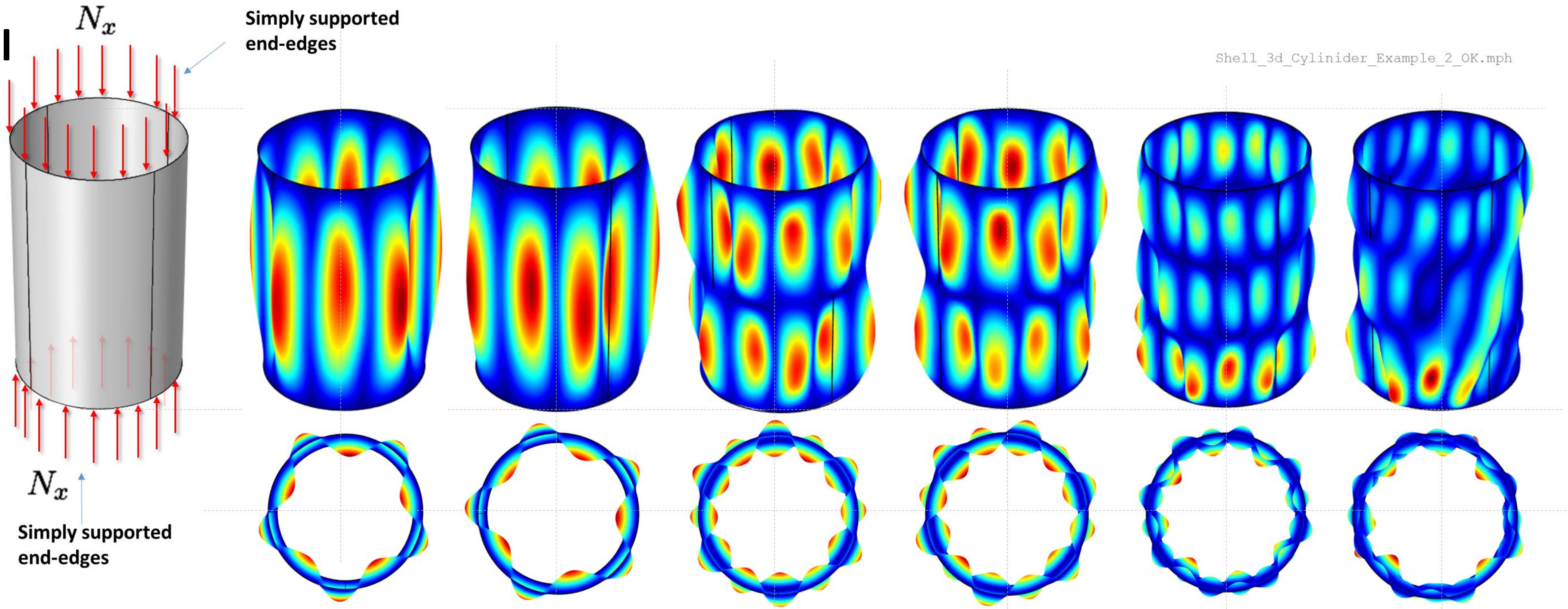
Name	Expression	Value	Description
Nx_post	1	1	Pa
param	1	1	param 0:3 post-buck
N_Reduct	10000	10000	Transverse perturb
F_perturb	1	1	N
u0_cr	1	1	critical strectch per e
xxx	0	0	******* BEGIN SHELL
rho_s	2700	2700	density
nu_s	0.33	0.33	Poisson ratio
E_s	70e9	7E10	[Pa] Young's modulu
R_shell	1	1	m shell-radius
h_shell	R_shell/20	0.05	shell wall-thick
L_shell	4*R_shell	4	length of the tube
P_shell_ref	1	1	N
R_inner	R_shell - h_shell	0.95	Inner Radius
h_shell_chek	R_shell-R_inner	0.05	m (croos-check)
A_thick	2*pi*(R_shell-R_inner)^2	0.015708	m**2
sigma_shell_ref	1	1	Pa
P_ref	sigma_shell_ref*A_thick	0.015708	N
u_ref	3*h_shell	0.15	m

Note how close to each other the Eigen-values are $\Longrightarrow$ **imperfection-sensitivity**

FE Computational example

A shorter shell

Axially compressed cylindrical shell



Buckling modes:	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	
$(N_{x,cr})_i/(N_{x,cr})_1 =$	1.0000	1.0007	1.0283	1.0331	1.0456	1.0501	
Shell theory: 215.14 MPa	268.47	268.67	276.06	277.35	280.7	281.93	**MPa**

Note how close to each other the Eigen-values are $\Longrightarrow$ **imperfection-sensitivity**

Effect of imperfection on post-buckling behaviour

Effect of imperfections

All **real structural** systems are **imperfect**

- ✓ in form,
- ✓ in material properties,
- ✓ in the sense of residual stresses
- ✓ in the way the loads are applied

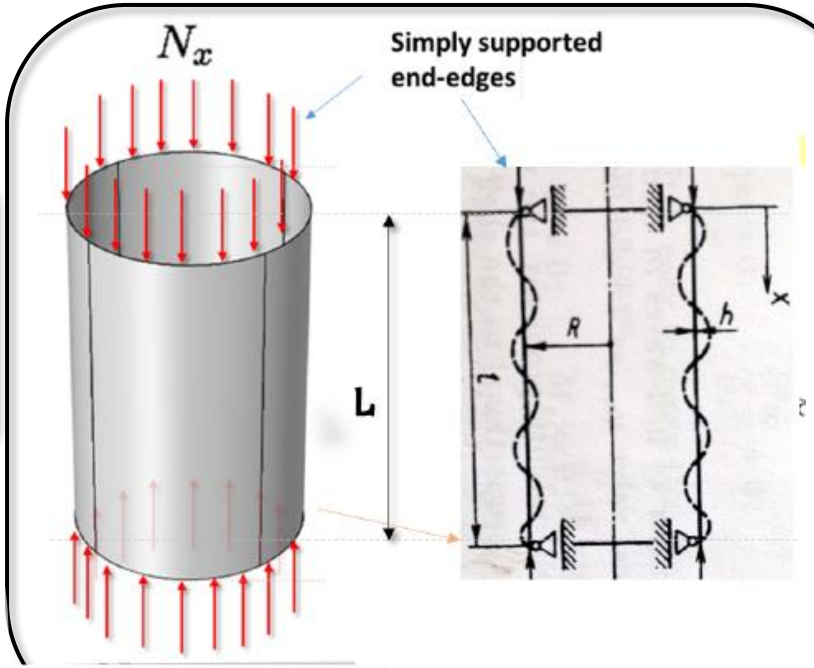
[12] *It may be safely said that all real structural systems are imperfect in form, imperfect in material properties, imperfect in the sense of residual stresses and imperfect in the way the loads are applied.* **Roorda** (1980)

Axisymmetric buckling of circular cylindrical shells under uniform axial compression

- consider an isotropic ideally perfect thin **cylindrical** shell of radius R under uniform axial compression ($N > 0$)
- consider **buckling** in an **axisymmetric mode** (ring mode)

($N > 0$, compression)

$$N_x^0 = -\frac{N}{2\pi R}, \quad N_{xs}^0 = N_s^0 = 0.$$



$$D\frac{d^4 w}{dx^4} + N\frac{d^2 w}{dx^2} + \frac{Eh}{R^2}w = 0,$$

Tim…

$\ldots/12(1-\nu^2)$,

Trial solution in the forr…

(kin. admissible)

$\ldots_0 \sin\frac{m\pi x}{\ell}$

(Ring patterns)

substi… solution into the ODE gives the critical load

$$N_{cr} \ldots D\left[\alpha_m^2 + \frac{Eh}{\alpha_m^2 R^2 D}\right] \quad \alpha_m = \frac{m\pi}{\ell}$$

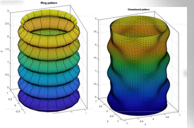
$$\Downarrow$$

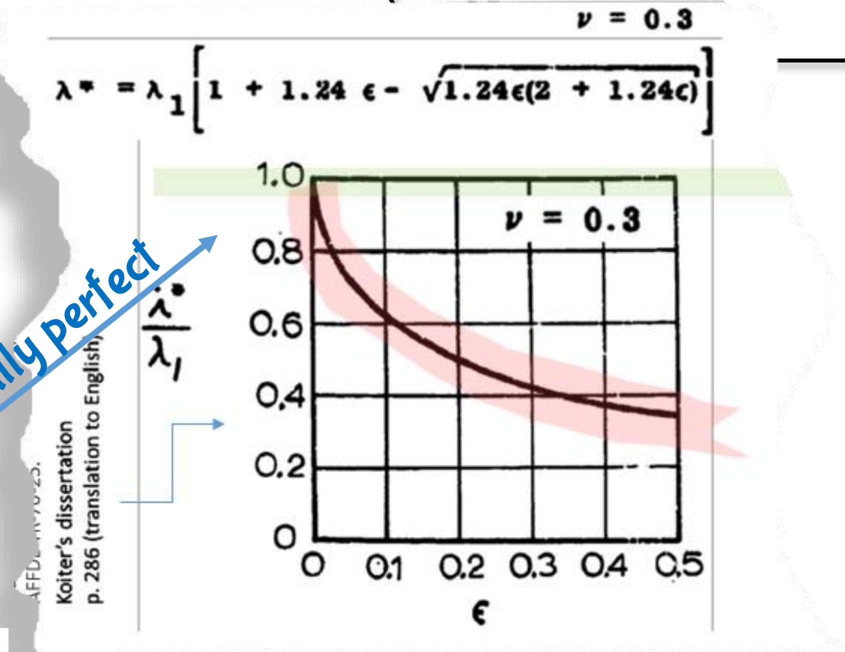
$$N_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h^2}{R} \rightarrow \sigma_{cr} = \frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}$$

= **0.6** (steel)

Effect of imperfection on post-buckling behaviour

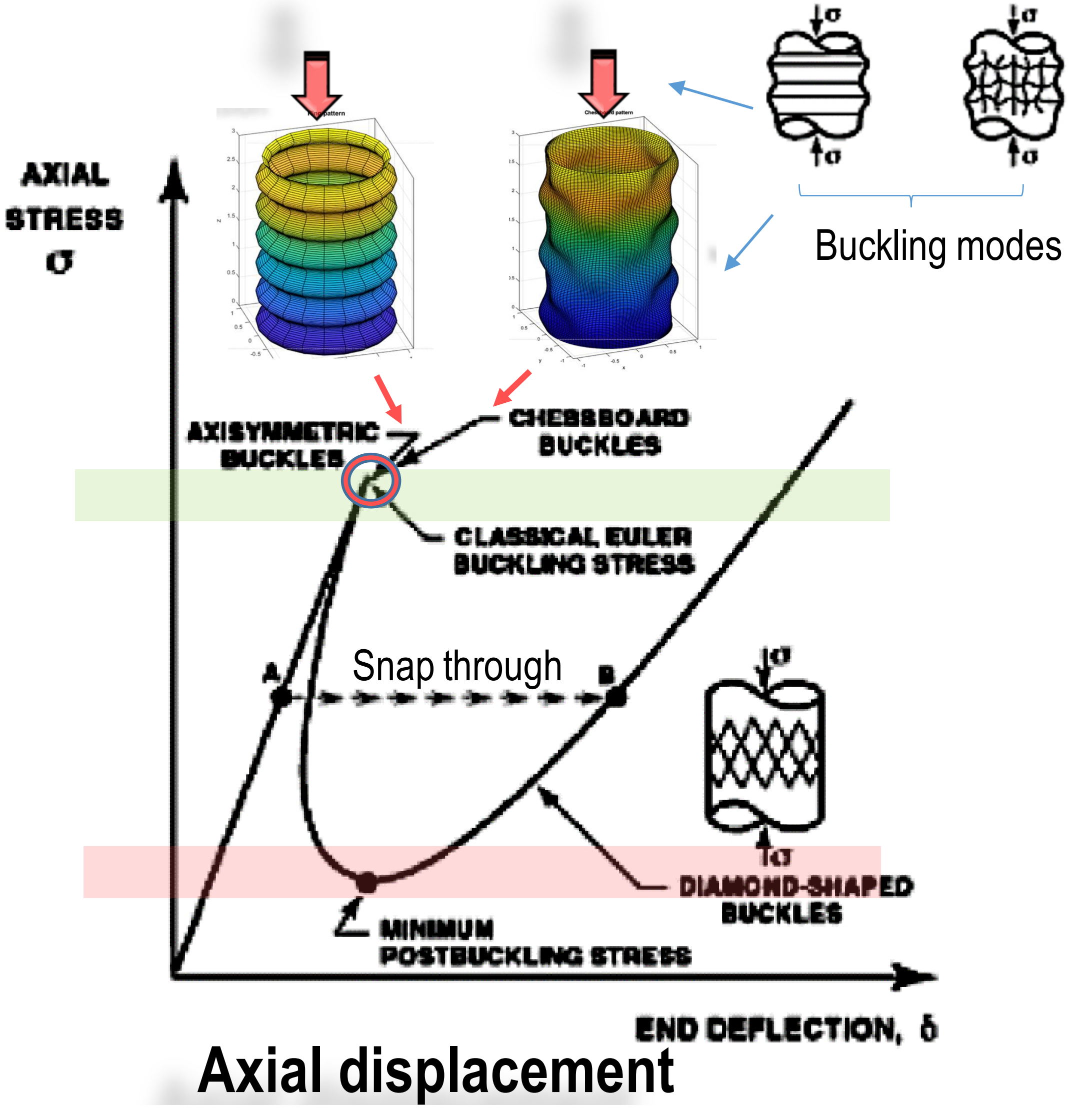
ideally perfect



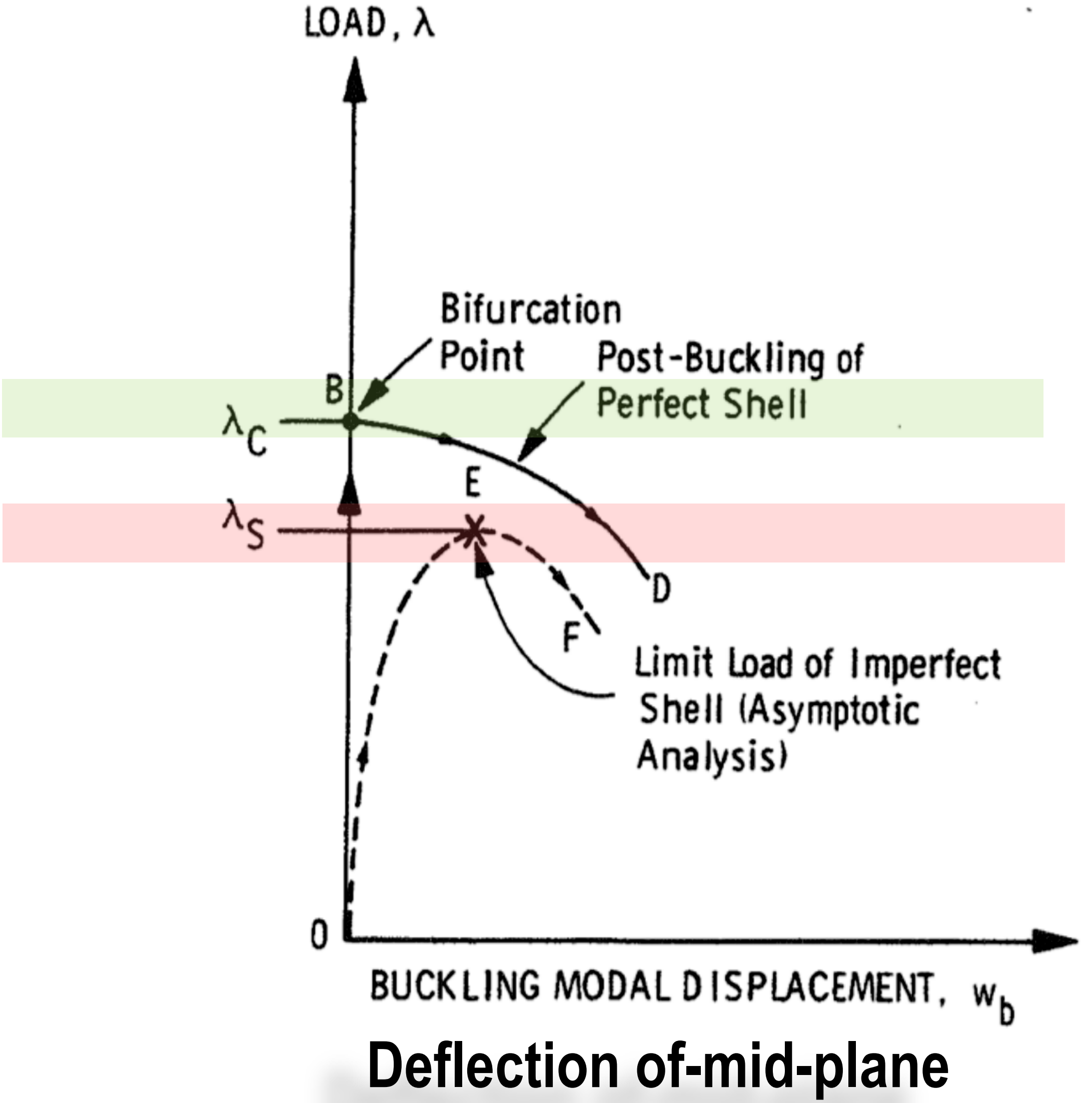


The great sensitivity of the buckling load for small deviations from the perfect cylindrical shell is clear from this last formula

Effect of imperfection on post-buckling behaviour



Typical post-buckling behaviour of axially compressed ideally perfect thin cylindrical shells (ref. Robert Jones, buckling of bars, plates and shells.).



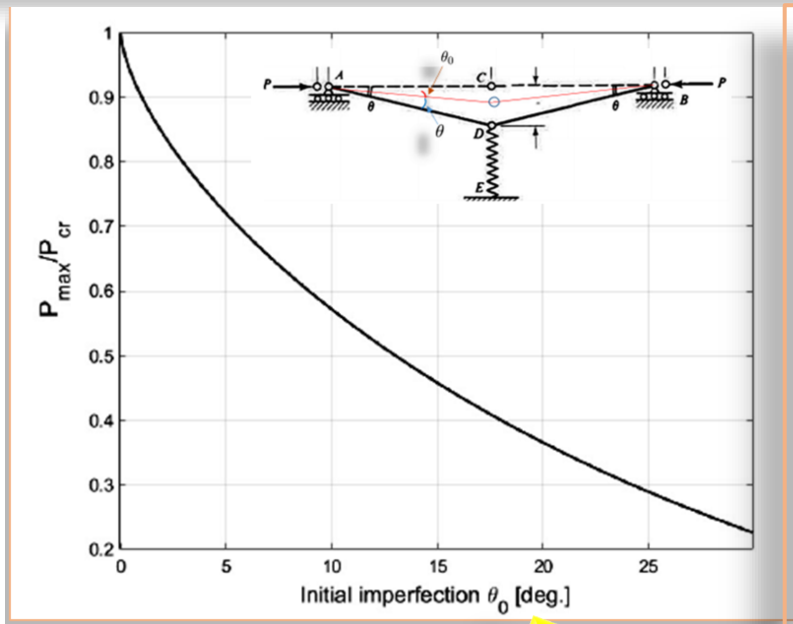
Unstable-symmetric bifurcation model

Axially loaded structure with imperfections:

- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

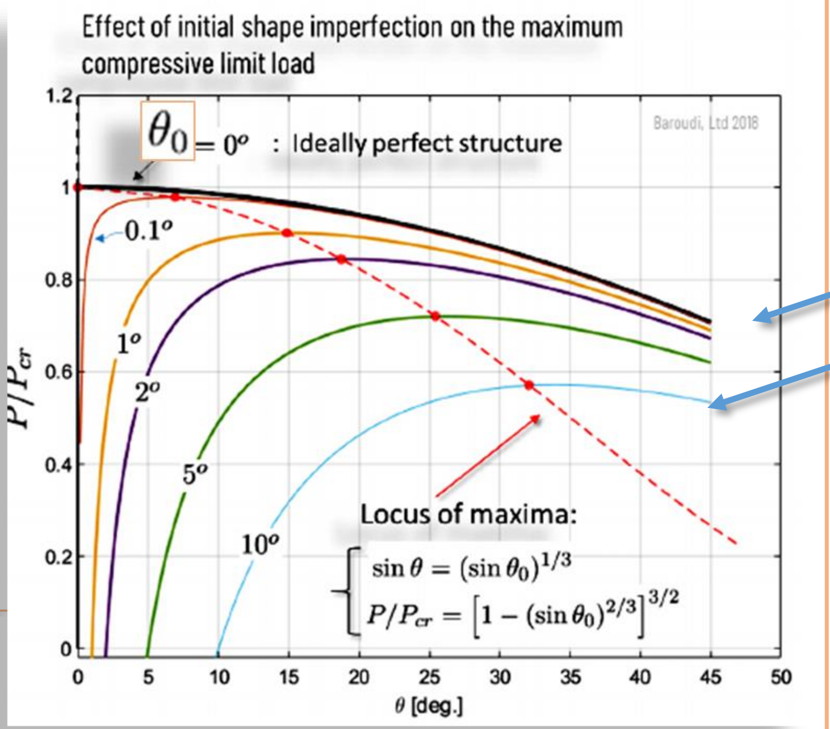
What behavior is common for these two structures?

Cylindrical shell

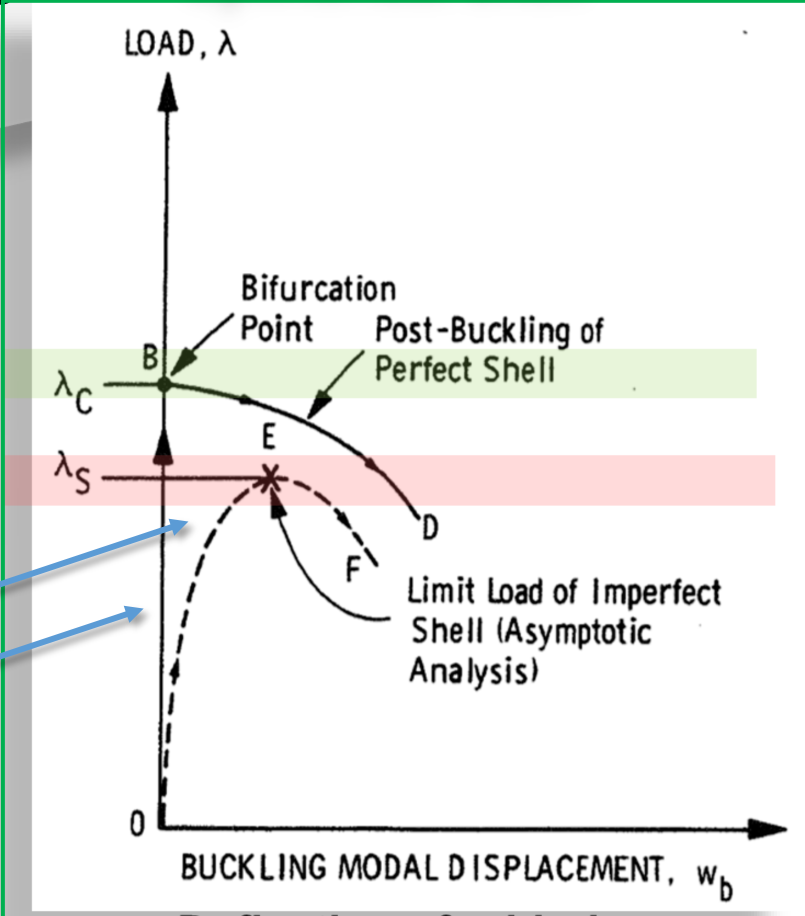


RECALL

Maximum axial force reduction with respect to the amplitude of initial imperfection. P_{cr} is the collapse or buckling load of the perfect structure.



Imperfect structure



Deflection of-mid-plane

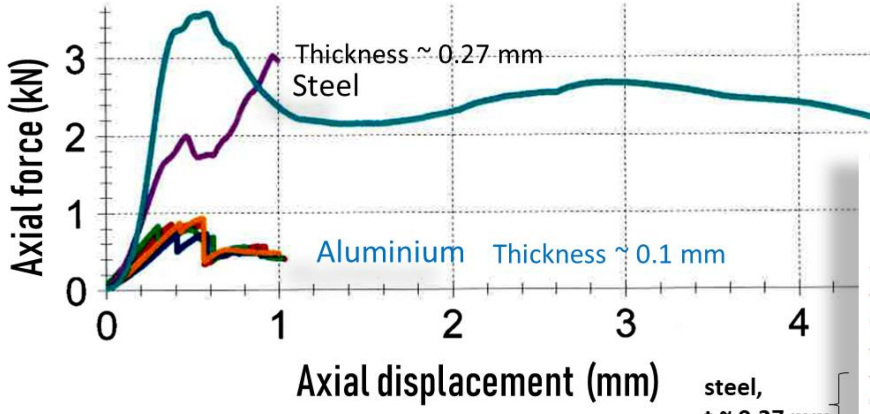
Shells are imperfection-sensitive structures

Tests by A. Niemi and V.A. Hakala & J. Piironen (Civil Engineering department, Otaniemi)

Buckling of cylindrical thin-walled shell

Experimental results

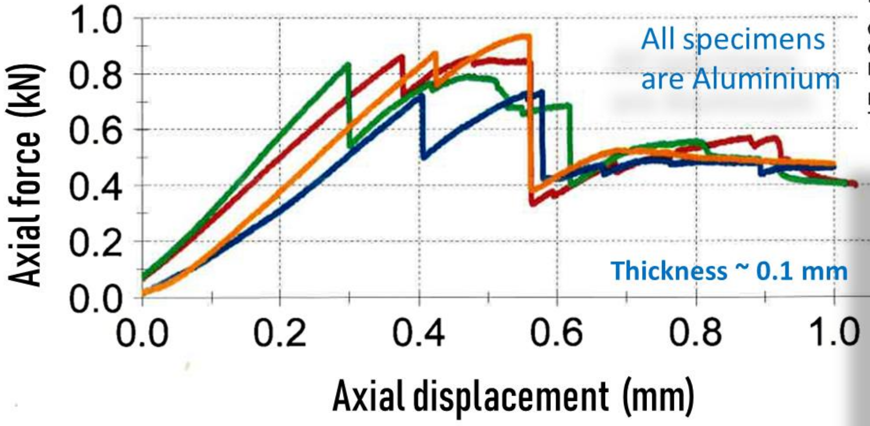
Aalto-yliopisto
Insinööritieteiden korkeakoulu
Rakennustekniikan laitos



Test results:

steel, t ~ 0.27 mm (specimens 5–6); Aluminium, t ~ 0.1 mm (specimens 1–4)

Legends	Nr	Specimen identifier	F_{max} kN	dL at F_{max} mm
(red)	1	Koff - Jouluolut, 0,33 l	0,86	0,4
(green)	2	Koff - Nikolai 0,33 l	0,83	0,3
(blue)	3	Koff - Karhu 0,50 l	0,73	0,6
(orange)	4	Koff - 0,50 l	0,93	0,6
(purple)	5	Teräslieriö H=150mmxD=66mm, t=0,27mm	3,03	1,0
(teal)	6	Teräslieriö, kokeen jatkaminen	3,58	0,6



Test report

Customer : Rak-54_3110 / Alumiinitölkit
Group : Esikokeet
Material : Alumiinitölkki
Specimen type : Alumiinitölkki 0,33 ja 0,5 l
Tester : Veli-Antti Hakala ja Jukka Piironen

Pre-load : 10 N
Test speed : 0,5 mm/min

Aalto-yliopisto
Insinööritieteiden korkeakoulu
Rakennustekniikan laitos

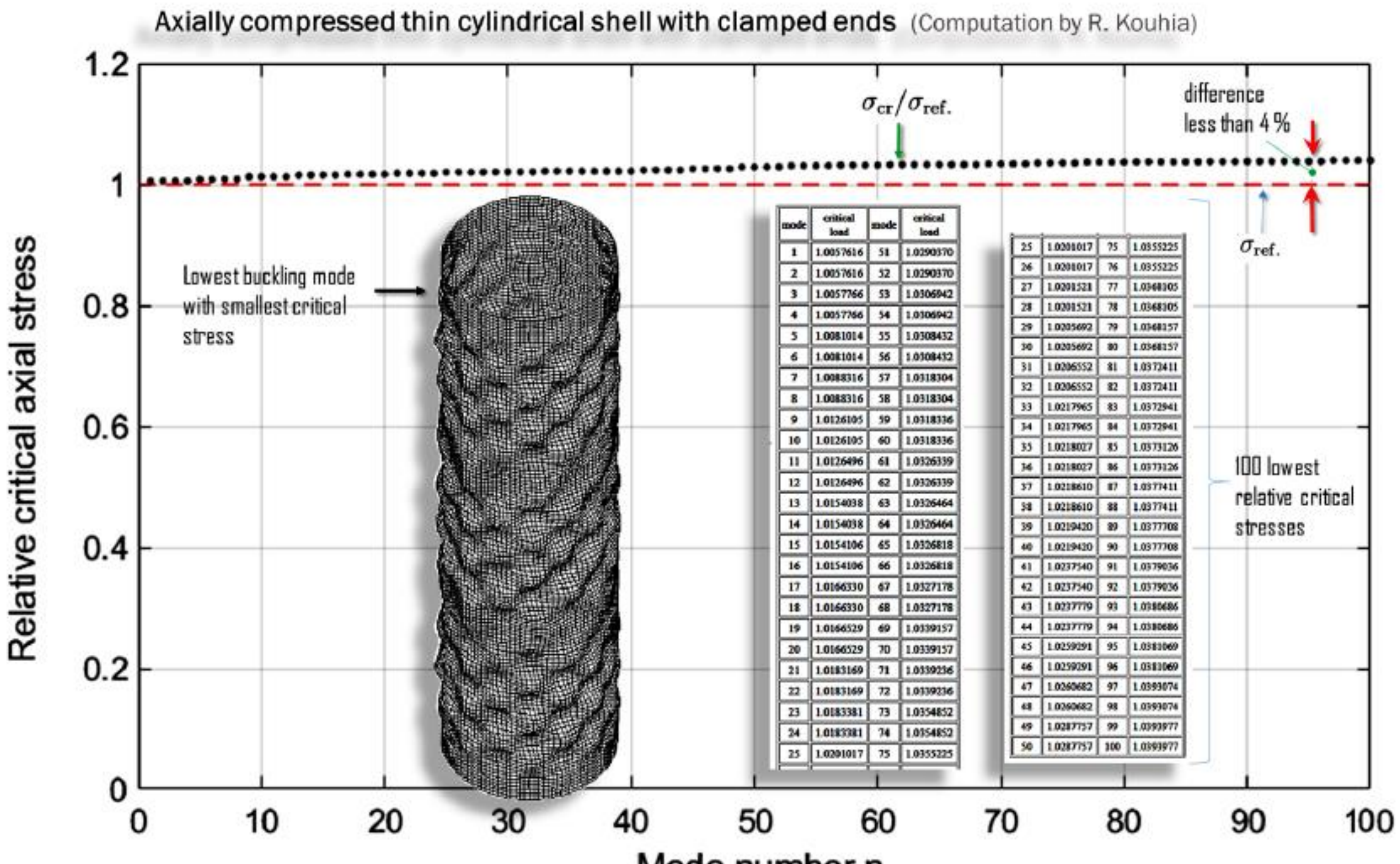
Test results: Aluminium, t ~ 0.1 mm

Legends	Nr	Specimen identifier	F_{max} kN	dL at F_{max} mm
(red)	1	Koff - Jouluolut, 0,33 l	0,86	0,4
(green)	2	Koff - Nikolai 0,33 l	0,83	0,3
(blue)	3	Koff - Karhu 0,50 l	0,73	0,6
(orange)	4	Koff - 0,50 l	0,93	0,6

Shells are imperfection-sensitive structures

Mode accumulation

makes **imperfection-** and **perturbation** sensitive



mode	critical load	mode	critical load
1	1.0057616	51	1.0290370
2	1.0057616	52	1.0290370
3	1.0057766	53	1.0306942
4	1.0057766	54	1.0306942
5	1.0081014	55	1.0308432
6	1.0081014	56	1.0308432
7	1.0088316	57	1.0318304
8	1.0088316	58	1.0318304
9	1.0126105	59	1.0318336
10	1.0126105	60	1.0318336
11	1.0126496	61	1.0326339
12	1.0126496	62	1.0326339
13	1.0154038	63	1.0326464
14	1.0154038	64	1.0326464
15	1.0154106	65	1.0326818
16	1.0154106	66	1.0326818
17	1.0166330	67	1.0327178
18	1.0166330	68	1.0327178
19	1.0166529	69	1.0339157
20	1.0166529	70	1.0339157
21	1.0183169	71	1.0339236
22	1.0183169	72	1.0339236
23	1.0183381	73	1.0354852
24	1.0183381	74	1.0354852
25	1.0201017	75	1.0355225

25	1.0201017	75	1.0355225
26	1.0201017	76	1.0355225
27	1.0201521	77	1.0368305
28	1.0201521	78	1.0368305
29	1.0205692	79	1.0368157
30	1.0205692	80	1.0368157
31	1.0206552	81	1.0372411
32	1.0206552	82	1.0372411
33	1.0217965	83	1.0372941
34	1.0217965	84	1.0372941
35	1.0218027	85	1.0373126
36	1.0218027	86	1.0373126
37	1.0218610	87	1.0377411
38	1.0218610	88	1.0377411
39	1.0219420	89	1.0377708
40	1.0219420	90	1.0377708
41	1.0237540	91	1.0379036
42	1.0237540	92	1.0379036
43	1.0237779	93	1.0380686
44	1.0237779	94	1.0380686
45	1.0259291	95	1.0381069
46	1.0259291	96	1.0381069
47	1.0260682	97	1.0393074
48	1.0260682	98	1.0393074
49	1.0287757	99	1.0393977
50	1.0287757	100	1.0393977

Ref: prof. **Reijo Kouhia**: http://www.tut.fi/rakmek/personnel/kouhia/rese/lectio/lectio.html

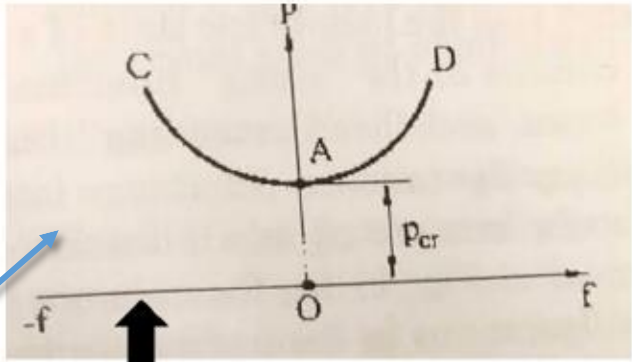
Finite Element Linear buckling analysis of an axially compressed thin cylindrical shell with clamped end. The. FEA shows that more than 100 buckling modes have corresponding critical loads which differ only by less than 4%!. (Reproduced with adaptation with permission of the author).

Shells are imperfection-sensitive structures

Plates and columns

Stable-symmetric bifurcation

Not imperfection sensitive structures

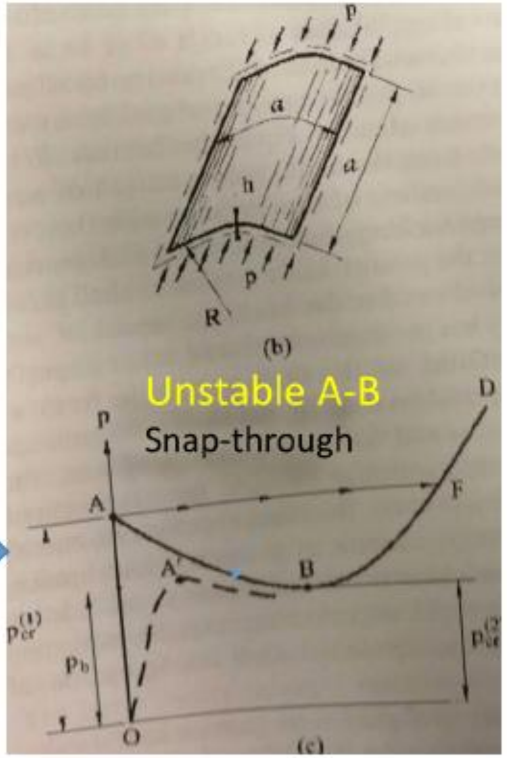


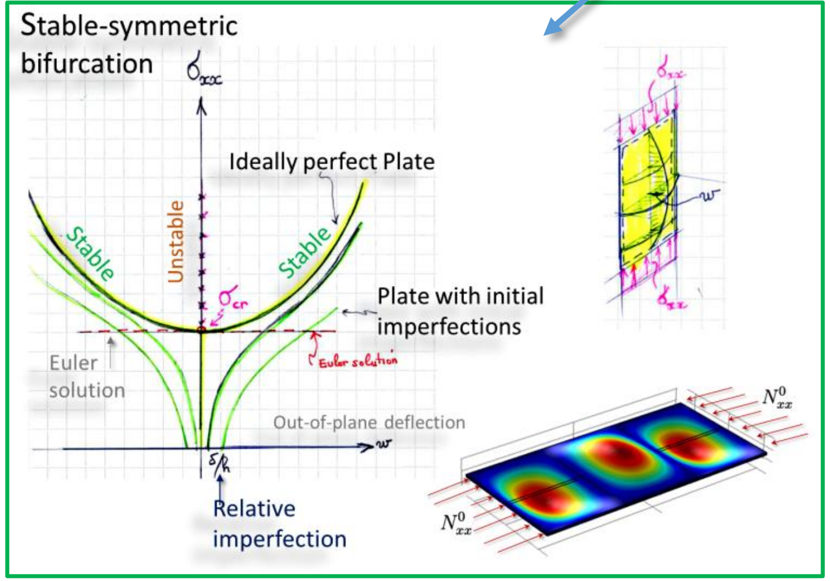
After bifurcation, a stable close neighborhood exists

Consequently, not imperfection sensitive structures

After bifurcation, the close neighborhood is unstable. Very far B-F-D, a stable branch exists

Consequently, imperfection-sensitive structures





what is the main and 'vital' difference, in terms of stability behaviour, between plates, columns and thin shells? Plates are imperfection-insensitive while shells are very sensitive to imperfections because 1) of unstable post-buckling behaviour and 2) some buckling modes are close to each other, mode interaction.

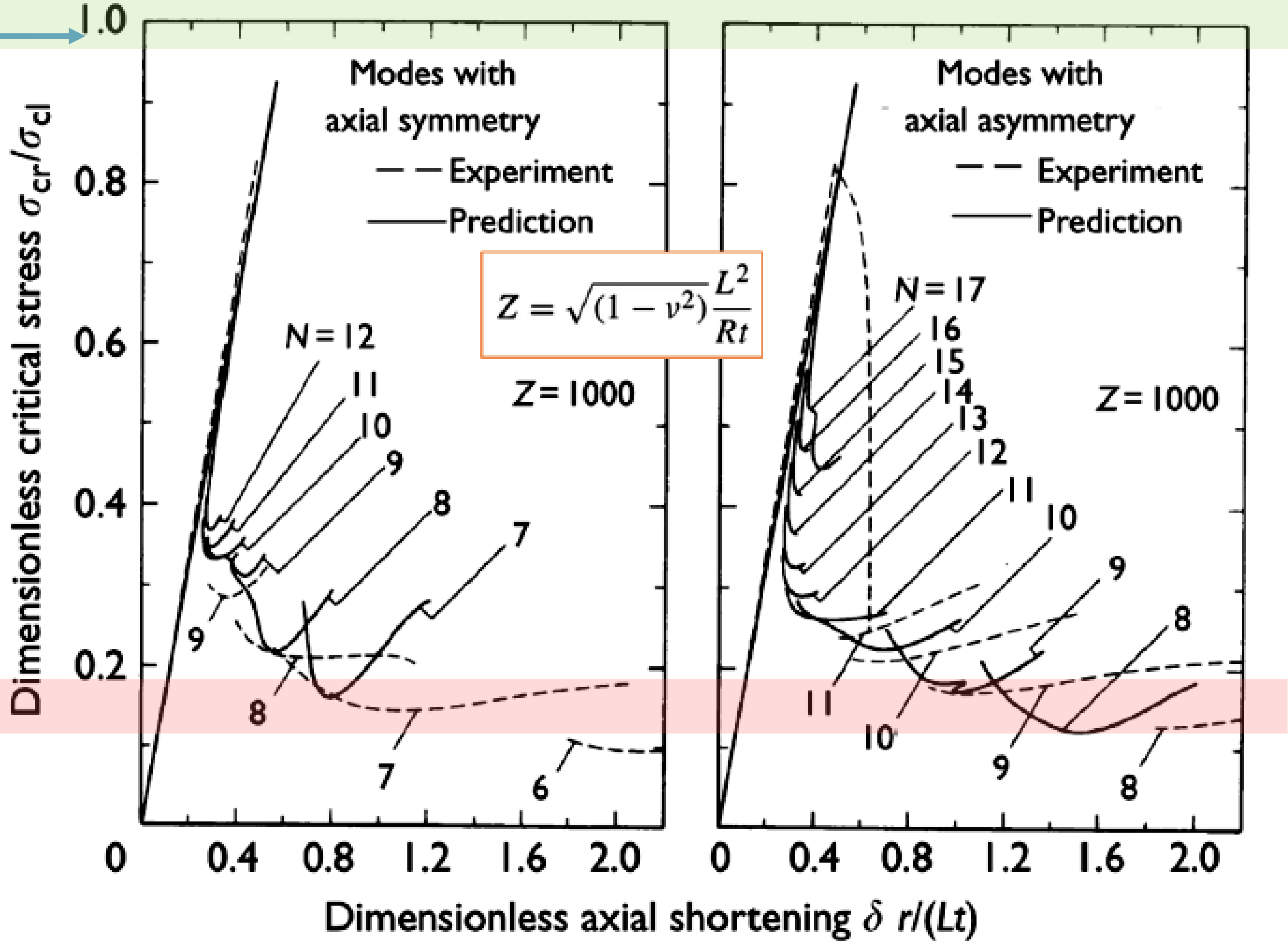
Shells are imperfection-sensitive structures

Experimental evidence

Mode accumulation

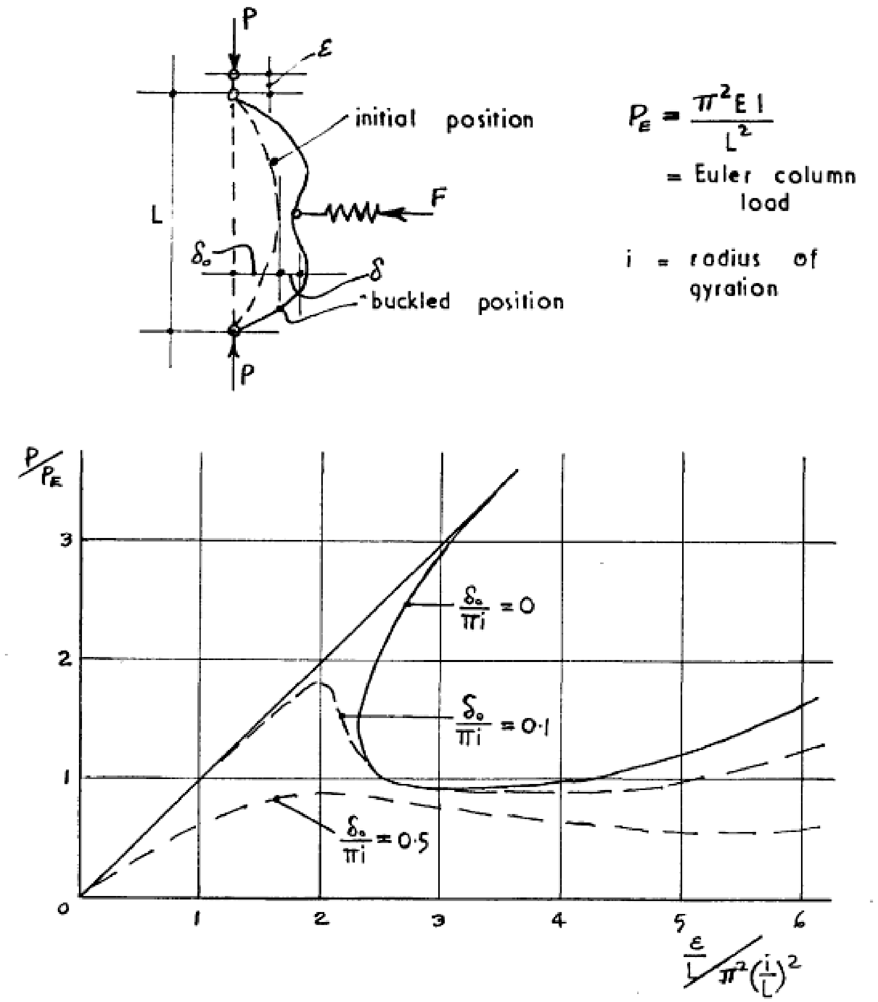
makes sensitive

Euler buckling stress for an ideally perfect shell



Collapse stress for real imperfect shell

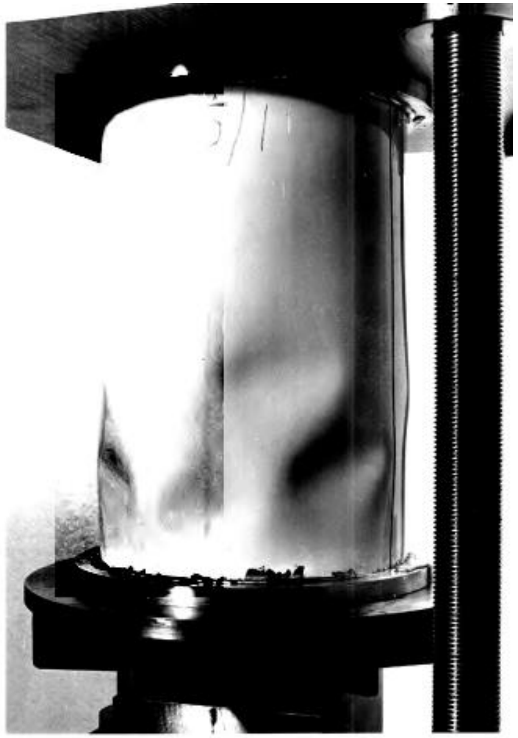
Effect of mode accumulation for thin-walled cylindrical shells. In this figure $t \equiv h$, σ_{cr} being the Euler critical stress and σ_{cl} being the collapse stress. figure reproduced from: N. Yamaki. *Elastic Stability of Circular Cylindrical Shells*. North-Holland (1984).



RELATION BETWEEN LOAD P AND END-SHORTENING ε FOR COLUMNS WITH NONLINEAR LATERAL SUPPORT AND VARYING AMOUNTS OF INITIAL DEFLECTION.

FIG. 3·12 COLUMN SUPPORTED LATERALLY BY NONLINEAR SPRING [REF. 133]

(133) Tsien, H.S. "The buckling of columns with non-linear supports" J. Aero. Sci., Vol. 9, p. 119, 1942.



THE STABILITY OF THIN SHELLS

by

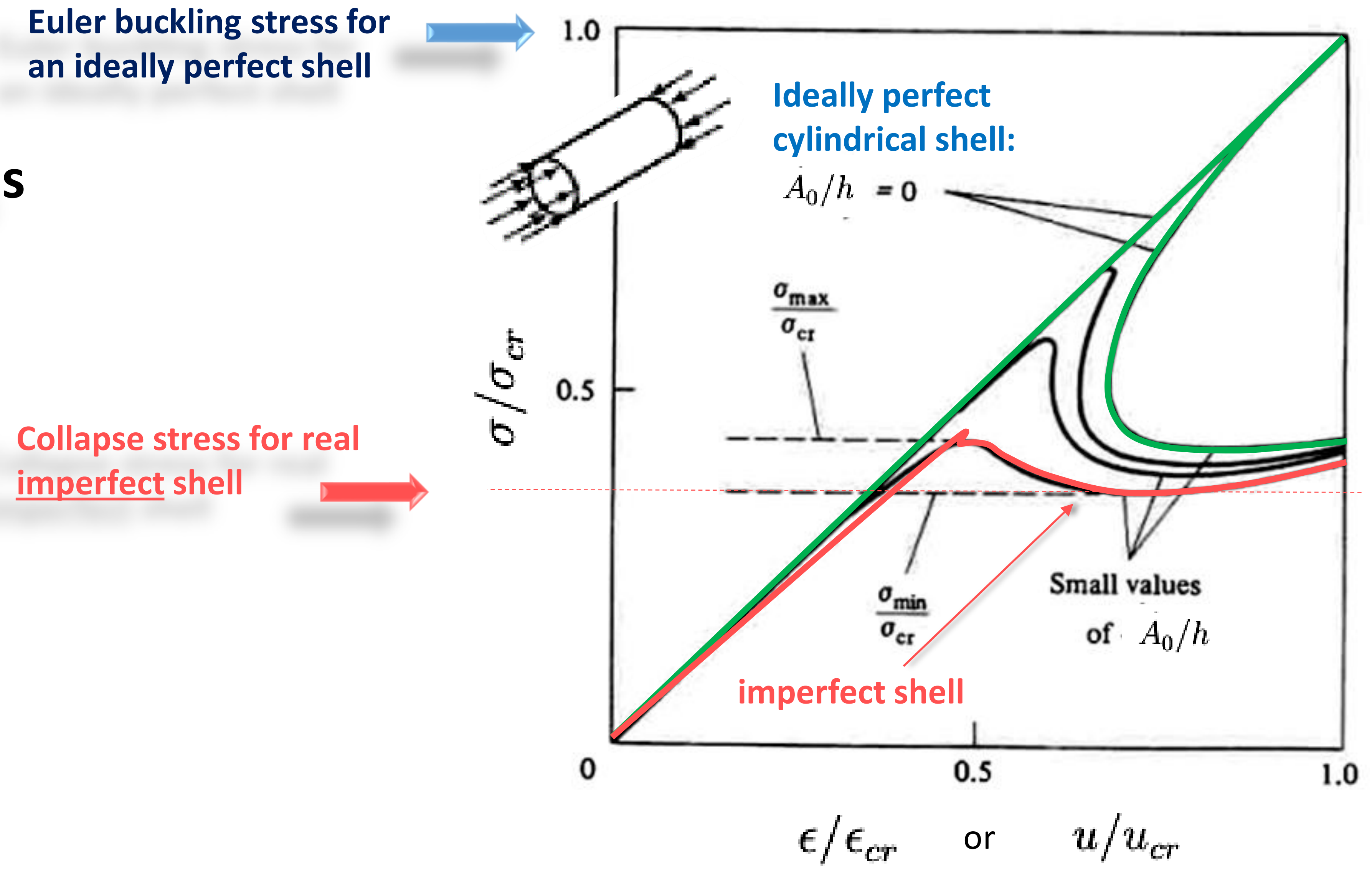
P.J. MOSS, B.E.(N.Z.), D.I.C.

A Thesis submitted for the Degree of Doctor of Philosophy in the Faculty of Engineering of the University of London

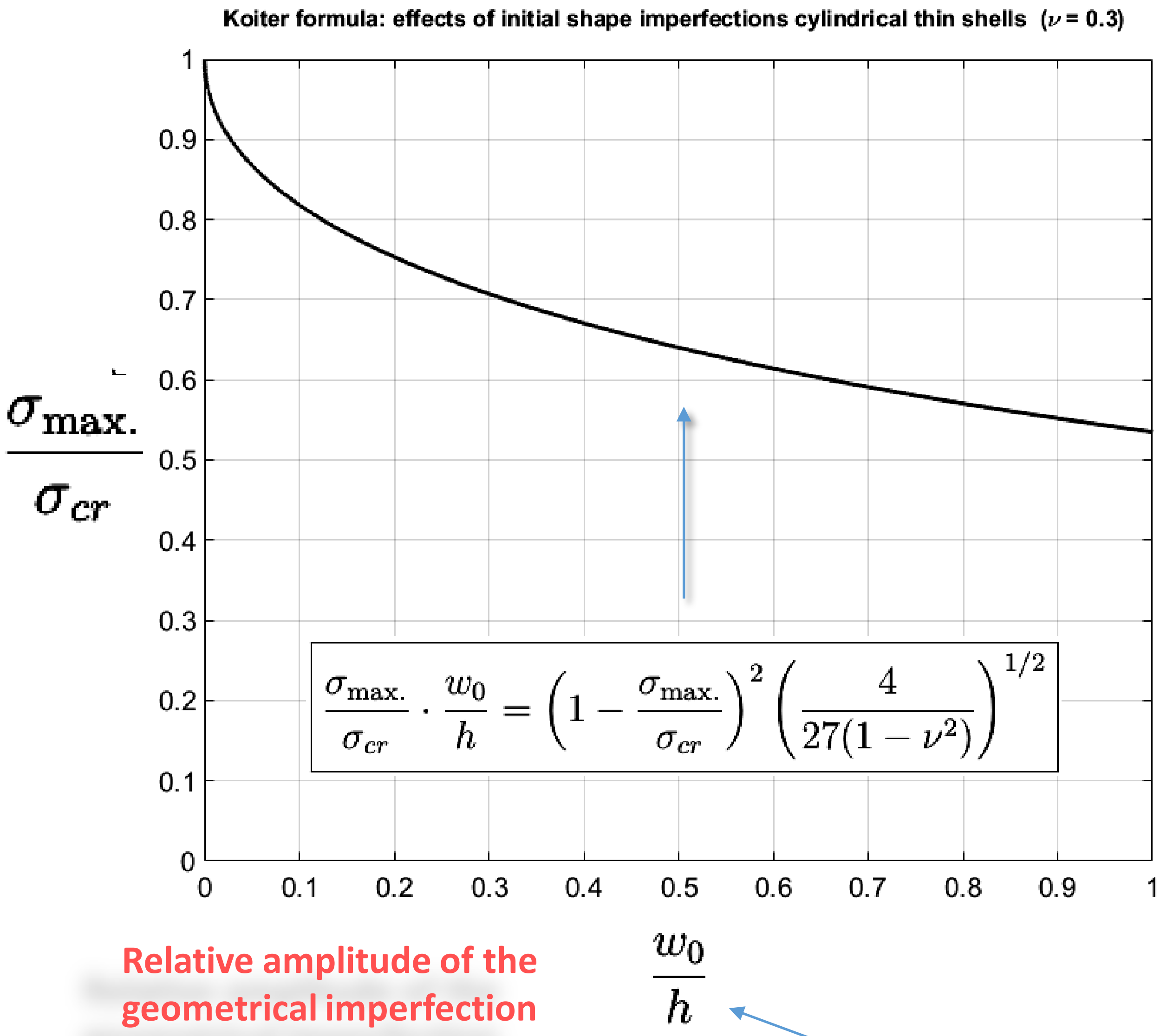
Civil Engineering Department,
Imperial College of Science and Technology,
London.

December 1964

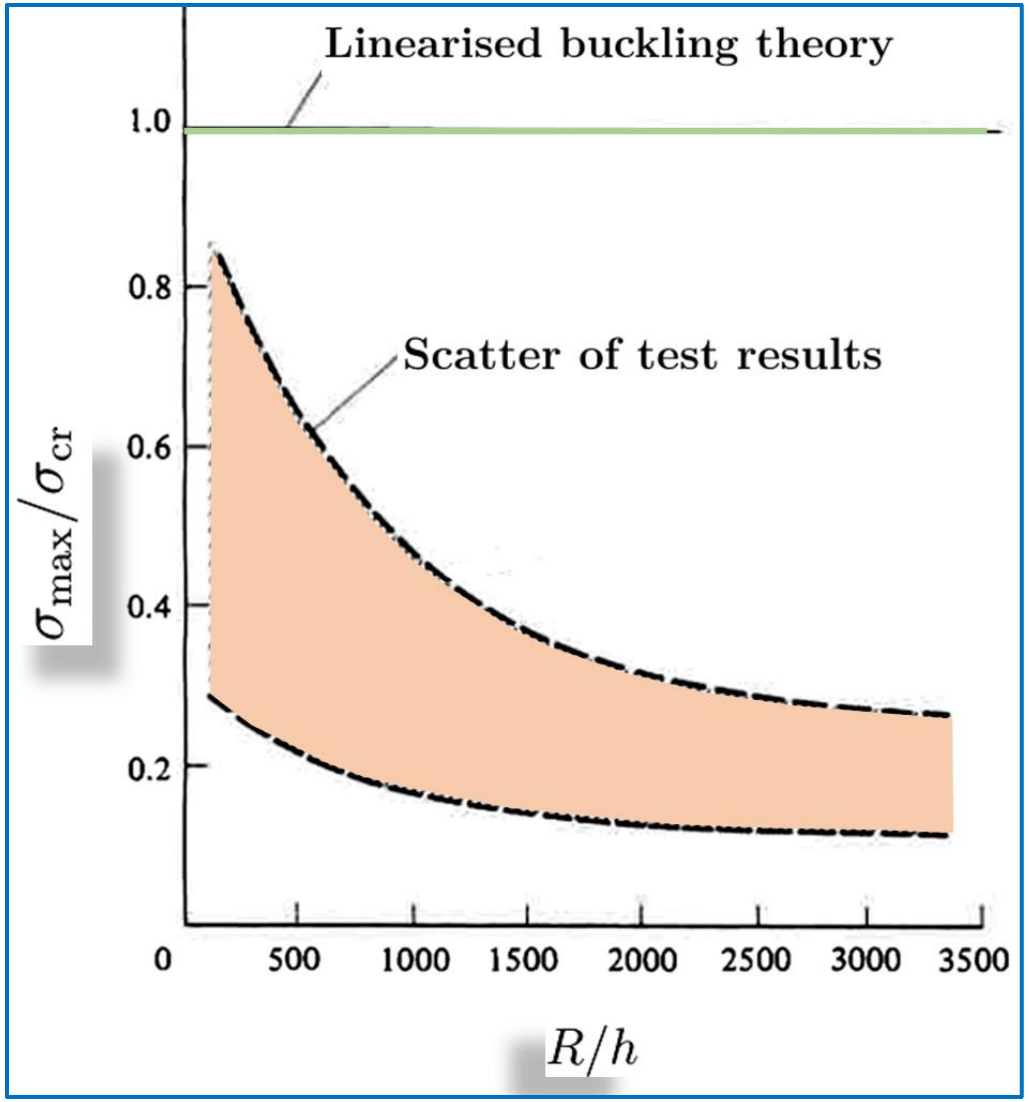
Shells are imperfection-sensitive structures



Effect of Initial Geometric Imperfections on Behavior of Axially Compressed Imperfect Circular Cylindrical Shells *(After Donnell and Wan [4-12])*



Relative amplitude of the geometrical imperfection



Koiter:

$$\left(1 - \frac{\sigma_{max.}}{\sigma_{cr}}\right)^2 - K\left(\frac{\sigma_{max.}}{\sigma_{cr}}\right) \cdot \left(\frac{w_0}{h}\right) = 0$$

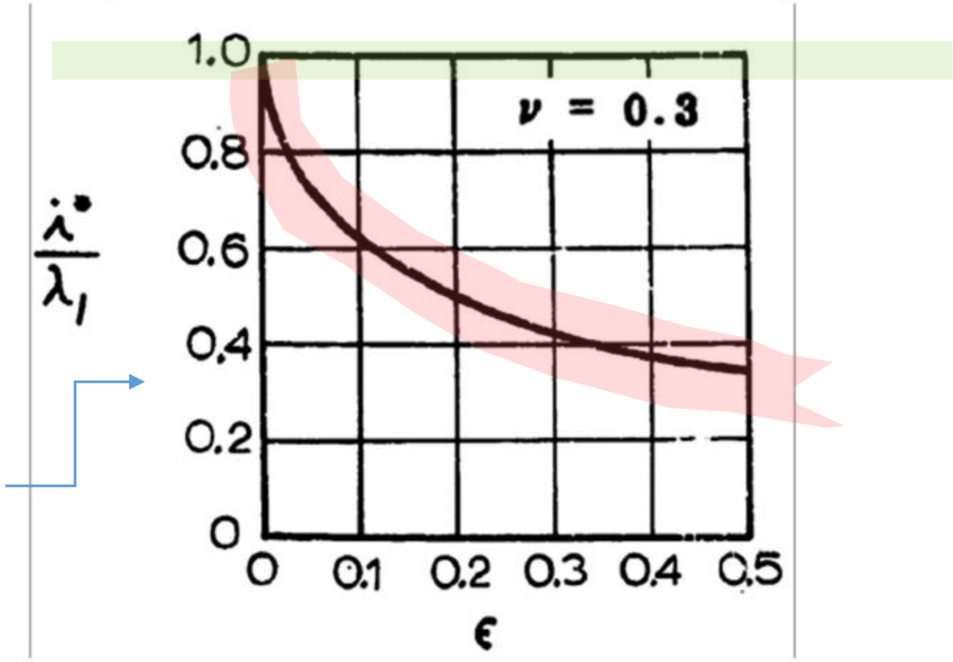
Effects of initial geometric imperfections on buckling load

Koiter, W.T., 1945. On the stability of elastic equilibrium. Thesis (in Dutch with English summary), Delft, H.J. Paris, Amsterdam. English translation, Air Force Flight Dym. Lab. Tech. Rep., AFFDL-TR-70-25.

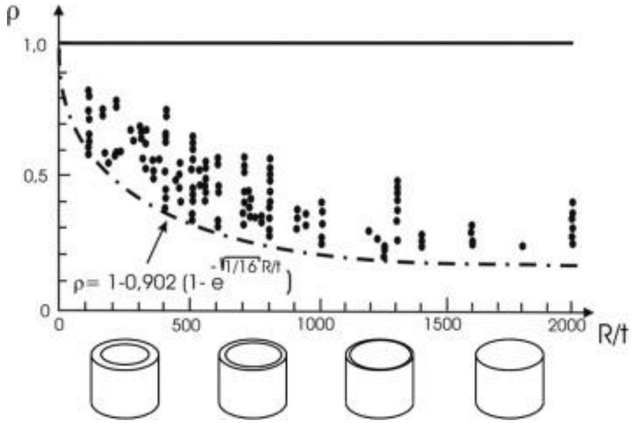
Koiter's dissertation p. 286 (translation to English)

$\nu = 0.3$

$$\lambda^* = \lambda_1 \left[1 + 1.24\,\epsilon - \sqrt{1.24\epsilon(2 + 1.24\epsilon)} \right]$$



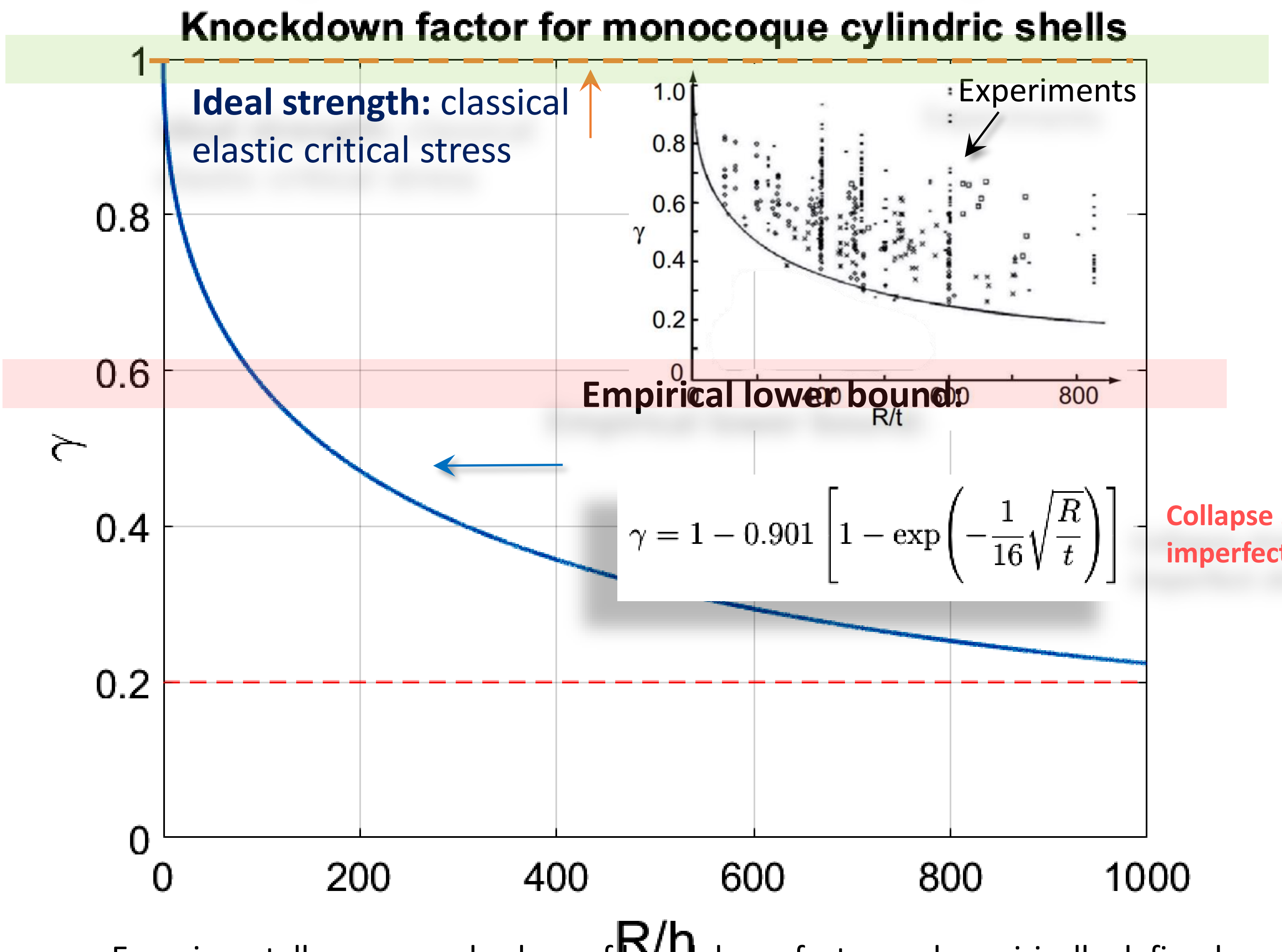
The great sensitivity of the buckling load for small deviations from the perfect cylindrical shell is clear from this last formula



REF: https://www.google.com/search?q=koiter+imperfection+effect&client=firefox-b-ab&source=lnms&tbm=isch&sa=X&ved=0ahUKEwiQ-YfLwJbhAhWt5aYKHcE4CQ8Q_AUIDigB&biw=1322&bih=894#imgdii=-0mMZY5cHanl2M:&imgrc=cbVas1JWSyUzdM:

Effect of initial geometrical imperfections on post-buckling behaviour of compressed thin-walled cylindrical shells as given by the original Koiter's formula. In this figure, $\epsilon \equiv w_0/h$ and $\lambda^*/\lambda \equiv \sigma_{max}/\sigma_{cr}$ Published also in: Koiter, W.T., 1963. The effect of axisymmetric imperfections on the buckling of cylindrical shells under axial compression. *Proc. K. Ned. Akad. Wet.*, Amsterdam, ser. B, vol. **6**; also, *Lockheed Missiles and Space Co.*, Rep. 6-90-63-86, Palo Alto, California..

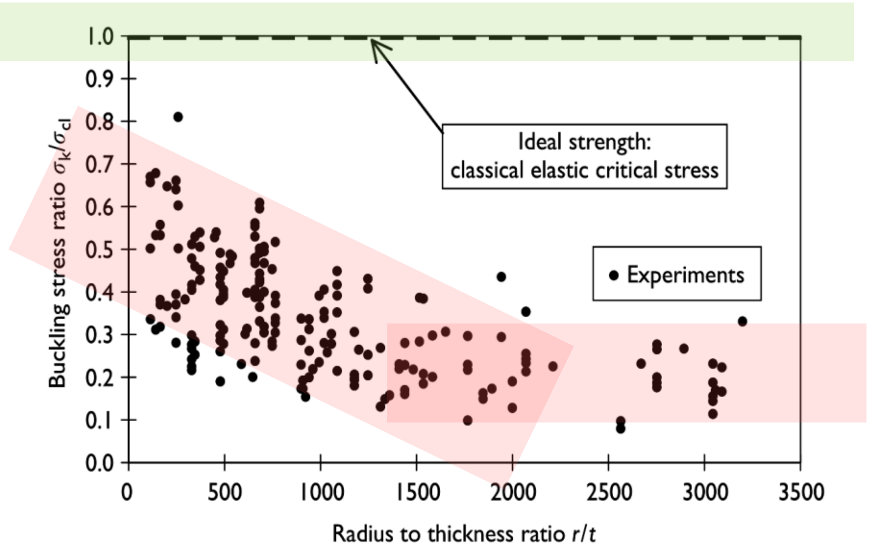
Effects of initial geometric imperfections on stability of thin shells



$$\gamma = 1 - 0.901\left[1 - \exp\left(-\frac{1}{16}\sqrt{\frac{R}{t}}\right)\right]$$

Collapse stress for real imperfect shell

Experimentally measured values of knockdown factor and empirically defined lower bound curve, as a function of the radius to thickness ratio (Jones, 2006).

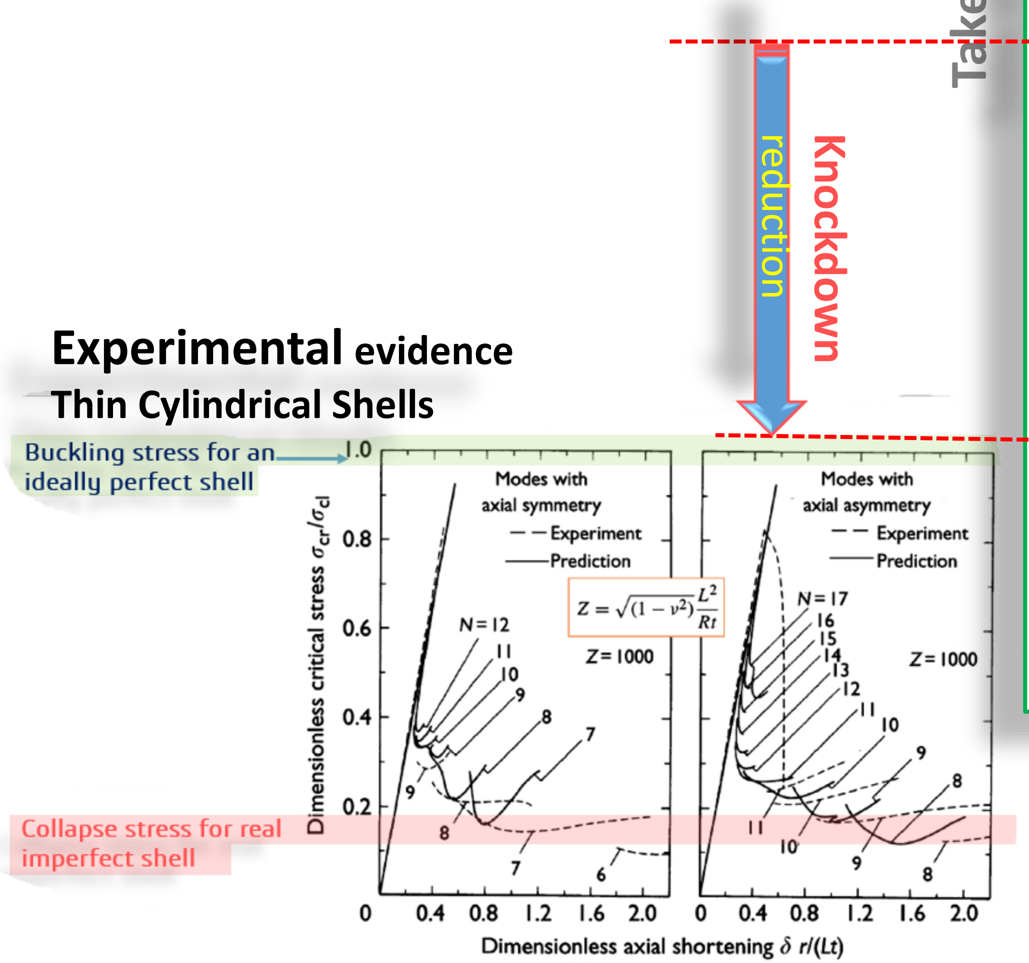


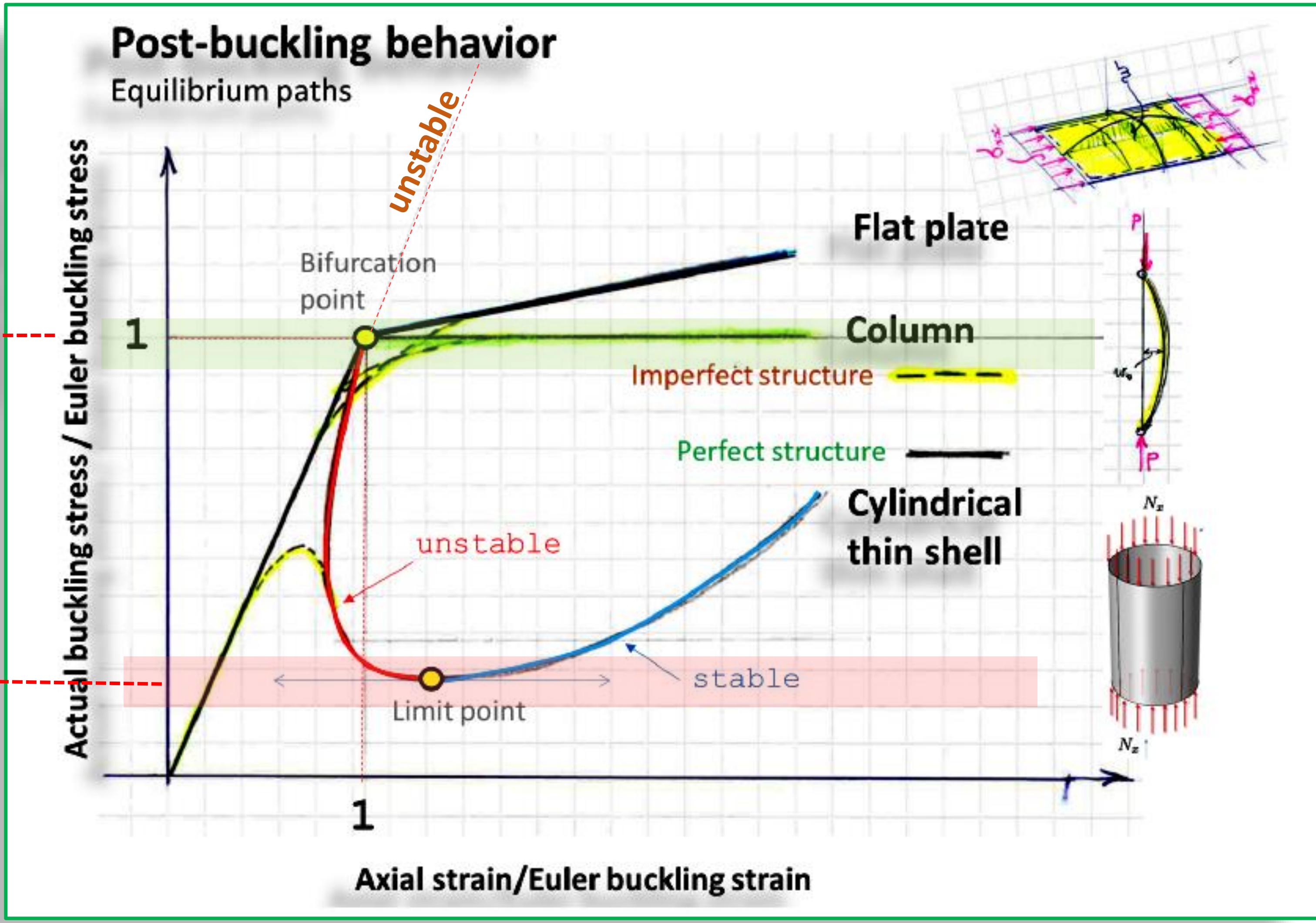
2.3 Experimental strengths of isotropic axially compressed cylinders (after Harris *et al.* 1957).

Example: cylindrical thin shell under uniform compression: lower bound:

$$\sigma_{cr} = \frac{\gamma E}{\sqrt{3(1-\nu^2)}}\frac{h}{R}$$

Shells are imperfection-sensitive structures



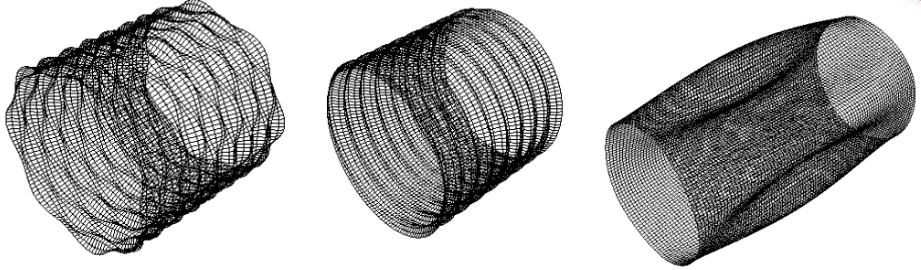


Schematic of fundamental characteristics of post-buckling behaviour for plates, columns and thin shells.

FE–based (non–linear) F.E.M.
analysis of imperfection sensitivity

- analysis the **post-buckling** behavior
- estimate the **lime-load**
- … and to **obtain knockdown** factors (reduction factor for imperfect structures)
- **non–linear: this course only geometrical**
- non–linear: for more realistic FE–analysis include material (plasticity , …) and all other relevant non– linearities as friction, …

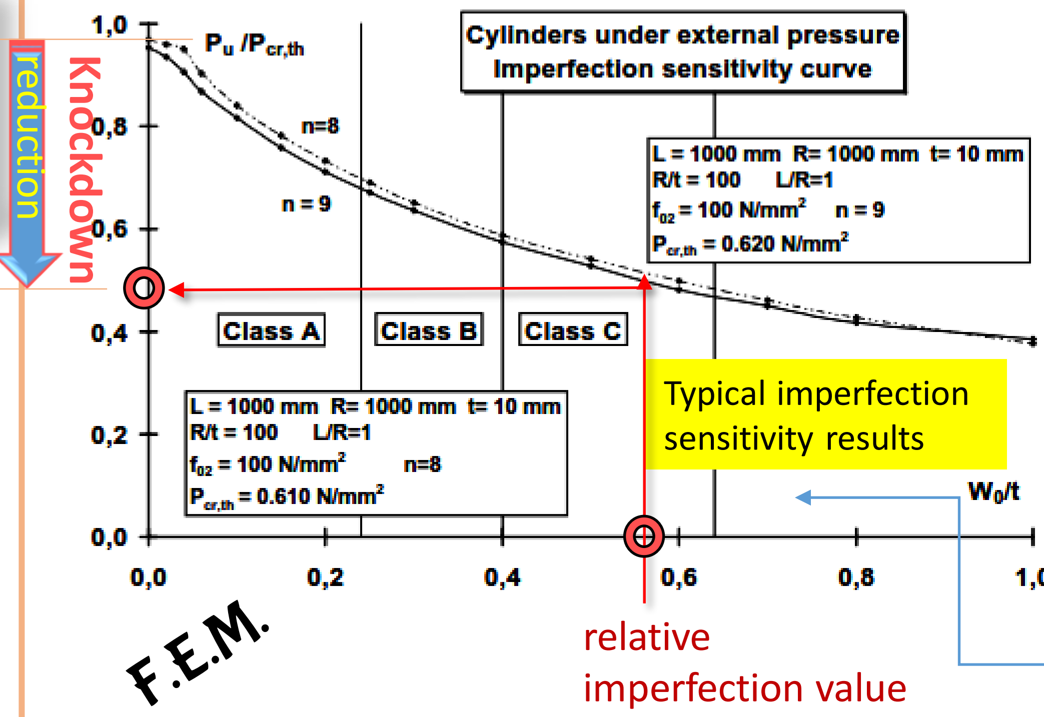
- To follow, for every choice of the initial imperfection pattern, the unstable post-buckling path after the limit-point an incremental static analysis FE-simulation a solid way: ABAQUS non-linear code is well-proven to do reliable job (RIKS algorithm). [of course other specialized software can do also a good job …. But I am not familiar with them]
- **Use the real geometry when available:** the real geometry can be our days obtained very accurately through direct laser scanning of real geometry when available or digital image correlation techniques.



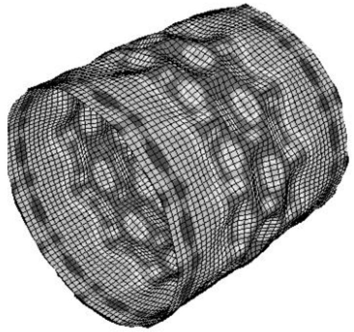
Example of initial shape imperfection patterns [Ref 1]
- as separate buckling modes or a combination of them

On this figure, geometric imperfections are amplified to render the visible to the reader

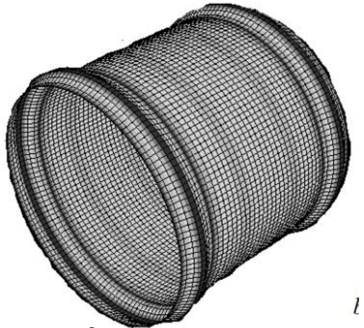
After experimental validation of the FE-model, it can be used to make predictions



FE-simulation: example of axial loading collapse



Elastic buckling



Plastic buckling

Ref 1:

BUCKLING OF ALUMINIUM SHELLS: PROPOSAL FOR EUROPEAN CURVES

[1] Department of Structural Analysis and Design, University of Naples Federico I Naples, Italy

[2] Department of Civil Engineering, Second University of Naples Aversa (CE), Italy

New course on material modelling – constitutive modellings

CIV-E4080 - Material Modelling in Civil Engineering L,

15.04.2019-27.05.2019

Constitutive Equations

Materiaalimallit – konstitutiiviset yhtälö

In physics & engineering:

A **constitutive equation** or **constitutive relation** is a material specific equation relating kinetic quantities and kinematic quantities

For ex., the classical ideal gas law which is an equation of state and also a constitutive equation too.

External action: fields or forces → A specific material → **response**

A Constitutive Equation approximates or models this response

Generic functional constitutive equation:

kinetic kinematic

$$\mathcal{F}\left(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}, T, \dot{T}\right) = 0$$

Ideal gas law

pressure

$$P = \frac{nRT}{V}$$

volum

n - amount of gas in mo

Hooke's law

stress

$$\sigma = E\epsilon$$

strain
stretch ratio

Ohm's law

$$I = \frac{U}{R}$$

voltage
current

Steady state creep

$$d\epsilon/dt = K_2\sigma^n \exp(-Q_c/R.T)$$

Plasticity

$$d\sigma = \mathbf{D}^{ep} d\varepsilon, \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

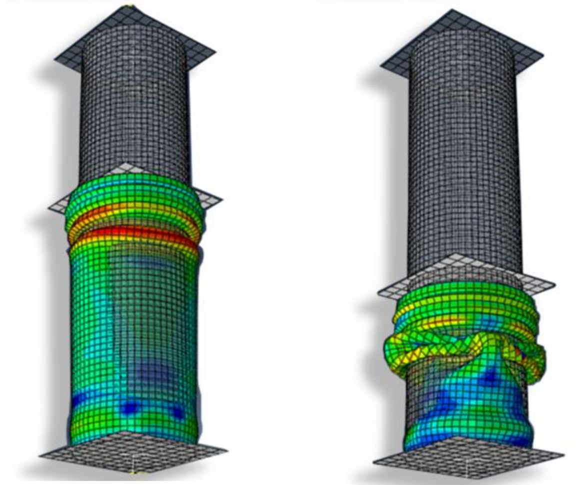
Elastic-plastic material stiffness (

A? MyCourses

- Material Modelling i
ering L,
05.2018

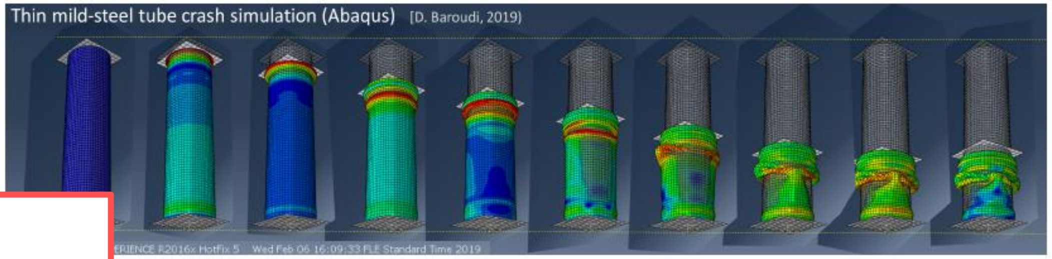
DO NOT MISS THIS COURSE

Advanced computational simulations



$$\bar{\sigma} = [A + B(\bar{\epsilon}^{pl})^N][1 + C\ln\frac{\dot{\bar{\epsilon}}^{pl}}{\dot{\epsilon}_0}][1 - \bar{\theta}^M]$$

In this particular one, a the continuous crash of a mild-steel tube is simulated (velocity control). The material behaviour is more complex than elastic: rate-dependent (visco-) plasticity with frictional contact.



Computational simulation of a crash of mild-steel tube s/Explicit). The colours code for von-Mises stresses.

0. INTRODUTION
1. ELASTICITY
2. VISCOELASTICITY (+ basics of creep)
3. PLASTICITY

https://mycourses.aalto.fi/course/view.php?id=20530

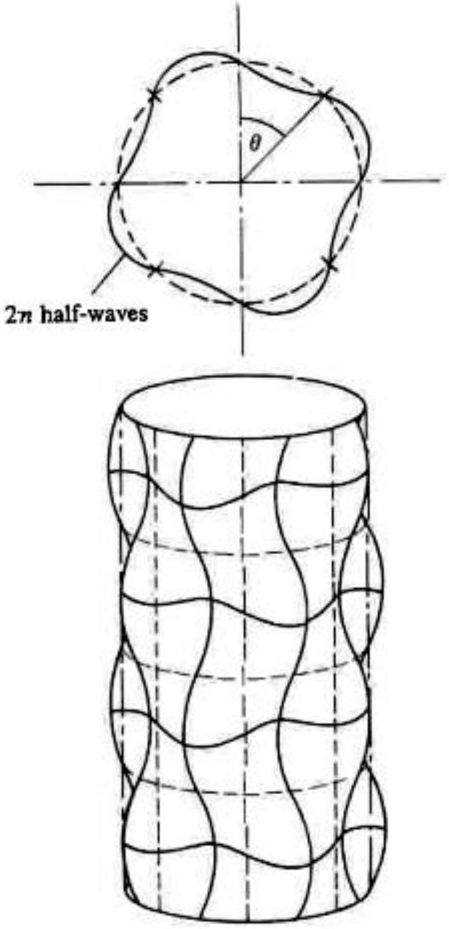
Lecturer: Dr. Djebar BAROUDI

Appendix

&

Miscellaneous

$$w = -A \sin \frac{m\pi x}{l} \cdot \sin n\theta,$$



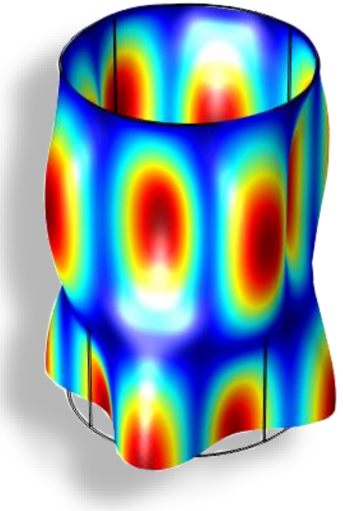
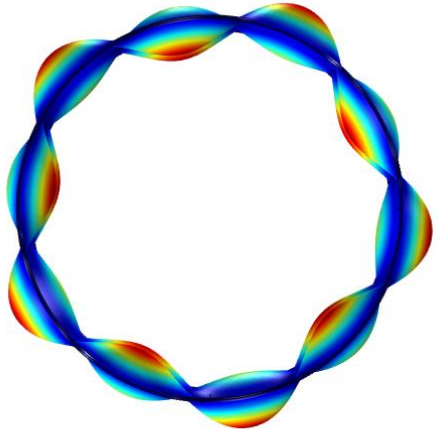
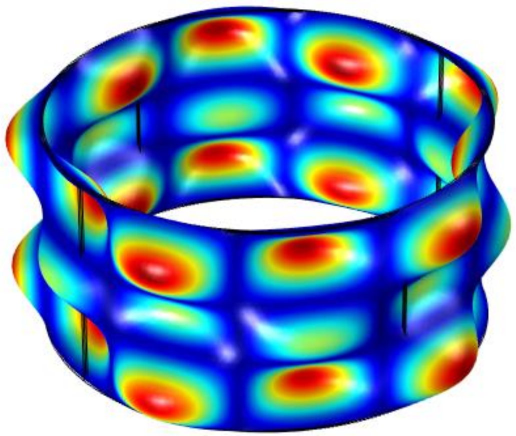


Fig. 7.4 Chessboard buckling.





$$w = -A \sin \frac{m\pi x}{l},$$

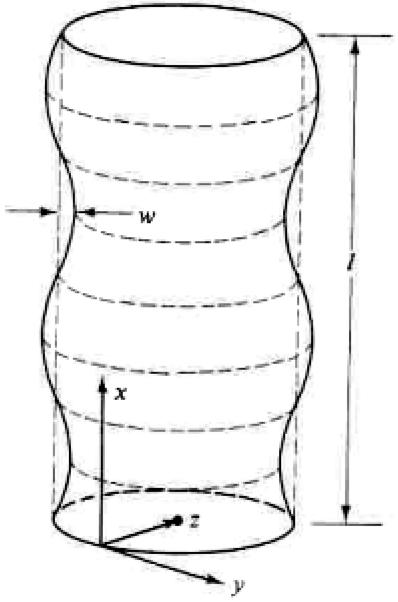
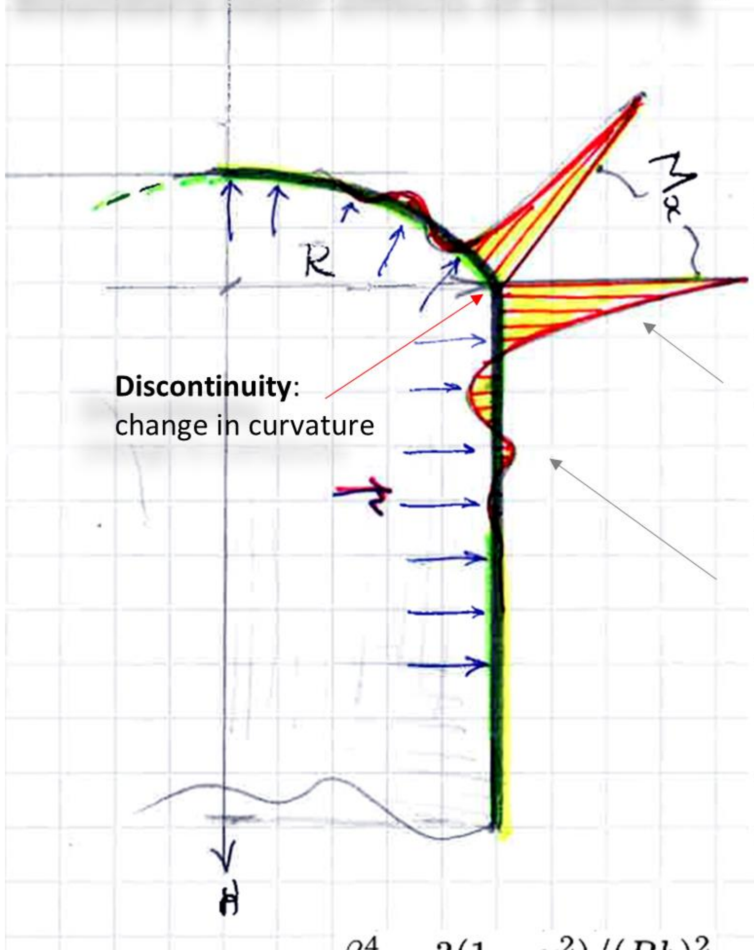


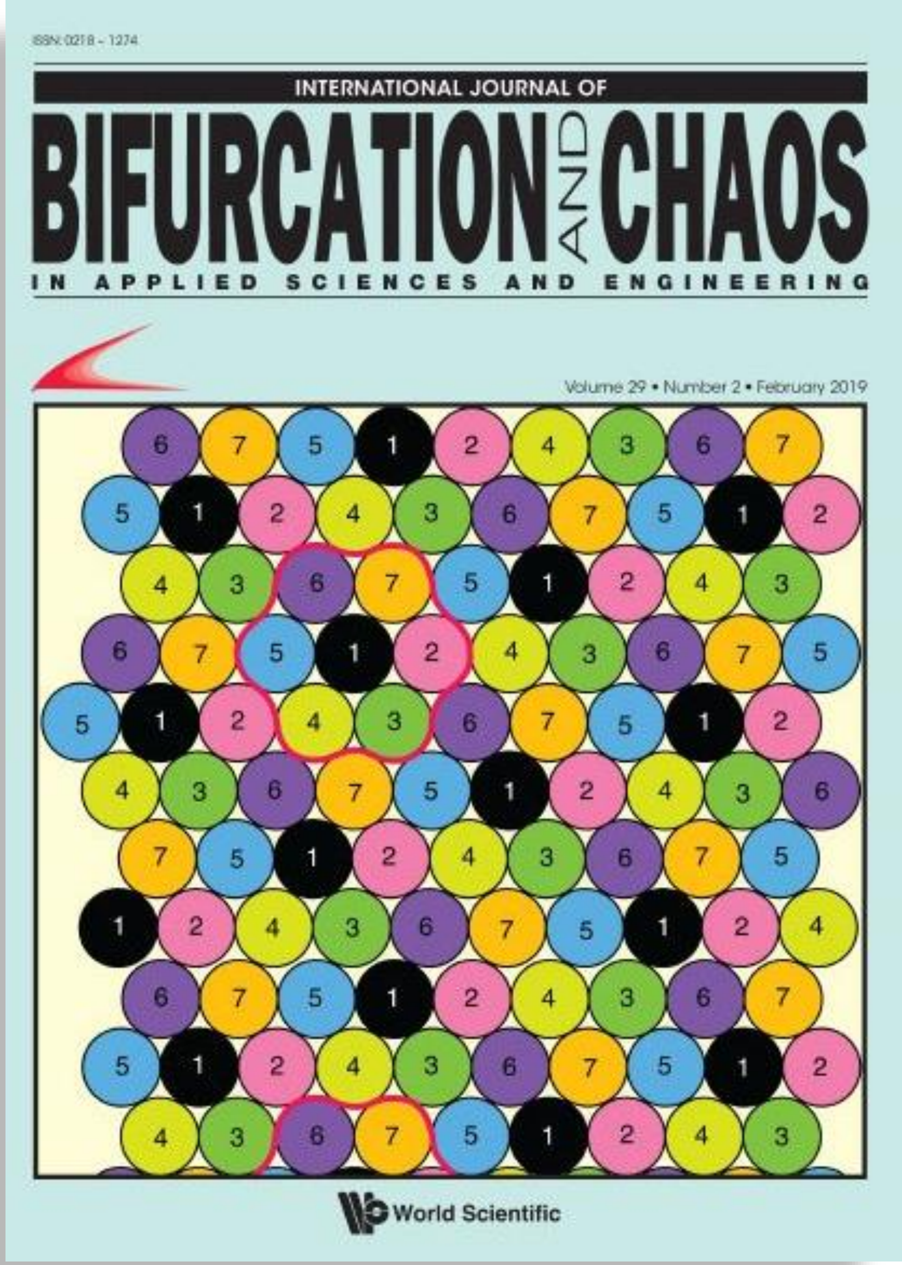
Fig. 7.3 Ring buckling.

Boundary-layer effects of bending



$$M_x(x) = Ap\exp(-\beta x)[\cos(\beta x) - \sin(\beta x)]$$

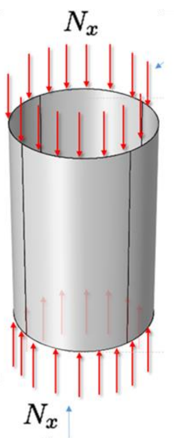
$$\beta^4 = 3(1-\nu^2)/(Rh)^2.$$

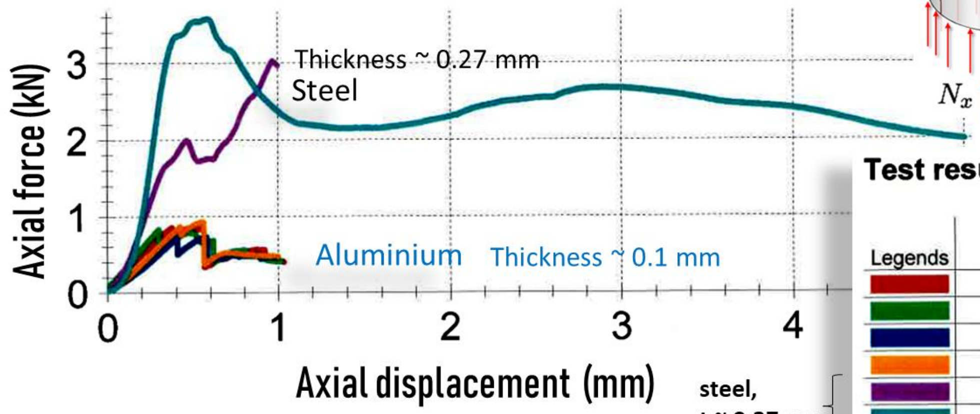


Buckling of cylindrical thin-walled shell

Experimental results

Aalto-yliopisto
Insinööritieteiden korkeakoulu
Rakennustekniikan laitos



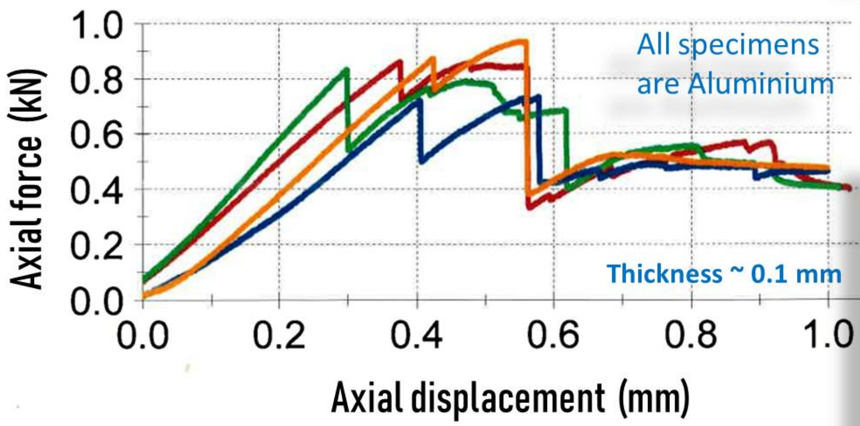


Test results:

Legends	Nr	Specimen identifier	F_{max} kN	dL at F_{max} mm
(red)	1	Koff - Jouluolut, 0,33 l	0,86	0,4
(green)	2	Koff - Nikolai 0,33 l	0,83	0,3
(blue)	3	Koff - Karhu 0,50 l	0,73	0,6
(orange)	4	Koff - 0,50 l	0,93	0,6
(purple)	5	Teräslieriö H=150mmxD=66mm, t=0,27mm	3,03	1,0
(teal)	6	Teräslieriö, kokeen jatkaminen	3,58	0,6

Nr 1–4: Aluminium, t ~ 0.1 mm

Nr 5–6: steel, t ~ 0.27 mm



Test report

Customer : Rak-54_3110 / Aluminitölkit
Group : Esikokeet
Material : Aluminitölkki
Pre-load : 10 N
Test speed : 0,5 mm/min
Specimen type : Aluminitölkki 0,33 ja 0,5 l
Tester : Veli-Antti Hakala ja Jukka Piironen

Aalto-yliopisto
Insinööritieteiden korkeakoulu
Rakennustekniikan laitos

Test results:

Aluminium, t ~ 0.1 mm

Legends	Nr	Specimen identifier	F_{max} kN	dL at F_{max} mm
(red)	1	Koff - Jouluolut, 0,33 l	0,86	0,4
(green)	2	Koff - Nikolai 0,33 l	0,83	0,3
(blue)	3	Koff - Karhu 0,50 l	0,73	0,6
(orange)	4	Koff - 0,50 l	0,93	0,6

```
% asymmetric chessboard bucklin of cylinder-shells
% Plotting only.
% Aythor: D. Baroudi, 2019
% ------------------------------------
% Z      - cylinder axis Z = [0, L]
% THETA - 0:2*pi
%% -------------------------------------------
R0 = 1;      % Radius
L = 3*R0;  % length
m = 6;      % nbre 1/2-waves in z-direction
n = 5;      % nbre 1/2-waves in theta direction
w0 = R0/10
NP = 4 * 30 ;
% ------------------------------
theta = linspace(0,2*pi, NP);
z      = linspace(0,L,   NP);

% Generating the mesh ---
Z = meshgrid(z);
[R , THETA] = meshgrid(z,  theta);

% The radial displacement w(z, theta)  at (z, theta) ---
w_z_theta = w0 * sin( m * pi * Z / L) .* sin(n  * THETA);
```

```
% Ploting the cylinder surface ---
% 1) Chessboard pattern
% ------------------------------------------------------------------------
figure
surf( (R0 + w_z_theta) .* cos(THETA), (R0 + w_z_theta) .* sin(THETA), Z);
axis square
grid on
box on
axis equal
title('Chessboard pattern')
xlabel('x')
ylabel('y')
zlabel('z')
```

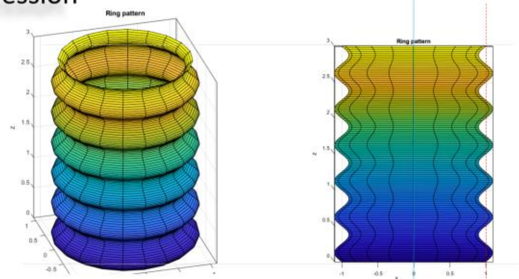
```
% -------
% 2) Ring pattern
figure
m = 8;
[X, Y, Z] = cylinder(R0 + w0 * sin( m * pi * z / L));
surf(X,Y,Z*L)
axis square
grid on
box on
axis equal
title('Ring pattern')
xlabel('x')
ylabel('y')
zlabel('z')
```

How to draw cylinders in Matlab

Buckling of axisymmetric cylindrical shells under uniform compression

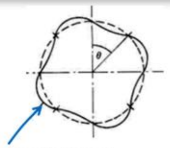
Ring patterns:

$$w(x) = w_0 \sin \frac{m\pi x}{\ell}$$

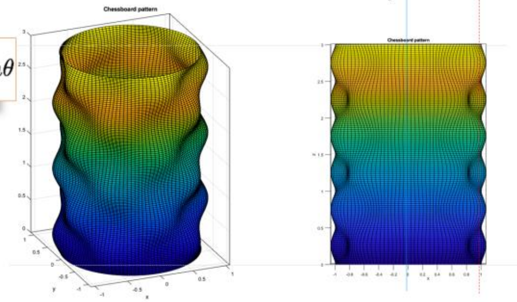


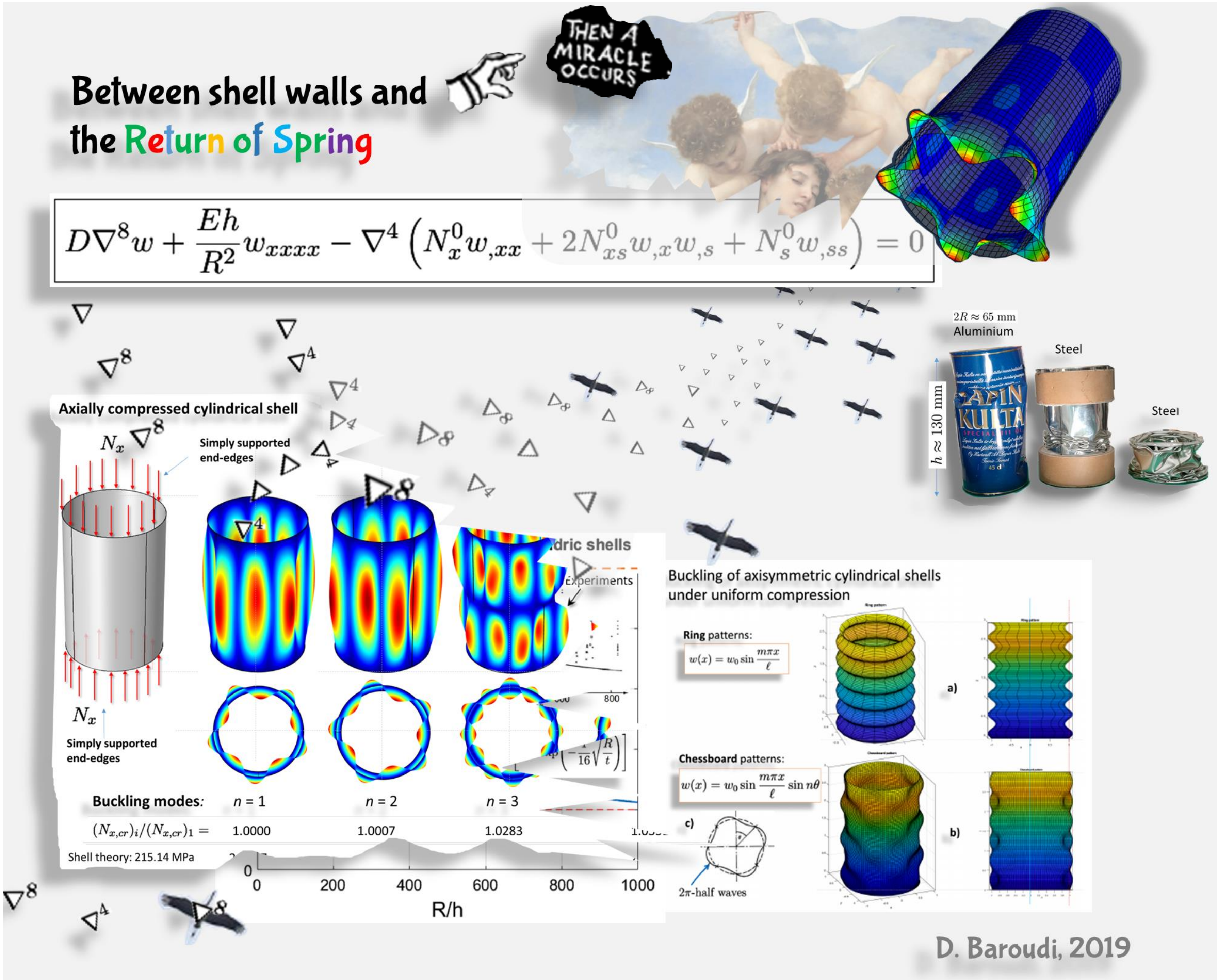
Chessboard patterns:

$$w(x) = w_0 \sin \frac{m\pi x}{\ell} \sin n\theta$$



2π-half waves





Enjoy for coming spring

I hope this course helped to you to learn what is about in stability and made you curious toward scientific based civil engineering

Enjoy for coming spring

Long before the beging of the begining, there was the endless sea of waves for eternity than came the bubble of our universe for a laps as long as is the life of the sparks.

The course Ends here ...