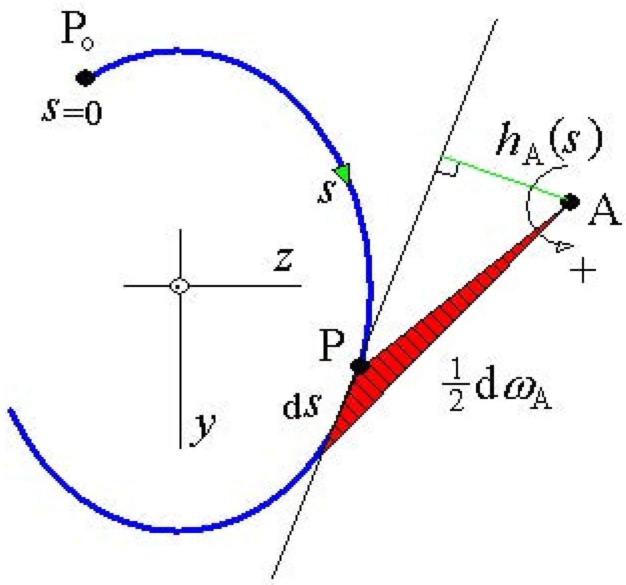


Lecture notes by:
Emir-prof. Juha Paavola

Kuva 2.17. A thin-walled cross-section.

2.9 Torsion of beams with open thin-walled cross-sections

2.9.1 Sectorial coordinate

Consider a thin walled beam cross-section of arbitrary shape where the wall thickness is small as compared to the other measures of the cross-section shown in Figure 2.17. The origin of the global coordinate system x, y, z is located at the center of gravity of the cross-section, and the axial coordinate x coincides the beam axis. The coordinate s is chosen to coincide with the centerline of the cross-section. Then, an arbitrary point P_o is chosen for the origin, and point A for an estimation for the shear center or a pole. The sectorial coordinate $\hat{\omega}_\mathrm{A}$ is defined according to Figure 2.17 as a gradient

$$\mathrm{d}\hat{\omega}_\mathrm{A} = \pm h_\mathrm{A}(s)\mathrm{d}s \tag{2.68}$$

where the subscript A refers to the pole and the superscript hat to the estimation of the sectorial coordinate, determined with respesct to the arbitrary point P_o. The definition of the gradient of the sectorial coordinate can be expressed also as the projections in the direction of the global coordinate axes

$$\mathrm{d}\hat{\omega}_\mathrm{A} = -(z - z_\mathrm{A})\mathrm{d}y + (y - y_\mathrm{A})\mathrm{d}z \tag{2.68a}$$

The same definition can still be given as the length of the vector parallell to the beam axis, which is obtained as the vector product of of the position vector from point A to point P_o, and the tangent vector of the centerline $\mathrm{d}\vec{s}$ on the cross-section plane

$$\mathrm{d}\vec{\hat{\omega}}_\mathrm{A} = (\vec{r}_\mathrm{P} - \vec{r}_\mathrm{A}) \times \mathrm{d}\vec{s} \tag{2.68b}$$

where the vectors utilized are $\vec{r}_\mathrm{P} - \vec{r}_\mathrm{A} = (y - y_\mathrm{A})\vec{j} + (z - z_\mathrm{A})\vec{k}$ ja $\mathrm{d}\vec{s} = \mathrm{d}y\vec{j} + \mathrm{d}z\vec{k}$. Or further, as the doubled area of the shaded triangle in the figure according to the definition of the vector product of two vectors.

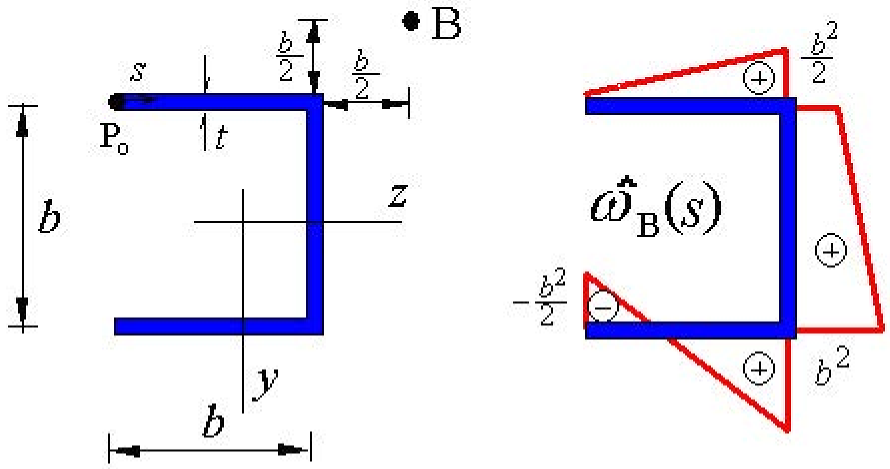


Figure 2.18. The sectorial coordinate $\hat{\omega}_{\mathrm{B}}(s)$.

The sectorial coordinate can be determined by utilizing the definition and direct integration and using the point P_o as the origin, when

$$\hat{\omega}_{\mathrm{A}} = \int_{\mathrm{P}_o}^{\mathrm{P}} \mathrm{d}\hat{\omega}_{\mathrm{A}} = \pm \int_{\mathrm{P}_o}^{\mathrm{P}} h_{\mathrm{A}}(s)\mathrm{d}s = -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{A}})\mathrm{d}y - (y - y_{\mathrm{A}})\mathrm{d}z] \tag{2.69}$$

As an **Example**, we define the distribution of the sectorial coordinate for a U-profile shown in Figure 2.18, when point B is used as a pole, and point P_o, the free end of the upper flange, as the origin of s coordinate. Sectorial coordinate at the interval $0 \leq s \leq 3b$ is

$$\hat{\omega}_{\mathrm{B}}(s) = \begin{cases} \displaystyle\int_0^s \frac{b}{2}\mathrm{d}s = \frac{b}{2}s, & \text{kun} \quad 0 \leq s \leq b \\ \displaystyle\hat{\omega}_{\mathrm{B}}(b) + \int_b^s \frac{b}{2}\mathrm{d}s = \frac{b}{2}s, & \text{kun} \quad b \leq s \leq 2b \\ \displaystyle\hat{\omega}_{\mathrm{B}}(2b) - \int_{2b}^s \frac{3b}{2}\mathrm{d}s = 4b^2 - \frac{3b}{2}s, & \text{kun} \quad 2b \leq s \leq 3b \end{cases}$$

Consider next, how the distribution of the sectorial coordinate will change, if the pole is relocated, for example at point A. Thus we can write very generally by applying directly the definition

$$\begin{aligned} \hat{\omega}_{\mathrm{A}} &= -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{A}})\mathrm{d}y - (y - y_{\mathrm{A}})\mathrm{d}z] \\ &= -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{B}} + z_{\mathrm{B}} - z_{\mathrm{A}})\mathrm{d}y - (y - y_{\mathrm{B}} + y_{\mathrm{B}} - y_{\mathrm{A}})\mathrm{d}z] \\ &= -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{B}})\mathrm{d}y - (y - y_{\mathrm{B}})\mathrm{d}z] - \int_{\mathrm{P}_o}^{\mathrm{P}} [(z_{\mathrm{B}} - z_{\mathrm{A}})\mathrm{d}y - (y_{\mathrm{B}} - y_{\mathrm{A}})\mathrm{d}z] \\ &= \hat{\omega}_{\mathrm{B}} - (z_{\mathrm{B}} - z_{\mathrm{A}})(y - y_o) + (y_{\mathrm{B}} - y_{\mathrm{A}})(z - z_o) \end{aligned} \tag{2.70}$$

Let's define now the sectorial coordinate for the U-profile by using as the pole point C, at the corner of the upper flange and the web. Thus, $z_{\mathrm{B}} - z_{\mathrm{C}} = \frac{b}{2}$ ja $y_{\mathrm{B}} - y_{\mathrm{C}} = -\frac{b}{2}$

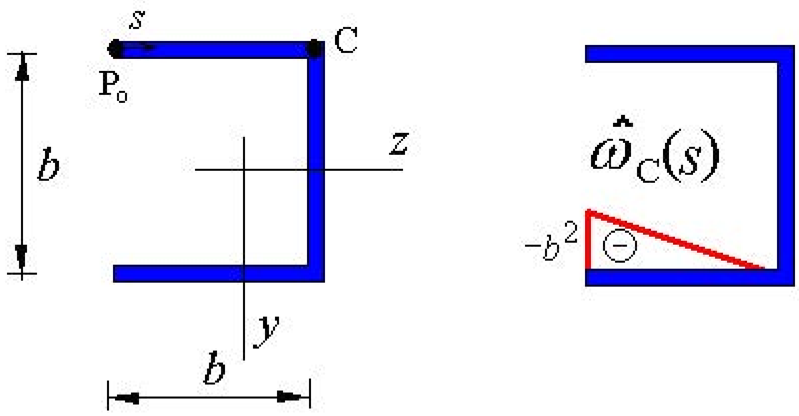


Figure 2.19. Sectorial coordinate $\hat{\omega}_{\mathrm{C}}$.

and we get

$$\hat{\omega}_{\mathrm{C}} = \begin{cases} \frac{b}{2}s - \frac{b}{2}0 + (-\frac{b}{2})s = 0, & \text{kun} \quad 0 \le s \le b \\ \frac{b}{2}s - \frac{b}{2}(s-b) + (-\frac{b}{2})b = 0, & \text{kun} \quad b \le s \le 2b \\ 4b^2 - \frac{3b}{2}s - \frac{b}{2}b + (-\frac{b}{2})(3b-s) = 2b^2 - bs, & \text{kun} \quad 2b \le s \le 3b \end{cases}$$

This distribution is shown in Figure 2.19.

2.9.2Axial displacement and stress distributions

V.Z. Vlasov derived already in the 1950s the theory for the torsion of open thin-walled beams or girders. He assumed that no shear strain will take place at the centerline of the walls, or

$$\gamma_{xs} = 0 \tag{2.71}$$

Consequently, no shear flow, similar to closed ring-type cross sections, doesn't appear. On the base of the definition of the shear strain it is obtained

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial u_s}{\partial x} = 0 \qquad \Longrightarrow \qquad \frac{\partial u}{\partial s} = -\frac{\partial u_s}{\partial x} \tag{2.72}$$

From this, it can be deduced that according to Figure 2.20, the displacement u_s is

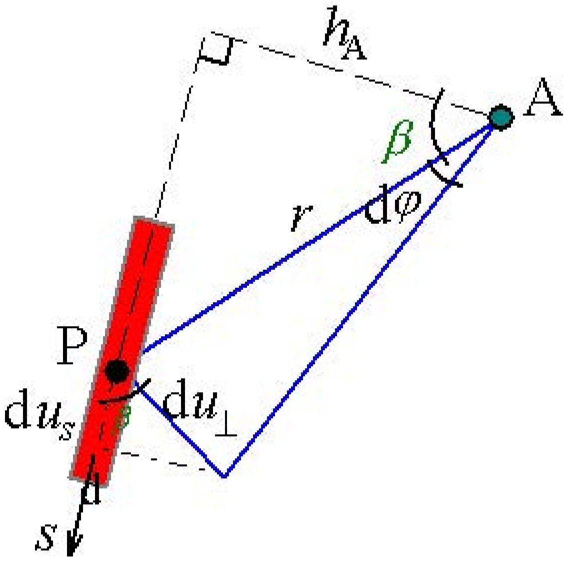
$$u_s = h_{\mathrm{A}}(s)\varphi(x)$$

and thus

$$\frac{\partial u}{\partial s} = -\frac{\partial h_{\mathrm{A}}(s)\varphi(x)}{\partial x} = -h_{\mathrm{A}}(s)\varphi'(x) \tag{2.73}$$

If at point P_o the axial displacemnet is zero, it is obtained by interating directly

$$u = \int_{\mathrm{P}_o}^{\mathrm{P}} \frac{\partial u}{\partial s}\mathrm{d}s = -\int_{\mathrm{P}_o}^{\mathrm{P}} h_{\mathrm{A}}(s)\varphi'(x)\mathrm{d}s = -\hat{\omega}_{\mathrm{A}}(s)\varphi'(x) \tag{2.74}$$



$$\frac{\mathrm{d}u_s}{\mathrm{d}u_\perp} = \frac{h_\mathrm{A}}{r}$$

$$\mathrm{d}u_\perp = r\mathrm{d}\varphi$$

Figure 2.20. Rotation of the cross-section.

The normal or axial strain is obtained according to the definition by differentiating

$$\epsilon_x = \frac{\partial u}{\partial x} = -\hat{\omega}_\mathrm{A}(s)\varphi''(x) \tag{2.75}$$

and the normal stress by adopting the one-dimensional linear elasticity as the material model

$$\sigma_{\hat{\omega}} = \sigma_x = E\epsilon_x = -E\hat{\omega}_\mathrm{A}(s)\varphi''(x) \tag{2.76}$$

When the loading of the beam consists of pure torsion only, the equilibrium equation states

$$N \equiv \int_A \sigma_{\hat{\omega}} \mathrm{d}A = 0 \quad \Longrightarrow \int_A \hat{\omega}_\mathrm{A} \mathrm{d}A = 0 \quad \Longleftrightarrow \quad S_{\omega_\mathrm{A}} = 0 \tag{2.77}$$

If the condition $S_{\omega_\mathrm{A}} = 0$ is valid, the distribution of the sectorial coordinate is called normalized. Further, according to the equilibrium equations it is obtained

$$\begin{aligned} M_y &\equiv \int_A \sigma_{\hat{\omega}} z \mathrm{d}A = 0 \quad \Longrightarrow \int_A \hat{\omega}_\mathrm{A} z \mathrm{d}A = 0 \quad \Longleftrightarrow \quad I_{\hat{\omega}_\mathrm{A} z} = 0 \\ M_z &\equiv \int_A \sigma_{\hat{\omega}} y \mathrm{d}A = 0 \quad \Longrightarrow \int_A \hat{\omega}_\mathrm{A} y \mathrm{d}A = 0 \quad \Longleftrightarrow \quad I_{\hat{\omega}_\mathrm{A} y} = 0 \end{aligned} \tag{2.78}$$

These two conditions determine the location of the shear center of the cross-section.

By inserting the expression of the sectorial coordinate, the location of the shear center will be calculated from the equations

$$\begin{aligned} I_{\hat{\omega}_\mathrm{A} y} &= \int_A \hat{\omega}_\mathrm{A} y \mathrm{d}A \\ &= \int_A \hat{\omega}_\mathrm{B} \mathrm{d}A - (z_\mathrm{B} - z_\mathrm{A})(\int_A y^2 \mathrm{d}A - y_o \int_A y \mathrm{d}A) + (y_\mathrm{B} - y_\mathrm{A})(\int_A yz \mathrm{d}A - z_o \int_A y \mathrm{d}A) \\ &= \; I_{\hat{\omega}_\mathrm{B} y} - (z_\mathrm{B} - z_\mathrm{A})(I_z - y_o S_z) + (y_\mathrm{B} - y_\mathrm{A})(I_{yz} - z_o S_z) \end{aligned} \tag{2.79a}$$

Correspondingly, the equation

$$I_{\hat{\omega}_\mathrm{A} z} = \; I_{\hat{\omega}_\mathrm{B} z} - (z_\mathrm{B} - z_\mathrm{A})(I_{yz} - y_o S_y) + (y_\mathrm{B} - y_\mathrm{A})(I_y - z_o S_y) \tag{2.79b}$$

is derived. When we will take still into account that the origin of the yz coordinates is located at the center of gravity of the cross-section, giving the conditions $S_y = S_z = 0$, it is obtained by requiring the disappearance of the two equations (2.79a,b) the coordinates of the shear centre

$$\begin{aligned} y_{\mathrm{A}} &= y_{\mathrm{B}} + \frac{I_z I_{\hat{\omega}_{\mathrm{B}} z} - I_{yz} I_{\hat{\omega}_{\mathrm{B}} y}}{I_y I_z - I_{yz}^2} \\ z_{\mathrm{A}} &= z_{\mathrm{B}} - \frac{I_y I_{\hat{\omega}_{\mathrm{B}} y} - I_{yz} I_{\hat{\omega}_{\mathrm{B}} z}}{I_y I_z - I_{yz}^2} \end{aligned} \tag{2.80}$$

If the yz-coordinates are in addition principal coordinates of the cross-section, when $I_{yz} = 0$, the simple expressions are obtained

$$\begin{aligned} y_{\mathrm{A}} &= y_{\mathrm{B}} + \frac{I_{\hat{\omega}_{\mathrm{B}} z}}{I_y} \\ z_{\mathrm{A}} &= z_{\mathrm{B}} - \frac{I_{\hat{\omega}_{\mathrm{B}} y}}{I_z} \end{aligned} \tag{2.81}$$

Normalization of the sectorial coordinate is done by applying the condition

$$S_{\omega_{\mathrm{A}}} = \int_A \omega_{\mathrm{A}} \mathrm{d}A = 0 \tag{2.82}$$

When we take into account that the surface integral in the definition of the sectorial coordinate can be split into pieces so that

$$\omega_{\mathrm{A}}(s) = \int_{\mathrm{P_o}}^{\mathrm{P}} \mathrm{d}\omega_{\mathrm{A}} = \int_{\mathrm{P}'_o}^{\mathrm{P}} \mathrm{d}\omega_{\mathrm{A}} - \int_{\mathrm{P}'_o}^{\mathrm{P}_o} \mathrm{d}\omega_{\mathrm{A}} = \hat{\omega}_{\mathrm{A}}(s) - \hat{\omega}_{\mathrm{A}}(s_o) \tag{2.83}$$

When this is inserted into the expression (2.82) it is obtained

$$S_{\omega_{\mathrm{A}}} = \int_A \hat{\omega}_{\mathrm{A}} \mathrm{d}A - \hat{\omega}_{\mathrm{A}}(s_o) A = S_{\hat{\omega}_{\mathrm{A}}} - \hat{\omega}_{\mathrm{A}}(s_o) A = 0 \tag{2.84}$$

From this , the location of the origin P_o of coordinate s can be determined

$$\hat{\omega}_{\mathrm{A}}(s_o) = \frac{S_{\hat{\omega}_{\mathrm{A}}}}{A}$$

and finally, the expression for the sectorial coordinate is reached.

$$\omega_{\mathrm{A}}(s) = \hat{\omega}_{\mathrm{A}}(s) - \frac{S_{\hat{\omega}_{\mathrm{A}}}}{A} \tag{2.85}$$

Illustrative example

The task is to determine the location of the shear center and the origin of s-coordinate for a U-profile, and then the distribution of the sectorial coordinate. In addition, we will determine the diagram of the sectorial static moment and the warping

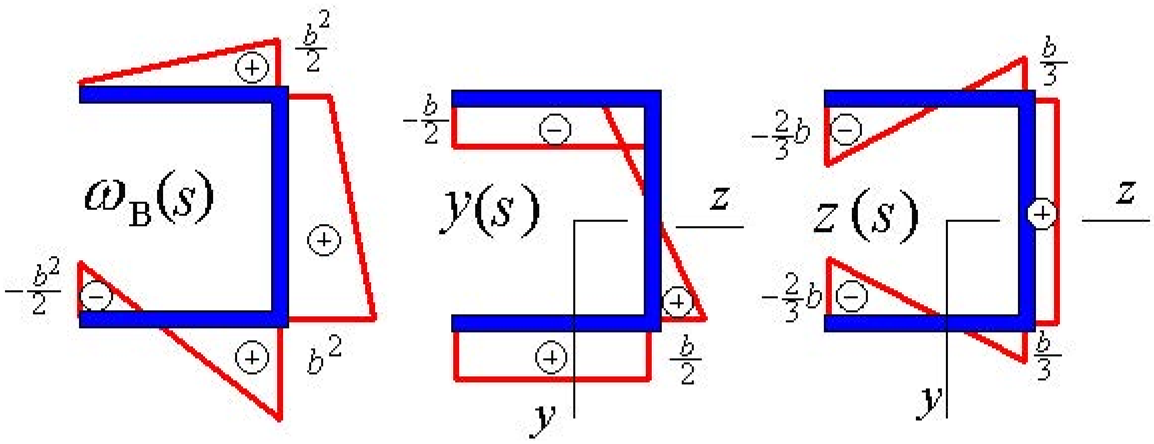


Figure 2.21. Determining the shear center of a thin-walled cross-section.

constant of the cross-section. The distance between the web of the cross-section and the centre of gravity is

$$3bt \cdot e = 2bt \cdot \tfrac{1}{2}\, b \quad \Longrightarrow \quad e = \tfrac{1}{3}\, b$$

The second moments or moments of inertia are correspondingly

$$\begin{aligned} I_y &= \tfrac{1}{3}\, b^3 t \\ I_z &= \tfrac{7}{12}\, b^3 t \end{aligned}$$

Next, we calculate the sectorial product moments

$$\begin{aligned} I_{\omega_{\rm B} y} &= \int_A \omega_{\rm B} y \mathrm{d}A = \int_s \omega_{\rm B}(s) y(s) t \mathrm{d}s \\ I_{\omega_{\rm B} z} &= \int_A \omega_{\rm B} z \mathrm{d}A = \int_s \omega_{\rm B}(s) y(s) t \mathrm{d}s \end{aligned}$$

In Figure 2.21, the diagrams of $\omega_{\rm B}(s)$, $y(s)$ ja $z(s)$ are presented. When calculating line-integrals, various practical tables can be used. We get

$$\begin{aligned} I_{\omega_{\rm B} y} &= \frac{1}{2} b \frac{b^2}{2}(-\frac{b}{2})t + \frac{b}{6}[\frac{b^2}{2}(-2\frac{b}{2} + \frac{b}{2}) + b^2(-\frac{b}{2} + 2\frac{b}{2})]t + \frac{b}{2}(b^2 - \frac{b^2}{2})\frac{b}{2}t = \frac{b^4 t}{24} \\ I_{\omega_{\rm B} z} &= \frac{1}{6} b \frac{b^2}{2}(-\frac{2b}{3} + 2\frac{b}{3})t + \frac{b}{2}(\frac{b^2}{2} + b^2)\frac{b}{3}t + \frac{b}{6}[b^2(2\frac{b}{3} - \frac{2b}{3}) - \frac{b^2}{2}(\frac{b}{3} - 2\frac{2b}{3})]t = \frac{b^4 t}{3} \end{aligned}$$

Now, y, z are the principal coordinates, when the coordinates of the shear centre are obtained by using the definitions

$$\begin{aligned} y_{\rm A} &= y_{\rm B} + \frac{I_{\omega_{\rm B} z}}{I_y} = -b + \frac{b^4 t/3}{b^3 t/3} = 0 \\ z_{\rm A} &= z_{\rm B} - \frac{I_{\omega_{\rm B} y}}{I_z} = \frac{5}{6} b - \frac{b^4 t/24}{7 b^3 t/12} = \frac{16}{21} b \end{aligned}$$

Now, when the location of the shear centre is known, the sectorial coordinate is determined by using the shear centre as the pole. By applying the definition directly, we

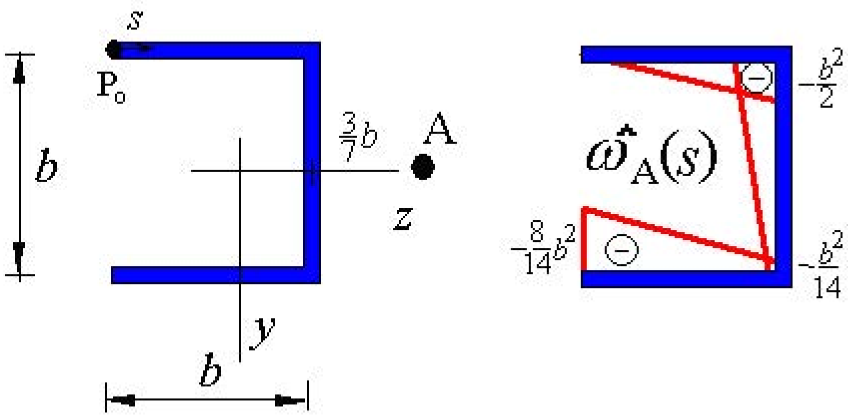


Figure 2.22. Sectorial coordinate $\hat{\omega}_{\mathrm{A}}$.

get

$$\hat{\omega}_{\mathrm{A}}(s)=\begin{cases}\int_0^s -\frac{b}{2}\mathrm{d}s=-\frac{b}{2}s, & \text{kun} \quad 0\le s\le b\\ \omega_{\mathrm{B}}(b)+\int_b^s \frac{3b}{7}\mathrm{d}s=-\frac{13b^2}{14}+\frac{3b}{7}s, & \text{kun} \quad b\le s\le 2b\\ \omega_{\mathrm{B}}(2b)-\int_{2b}^s \frac{3b}{2}\mathrm{d}s=\frac{13b^2}{14}-\frac{b}{2}s, & \text{kun} \quad 2b\le s\le 3b\end{cases}$$

which is depicted in Figure 2.22.

The integral due to the sectorial first moment over the whole cross-section takes the value

$$\begin{aligned}S_{\hat{\omega}_{\mathrm{A}}} &= \int_A \hat{\omega}_{\mathrm{A}}\mathrm{d}A=\int_s \omega_{\mathrm{A}}(s)t\mathrm{d}s\\ &= \frac{1}{2}b(-\frac{b^2}{2})t+\frac{1}{2}b(-\frac{b^2}{2}-\frac{b^2}{14})t+\frac{1}{2}b(-\frac{b^2}{14}-\frac{8b^2}{14})t=-\frac{6b^3t}{7}\end{aligned}$$

Now, the final diagram of the sectorial coordinate is obtained by normalizing

$$\omega_{\mathrm{A}}=\hat{\omega}_{\mathrm{A}}-\frac{S_{\hat{\omega}_{\mathrm{A}}}}{A}=\hat{\omega}_{\mathrm{A}}-\frac{-6b^3t/7}{3bt}=\hat{\omega}_{\mathrm{A}}+\frac{2}{7}b^2$$

and the sectorial coordinate diagram is finally

$$\omega_{\mathrm{A}}(s)=\begin{cases}\frac{2}{7}b^2-\frac{b}{2}s, & \text{kun} \quad 0\le s\le b\\ -\frac{9}{14}b^2+\frac{3b}{7}s, & \text{kun} \quad b\le s\le 2b\\ \frac{17}{14}b^2-\frac{b}{2}s, & \text{kun} \quad 2b\le s\le 3b\end{cases}$$

It is presented in Figure 2.23.

We determine still the distribution of the sectorial first moment of the cross section

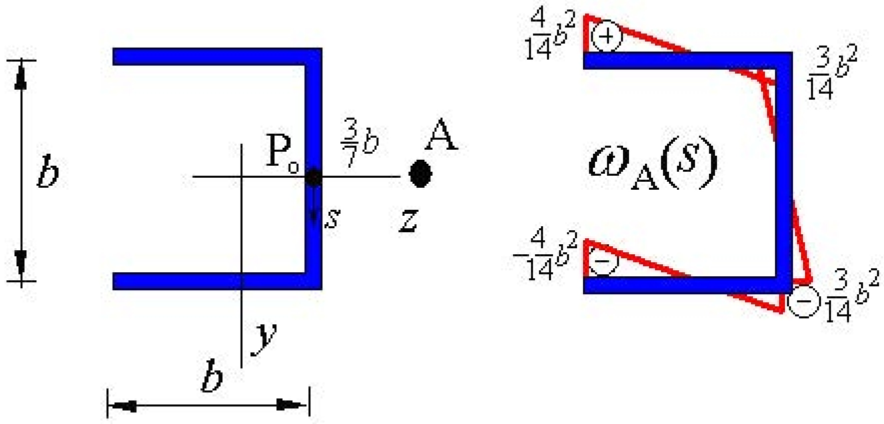


Figure 2.23. Sectorial coordinate ω_A.

S_{ω_A}. It is obtained by direct integration

$$
\begin{aligned}
S_{\omega_\mathrm{A}}(s) &= \int_0^s \omega_\mathrm{A}(s)t\mathrm{d}s \\
&= \begin{cases}
\int_0^s (\frac{2}{7}b^2 - \frac{1}{2}bs)t\mathrm{d}s = \frac{2}{7}b^2ts - \frac{1}{4}bts^2, & \text{kun} \;\; 0 \le s \le b \\
\frac{1}{28}b^3t - \int_b^s (\frac{9}{14}b^2 - \frac{3}{7}bs)t\mathrm{d}s = \frac{13}{28}b^3t - \frac{9}{14}b^2ts + \frac{3}{14}bts^2, & \text{kun} \;\; b \le s \le 2b \\
\frac{1}{28}b^3t + \int_{2b}^s (\frac{17}{14}b^2 - \frac{1}{2}bs)t\mathrm{d}s = -\frac{39}{28}b^3t + \frac{17}{14}b^2ts - \frac{1}{4}bts^2, & \text{kun} \;\; 2b \le s \le 3b
\end{cases}
\end{aligned}
$$

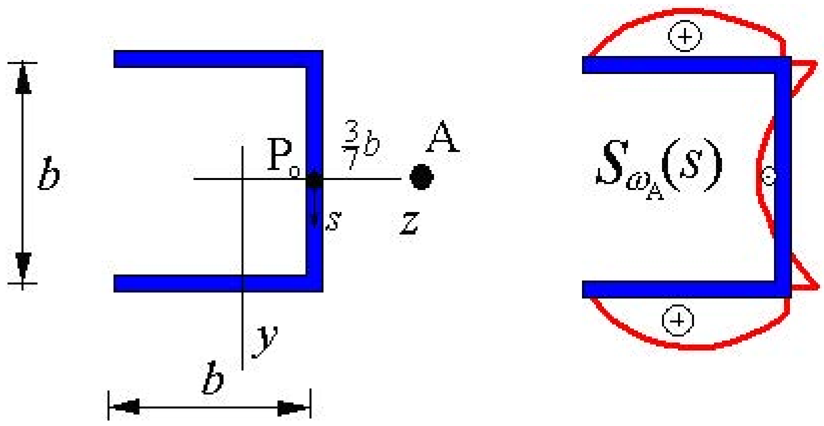


Figure 2.24. Sectorial first momenti $S_{\omega_\mathrm{A}}(s)$.

and it is depicted in Figure 2.24. The warping constant I_ω is

$$
I_\omega = \int_A \omega_\mathrm{A}^2 \mathrm{d}A = \int_0^{3b} \omega_\mathrm{A}^2(s)t\mathrm{d}s = \frac{5}{84}b^5t
$$

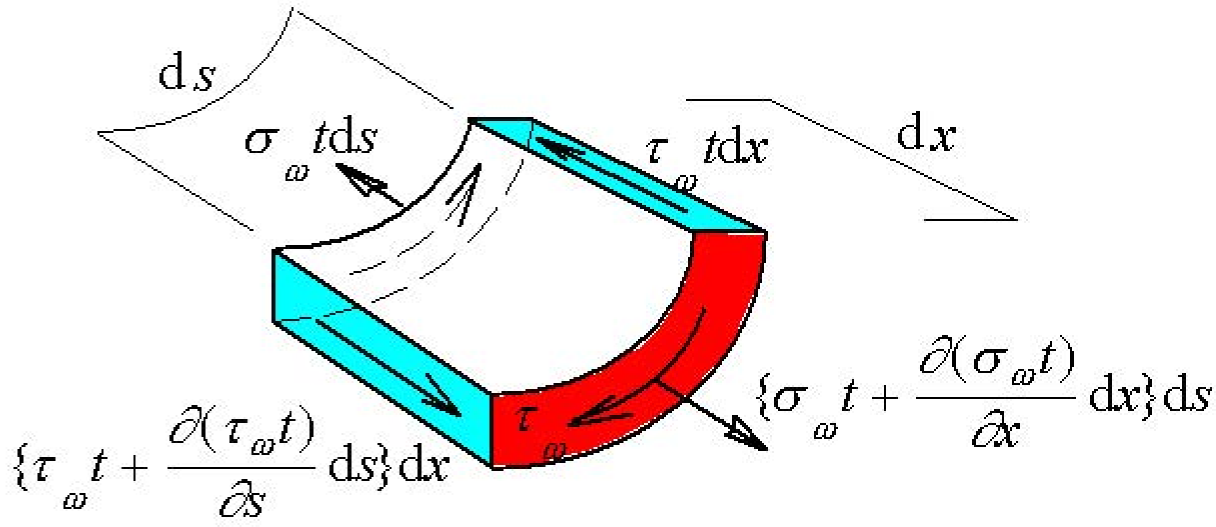


Figure 2.25. Equilibrium of a wall element.

2.9.4 Sectorial shear stresses

Consider an element cut out of the wall of an open thin-walled girder under torsion without volume forces, presented in Figure 2.25. In figure, all the stress components parallel to the beam axis are presented. The equilibrium between them gives

$$(\sigma_\omega t + \frac{\partial(\sigma_\omega t)}{\partial x}\mathrm{d}x)\mathrm{d}s - \sigma_\omega t\mathrm{d}s + (\tau_\omega t + \frac{\partial(\tau_\omega t)}{\partial s}\mathrm{d}s)\mathrm{d}x - \tau_\omega t\mathrm{d}x = 0$$

The shear stresses take the form

$$\frac{\partial(\tau_\omega t)}{\partial s} = -\frac{\partial(\sigma_\omega t)}{\partial x} = E\omega_{\mathrm{A}}(s)\varphi'''(x)t(s)$$

of which we get by integrating

$$\tau_\omega t - \underbrace{\tau_\omega(0)}_{=0}t(0) = E\varphi'''(x)\int_0^s \omega_{\mathrm{A}}(s)t(s)\mathrm{d}s = E\varphi'''(x)S_{\omega_{\mathrm{A}}}(s)$$

Consequently,

$$\tau_\omega(s) = E\varphi'''(x)S_{\omega_{\mathrm{A}}}(s)\frac{1}{t(s)} \tag{2.86}$$

The shear stresses due to Saint Venant eli vapaan väännön leikkausjännitykset ovat puolestaan

$$\tau_t(s) = \pm G\varphi'(x)t(s) \tag{2.87}$$

and the total shear stress distribution is the sum of both of these

$$\tau(s) = \tau_\omega(s) + \tau_t(s) \tag{2.88}$$

Under torsion only, the stress resultants

$$N = Q_y = Q_z = M_y = M_z \equiv 0$$

must disappear. Instead, the external torque loading M_x has to be in equilibrium with the internal shear stress resultants. The external loading will generate in the beam shear stresses of Saint Venant and warping analyses. The torque resultant for warping stresses M_ω is obtained by considering the equilibrium at the cross section plane and utilizing the expression (2.86) when we get

$$M_\omega = \int_A \tau_\omega h_{\mathrm{A}} \mathrm{d}A = E\varphi'''(x) \int_0^{s_\ell} S_{\omega_{\mathrm{A}}}(s) h_{\mathrm{A}}(s) \mathrm{d}s \tag{2.89}$$

Performing once the integration by parts with

$$\frac{\mathrm{d}\omega_{\mathrm{A}}(s)}{\mathrm{d}s} = h_{\mathrm{A}}(s)$$
$$\frac{\mathrm{d}S_{\omega_{\mathrm{A}}}(s)}{\mathrm{d}s} = \omega_{\mathrm{A}}(s)$$

we get

$$\begin{aligned} M_\omega &= E\varphi'''(x) \left[\underbrace{\Big[S_{\omega_{\mathrm{A}}}(s)\omega_{\mathrm{A}}(s) \Big]_{s=0}^{s=s_\ell}}_{=0} - \int_0^{s_\ell} \omega_{\mathrm{A}}(s) t(s) \omega_{\mathrm{A}}(s) \mathrm{d}s \right] \\ &= -E\varphi'''(x) \int_A \omega_{\mathrm{A}}^2(s) \mathrm{d}A = -EI_\omega \varphi'''(x) \end{aligned} \tag{2.90}$$

because $S_{\omega_{\mathrm{A}}}(0) = S_{\omega_{\mathrm{A}}}(s_\ell) = 0$. Thus, the shear stresses due to warping torsion are of the form

$$\tau_\omega(s) = -\frac{M_\omega S_{\omega_{\mathrm{A}}}(s)}{I_\omega t(s)} \tag{2.91}$$

Saint Venant shear stress resultant is correspondingly

$$M_t = GI_t \varphi'(x)$$

and shear stress distribution

$$\tau_t(s) = \pm \frac{M_t t(s)}{I_t} \tag{2.92}$$

The total resultant, equilibrating the external loading, is thus the sum of these

$$M_x = M_\omega + M_t = -EI_\omega \varphi'''(x) + GI_t \varphi'(x) \tag{2.93}$$

For the bimoment B, the definition is obtained on the base of equation (2.76)

$$B = \int_A \sigma_\omega \omega_{\mathrm{A}} \mathrm{d}A = -E\varphi''(x) \int_A \omega_{\mathrm{A}}^2 \mathrm{d}A = -EI_\omega \varphi''(x) \tag{2.94}$$

and from this further, the expression for the axial or warping stresses as expressed by the bimoment

$$\sigma_\omega(s) = -\frac{B}{I_\omega} \omega_{\mathrm{A}}(s) \tag{2.95}$$

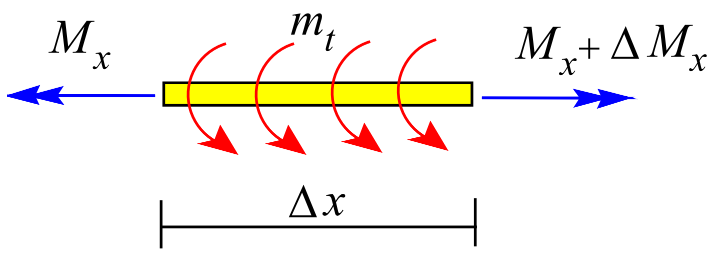


Figure 2.26. Equilibrium of a beam element.

Finally, it is useful to observe that the connection between the bimoment and the torque due to warping torsion is

$$M_\omega = \frac{\mathrm{d}B}{\mathrm{d}x} \tag{2.96}$$

2.9.4 Equilibrium of a beam element

Consider generally an element of length $\mathrm{d}x$ cut out from a beam under torsional loading, shown in Figure 2.26. The equilibrium of this element states

$$M_x + \frac{\mathrm{d}M_x}{\mathrm{d}x}\mathrm{d}x - M_x + m_t(x)\mathrm{d}x = 0$$

when we get

$$\frac{\mathrm{d}M_x}{\mathrm{d}x} + m_t = 0 \tag{2.97}$$

According to (2.93), the total torque is the sum

$$M_x = M_\omega + M_t = -EI_\omega\varphi'''(x) + GI_t\varphi'(x)$$

and by inserting this into the equilibrium equation it is obtained

$$EI_\omega\varphi''''(x) - GI_t\varphi''(x) - m_t(x) = 0 \tag{2.98}$$

Dividing the equation by the warping stiffness, the equation takes the form

$$\varphi''''(x) - \frac{k^2}{\ell^2}\varphi''(x) = f(x) \tag{2.99}$$

where

$$k = \ell\sqrt{\frac{GI_t}{EI_\omega}} \qquad \text{ja} \qquad f(x) = \frac{m_t(x)}{EI_\omega}$$

Equation (2.99) can be expressed by utilizing the definition of the bimoment also in the form

$$B''(x) - \frac{k^2}{\ell^2}B(x) = m_t(x) \tag{2.100}$$

Solving these equations is categorized into three different cases depending on the mutual value of the torsional stiffness due to Saint Venant and warping torsion. If the coefficient $k > 10(...20)$, Saint Venant torsion is more dominant and the first term due to warping torsion can be dropped out of the equations. This corresponds to beams with massive solid and box-type cross-sections. The differential equation is simplified to the form

$$GI_t\varphi''(x) = -m_t(x) \tag{2.101}$$

If $k < \frac{1}{2}$, Saint Venant torsion is more or less meaningless. Thus we have pure warping torsion, and it is often met with very thin-walled coldformed steel cross-sections. The differential equation is then of the form

$$EI_\omega\varphi''''(x) = m_t(x) \tag{2.102}$$

The complete equation(2.99) with both parts covers the range $\frac{1}{2} < k < 10(...20)$. In this group, there are the hot-rolled steel profiles and thin-walled reinforced concrete girders.

2.9.5 Solution for the differential equation

The solution for the homogeneus equation

$$\varphi''''(x) - \frac{k^2}{\ell^2}\varphi''(x) = 0 \tag{2.103}$$

is searched by a trial function $\varphi(x) = \exp(rx)$. Inserting this to the differential equation gives the characteristic equation of the problem

$$(r^4 - \frac{k^2}{\ell^2}r^2)\exp(rx) = 0$$

Its roots

$$\begin{aligned} r_{1,2} &= 0 \\ r_{3,4} &= \pm\frac{k}{\ell} \end{aligned}$$

are all real numbers, and one of them is double root. Thus, the general solution of the homogeneous equation is

$$\begin{aligned} \varphi(x) &= c_1 + c_2x + c_3\exp(\frac{k}{\ell}x) + c_4\exp(-\frac{k}{\ell}x) \\ &= c_1 + c_2x + \frac{c_3 - c_4}{2}(\exp(\frac{k}{\ell}x) - \exp(-\frac{k}{\ell}x)) + \frac{c_3 + c_4}{2}(\exp(\frac{k}{\ell}x) + \exp(-\frac{k}{\ell}x)) \end{aligned}$$

Because the hyperbolic functions are defined by

$$\begin{aligned} \sinh(x) &= \frac{1}{2}(\exp(x) - \exp(-x)) \\ \cosh(x) &= \frac{1}{2}(\exp(x) + \exp(-x)) \end{aligned}$$

we can write further

$$\varphi(x) = C_1 + C_2x + C_3\sinh(\frac{k}{\ell}x) + C_4\cosh(\frac{k}{\ell}x) \tag{2.104}$$

The complete solution for the differential equation is obtained by combining this with the particular solution corresponding to the cureent loading $\bar{\varphi}(x)$, or

$$\varphi(x) = C_1 + C_2 x + C_3 \sinh\left(\frac{k}{\ell}x\right) + C_4 \cosh\left(\frac{k}{\ell}x\right) + \bar{\varphi}(x) \tag{2.105}$$

The constants C_1, C_2, C_3 and C_4 are solved from the boundary conditions at each end of the beam. The stress rsultants can then be differentiated from the solution as

$$M_t = GI_t\varphi'(x) = GI_t[C_2 + C_3\frac{k}{\ell}\cosh\left(\frac{k}{\ell}x\right) + C_4\frac{k}{\ell}\sinh\left(\frac{k}{\ell}x\right) + \bar{\varphi}'(x)] \qquad (a)$$

$$B = -EI_\omega\varphi''(x) = -GI_t[C_3\sinh\left(\frac{k}{\ell}x\right) + C_4\cosh\left(\frac{k}{\ell}x\right) + \left(\frac{\ell}{k}\right)^2\bar{\varphi}''(x)] \qquad (b)$$

$$M_\omega = B' = -EI_\omega\varphi'''(x) = -GI_t[C_3\frac{k}{\ell}\cosh\left(\frac{k}{\ell}x\right) + C_4\frac{k}{\ell}\sinh\left(\frac{k}{\ell}x\right) + \left(\frac{\ell}{k}\right)^2\bar{\varphi}'''(x)] \quad (c)$$

$$M_x = M_t + M_\omega = GI_t[C_2 + \bar{\varphi}'(x) - \left(\frac{\ell}{k}\right)^2\bar{\varphi}'''(x)] \qquad (2.106d)$$

2.9.6 Some particular solutions

When the beam is loaded by an evenly along the length distributed torque m_t^o, the inhomogeneous term on the right hand side of the differential equation

$$\varphi''''(x) - \frac{k^2}{\ell^2}\varphi''(x) = f(x)$$

is of the form

$$f(x) = \frac{m_t^o}{EI_\omega}$$

The particular solution can be searched by a trial function $\bar{\varphi}(x) = Cx^2$. Inserting this into the differential equation results in the value for C as

$$C = -\frac{m_t^o}{2GI_t}$$

and the particular solution takes the form

$$\bar{\varphi}(x) = -\frac{1}{2}\frac{m_t^o}{GI_t}x^2 \tag{2.107}$$

If the loading torque is distributed along linearly increasing function $m_t^o x/\ell$ the inhomogeneous term in the differential equation is of the form

$$f(x) = \frac{m_t^o}{EI_\omega}\frac{x}{\ell}$$

and the particular solution is searched bu the trial function of $\bar{\varphi}(x) = Cx^3$. Constant C takes the value

$$C = -\frac{m_t^o}{6GI_t\ell}$$

and the particular solution is

$$\bar{\varphi}(x) = -\frac{1}{6}\frac{m_t^o}{GI_t\ell}x^3 \tag{2.108}$$

A concentrated point torque as loading M_t^o at $x = a$ corresponds to the particular solution

$$\bar{\varphi}(x) = \begin{cases} 0, & \text{kun } x \leq a \\ \dfrac{M_t^o\ell}{GI_t}[\dfrac{1}{k}\sinh(\dfrac{k}{\ell}(x-a)) - \dfrac{1}{\ell}(x-a)], & \text{kun } \quad x > a \end{cases} \tag{2.109}$$

and a concentrated point bimoment B_o at $x = a$ correspondingly to the solution

$$\bar{\varphi}(x) = \begin{cases} 0, & \text{kun } x \leq a \\ \dfrac{B_o}{GI_t}[\cosh(\dfrac{k}{\ell}(x-a)) - 1], & \text{kun } \quad x > a \end{cases} \tag{2.110}$$

2.9.7 Boundary conditions

The solution of a fourth order ordinary differential equation assumes four integration constants to be solved on the base of boundary conditions. Two conditions are usually adopted from each end. The most general restrictions are:
1. Rotation of the cross-section plane about the beam axis is prevented i.e. the angle of torsion is zero

$$\varphi = 0$$

2. Warping of the cross section plane is prevented

$$u \equiv -\omega_{\mathrm{A}}\varphi' = 0 \quad \Longrightarrow \quad \varphi' = 0$$

3. The known external torque is M_o at one end of the beam (can be also zero)

$$M_x \equiv M_\omega + M_t = -EI_\omega\varphi''' + GI_t\varphi' = M_o$$

4. The known external bimoment B_o at one end of the beam (can be also zero)

$$B = -EI_\omega\varphi'' = B_o$$

The traditional suports in the practice:

Fixed end with respect to the twisting

$$\varphi = 0, \qquad \varphi' = 0 \tag{2.111}$$

Fully free end of the beam:

$$M_x = 0, \qquad B = 0 \tag{2.112}$$

Simply supported end of the beam:

$$\varphi = 0, \qquad B = 0 \tag{2.113}$$

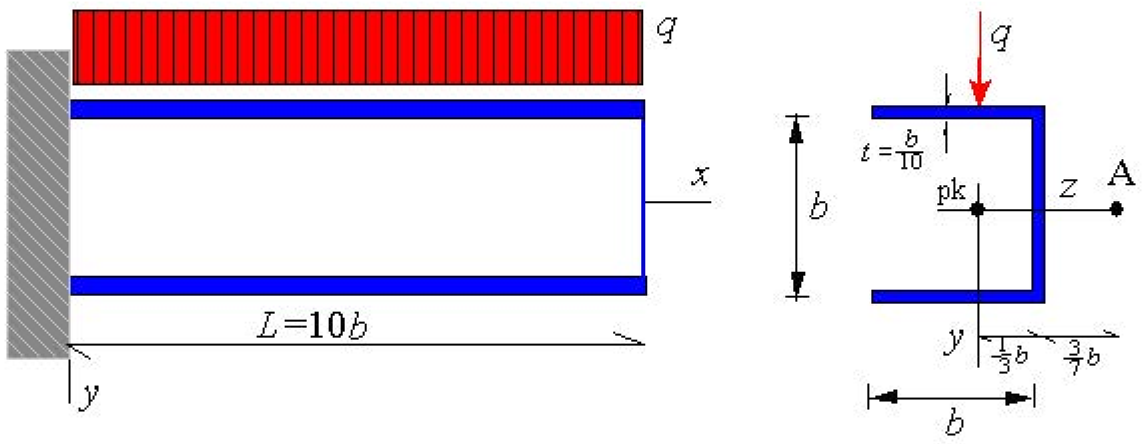


Figure 2.27. Beam of the example.

Illustrative example

Consider a cantilever beam, shown in Figure 2.27, which is loaded by a uniformly distributed vertical load q along the beam axis. The load passes through the centre of gravity of the cross section. The geometry of the beam and cross-section is given by the mutual ratios of the measures $\ell : b : t = 100 : 10 : 1$. Poisson's ratio $\nu = 0.3$. We have been considering the cross-section before and determined the location of the shear center and all the moment of inertia (second moments). The distibuted loading through the centre of gravity generates a distributed torque with the intensity of the load value times the distance between the shear centre and the centre of gravity of the cross-section, or

$$m_t^o = q(\tfrac{1}{3}\,b + \tfrac{3}{7}\,b) = \tfrac{16}{21}\,qb$$

The second moments are

$$\begin{aligned} I_\omega &= \tfrac{5}{84}\,b^5 t = \tfrac{5}{840}\,b^6 \\ I_t &= 3\,\tfrac{1}{3}\,bt^3 = \tfrac{1}{1000}\,b^4 \\ I_z &= \tfrac{7}{12}\,b^3 t = \tfrac{7}{120}\,b^4 \end{aligned}$$

When $\nu = 0.3$, $G = E/2.6$, the constant k is

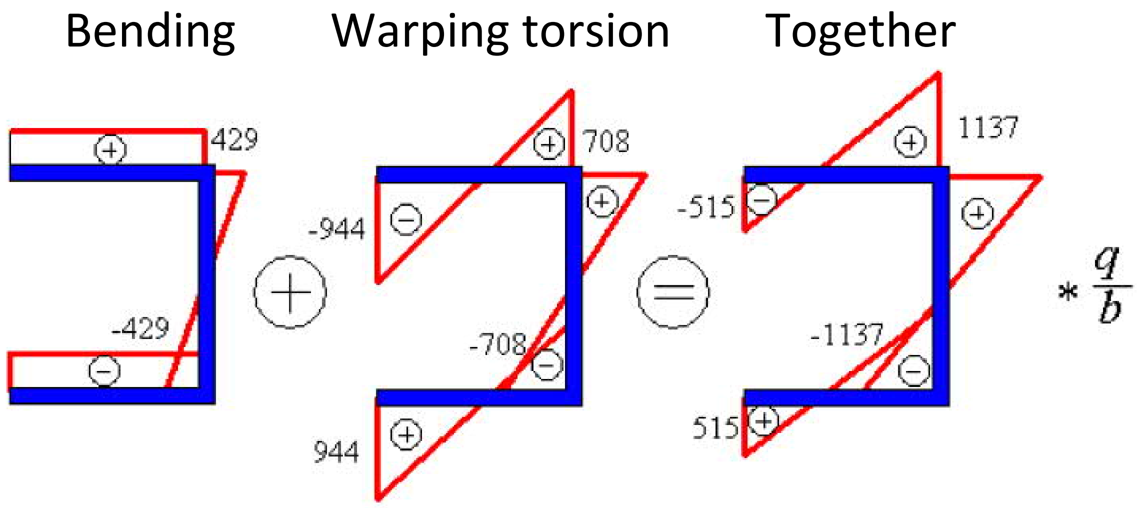
$$k = \ell\sqrt{\frac{GI_t}{EI_\omega}} = 2.542$$

The full solution of the differential equation is the sum of the solution of the homogeneous equation and the particular solution

$$\varphi(x) = C_1 + C_2 x + C_3 \sinh(\frac{k}{\ell}x) + C_4 \cosh(\frac{k}{\ell}x) - \frac{m_t^o}{2GI_t}x^2$$

From the boundary conditions which are in this case

$$\begin{aligned} \varphi(0) = 0 \quad &\Longrightarrow \quad C_1 + C_4 = 0 \\ \varphi'(0) = 0 \quad &\Longrightarrow \quad C_2 + \frac{k}{\ell}C_3 = 0 \\ M_x(\ell) = 0 \quad &\Longrightarrow \quad GI_t C_2 + m_t^o \ell = 0 \\ B(\ell) = 0 \quad &\Longrightarrow \quad -GI_t(C_3 \sinh(k) + C_4 \cosh(k)) - (\frac{\ell}{k})^2 m_t^o = 0 \end{aligned}$$



Kuva 2.28. Normal stress distributions.

we get for the integration constants C_1, C_2, C_3 and C_4 the values

$$C_1 = -C_4 = \frac{m_t^o \ell^2}{GI_t k^2}(k \tanh(k) + \cosh^{-1}(k))$$

$$C_2 = -\frac{k}{\ell} C_3 = \frac{m_t^o \ell^2}{kGI_t}$$

the angle of rotation and the force quatities obtained by differentiating are thus

$$\varphi(x) = \frac{m_t^o \ell^2}{kGI_t}\left[\frac{x}{\ell} - \sinh(\frac{k}{\ell}x) + (\tanh(k) + k^{-1}\cosh^{-1}(k))(\cosh(\frac{k}{\ell}x) - 1) - k(\frac{x}{\ell})^2\right]$$

$$M_t(x) = m_t^o \ell\left[1 - \frac{x}{\ell} - \cosh(\frac{k}{\ell}x) + (\tanh(k) + k^{-1}\cosh^{-1}(k))\sinh(\frac{k}{\ell}x)\right]$$

$$M_\omega(x) = m_t^o \ell\left[\cosh(\frac{k}{\ell}x) - (\tanh(k) + k^{-1}\cosh^{-1}(k))\sinh(\frac{k}{\ell}x)\right]$$

$$B(x) = \frac{m_t^o \ell^2}{k^2}\left[1 + k\sinh(\frac{k}{\ell}x) - (k\tanh(k) + \cosh^{-1}(k))\cosh(\frac{k}{\ell}x)\right]$$

We determine at first the maximum values for the normal stresses due to bending and warping. Both the bimoment and the bending moment take the maximum values at the fixed end of the beam. The maximum value of the bending moment is

$$M_z^{\max} = M_z(0) = \frac{1}{2}q\ell^2$$

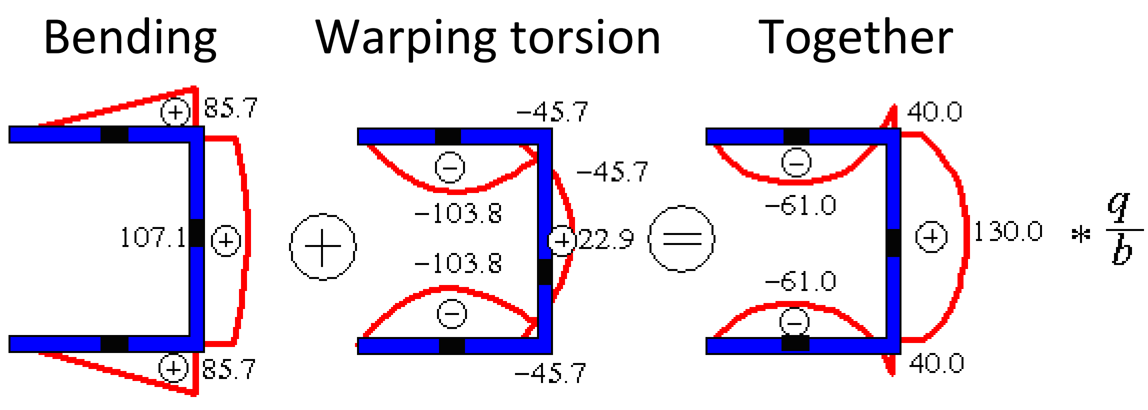
Correspondingly, the maximum value of the bimoment is

$$B^{\max} = B(0) = \frac{m_t^o \ell^2}{k^2}\left[1 - (k\tanh(k) + \cosh^{-1}(k))\right] = -0.2580 m_t^o \ell^2 = -19.66 q b^3$$

The normal and warping stresses are calculated from the equation

$$\sigma_x = \frac{M_z}{I_z}y(s) + \frac{B}{I_\omega}\omega(s) = -857.1\frac{q}{b^2}y(s) - 3302\frac{q}{b^3}\omega(s)$$

Figure 2.28 shows the normal stress distributions.



Kuva 2.29. Shear stress distribution.

The shear stresses are also a combination of the stresses due to bending and due to torsion. The shear stresses due to bending are distributed according to the first moment of the cross sectioni S_z, when the wall-thickness t is constant.The distribution of the first moment on the cross-section is

$$
\begin{aligned}
S_z(s) &= \int_0^s y(s)t\mathrm{d}s = \int_0^s -\frac{b}{2}t\mathrm{d}s = -\frac{bt}{2}s, \quad \text{kun} \quad 0 \le s \le b \\
&= S_z(b) + \int_b^s y(s)t\mathrm{d}s = -\frac{b^2t}{2} + \int_b^s (s - \frac{3b}{2})t\mathrm{d}s = \frac{b^2t}{2} - \frac{3bt}{2}s + \frac{t}{2}s^2, \ \text{kun} \ b \le s \le 2b \\
&= S_z(2b) + \int_{2b}^s y(s)t\mathrm{d}s = \int_{2b}^s \frac{b}{2}t\mathrm{d}s = -\frac{3b^2t}{2} + \frac{bt}{2}s, \quad \text{kun} \quad 2b \le s \le 3b
\end{aligned}
$$

The shear force takes its maximum value also at the fixed end of the beam, and it is

$$Q_{\max} = Q_y(0) = q\ell = 10qb$$

The shear stresses due to this are

$$\tau = -\frac{Q_y S_z(s)}{I_z t} = -1714\frac{q}{b^4}S_z(s)$$

The torque due to warping M_ω takes also its maximum value at the fixed end

$$M_\omega^{\max} = m_t^o\ell = \frac{160}{21}qb^2$$

when the maximum shear stresses due to sectorial torsion are

$$\tau_\omega = -\frac{M_\omega S_\omega(s)}{I_\omega t} = -12800\frac{q}{b^5}S_\omega(s)$$

When the moment resultant due to Saint Venant torsion vanishes at the fixed end, the total shear stresses there consist only of the shear stresses due to bending and warping torsion, and are

$$\tau(s) = -\frac{Q_y S_z(s)}{I_z t} - \frac{Q_y S_z(s)}{I_z t} = -1714\frac{q}{b^4}S_z(s) - 12800\frac{q}{b^5}S_\omega(s)$$

Figure 2.29 shows the distributions separately and as summed together.

Instead on the other sections of the beam, the shear stresses due to Saint Venant torsion do not vanish. The torque $M_t(x)$ reaches its maximum value at $x = 0.467\ell$

$$M_t^{\max} = 0.301 m_t^o \ell = 2.293 q b^2$$

At this section the maximum shear stress values are

$$\tau_t = \pm \frac{M_t t}{I_t} = 229.3 \frac{q}{b}$$

These stresses are distributed linearly over the wall-thickness and are zero on the centre line of the wall according to Vlasov's assumption. At the same section, the shear stresses due to warping torsion are constant over the wall-thickness, and are ($M_\omega = 0.232 m_t^o \ell$)

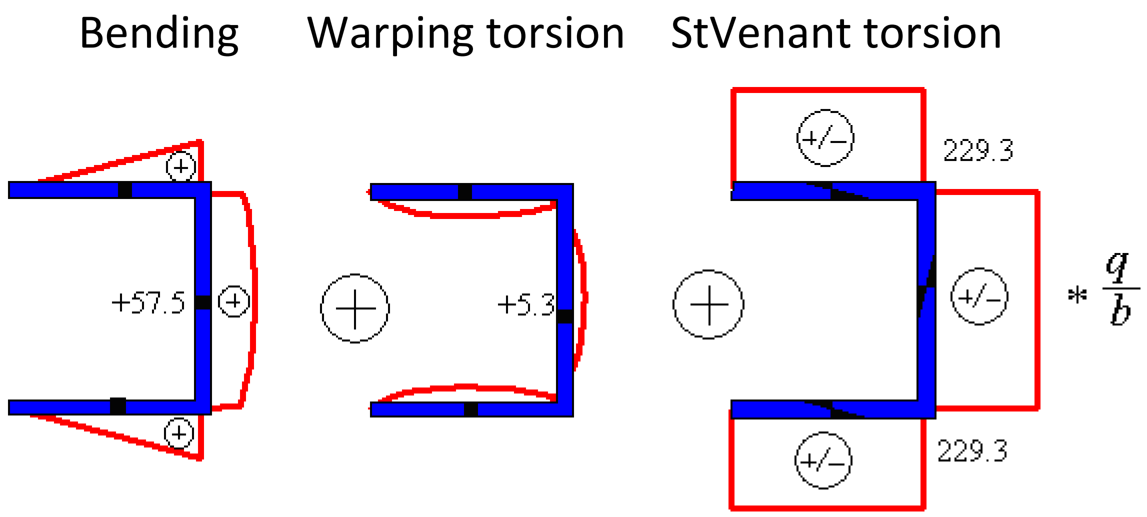
$$\tau_\omega(0.467\ell) = -\frac{M_\omega S_\omega(s)}{I_\omega t} = 2968 \frac{q}{b^5} S_\omega(s)$$

and also due to bending

$$\tau(0.467\ell) = -\frac{0.533 q\ell S_z(s)}{I_z t} = -913.6 \frac{q}{b^4} S_z(s)$$

Figure 2.30 shows the corresponding shear stress distributions. It can be observed that the shear stresses due to Saint Venant torsion have clearly higher values as compared to the other effects. For the maximum value of existing shear stresses we get the value

$$\tau_{\max} = (57.5 + 5.3 + 231.9) \frac{q}{b} = 294.7 \frac{q}{b}$$



Kuva 2.30. Shear stress distributions.

2.9.8 Analogy between the solution procedures of beam in combined tension and bending and alternatively of combined torsion of a thin-walled beam

The formulation of the problem of combined torsion of a thin-walled beam is analogous to the analysis of the beam under combined bendig and tension/compression, and results in similar ordinary differential equation. Thus each quantity here has a corresponding one in the bending analysis. The table besides shows the analogy between these two theories.

Bending	**Torsion**
$EI_z v'''' - Nv'' = q(x)$	$EI_\omega \varphi'''' - GI_t \varphi'' = m_t(x)$
deflection v	angle of torsion φ
slope v'	warping displacement φ'
coordinate y	sectorial coordinate ω
first moment of the cross section $S_z(s)$	sectorial first moment of the cross-section $S_\omega(s)$
moment of inertia (second moment) I_z	warping constant I_ω
intensity of distributed loading $q(x)$	intensity of distributed torque $m_t(x)$
concentarted load $F = \lim\limits_{\Delta x \to 0} \int_{\Delta x} q(x) \mathrm{d}x$	concentrated torque $M = \lim\limits_{\Delta x \to 0} \int_{\Delta x} m(x) \mathrm{d}x$
axial normal force N	Saint Venant torsion stiffness GI_t
normal force versus shear force Nv'	torque due to Saint Venant torsion $M_t = GI_t \varphi'$
bending moment $M_z = -EI_z v''$	bimoment $B = -EI_\omega \varphi''$
normal stress distribution $\sigma_x = \dfrac{M_z}{I_z} y$	warping normal stress distribution $\sigma_\omega = \dfrac{B}{I_\omega} \omega$
shear force $Q = M_z' = -EI_z v''$	warping torque $M_\omega = B' = -EI_\omega \varphi''$
shear stress distribution $\tau = -\dfrac{Q_y(x) S_z(s)}{I_z t}$	shear stress distribution due to sectorial torsion $\tau_\omega = -\dfrac{M_\omega(x) S_\omega(s)}{I_\omega t}$
vertical shear force component $V = Nv' + Q = Nv' - EI_z v''$	total internal torque $M_x = GI_t \varphi' - EI_\omega \varphi'''$

Taulukko 2.1 Analogy between various quantities and concept in bending and torsions.