MS-A0503 First course in probability and statistics

6A Hypothesis testing

Jukka Kohonen

Deparment of mathematics and systems analysis
Aalto SCI

Academic year 2020–2021
Period III

Contents

Paul the Octopus

By choosing from two food boxes (with national flags), Paul predicted the winner of football matches. In 2008, correct 4/6 times. In 2010, correct 7/7 times.

Opponent	Tournament	Stage	Date	Prediction	Result	Outcome
Poland	Euro 2008	group stage	8 June 2008	Germany	2–0	Correct
Croatia	Euro 2008	group stage	12 June 2008	Germany[3][20]	1–2	Incorrect
Austria	Euro 2008	group stage	16 June 2008	Germany	1–0	Correct
Portugal	Euro 2008	quarter-finals	19 June 2008	Germany	3–2	Correct
Turkey	Euro 2008	semi-finals	25 June 2008	Germany	3–2	Correct
Spain	Euro 2008	final	29 June 2008	Germany[3]	0–1	Incorrect
Australia	World Cup 2010	group stage	13 June 2010	Germany[31]	4–0	Correct
Serbia	World Cup 2010	group stage	18 June 2010	Serbia[31]	0–1	Correct
Ghana	World Cup 2010	group stage	23 June 2010	Germany[31]	1–0	Correct
England	World Cup 2010	round of 16	27 June 2010	Germany[32]	4–1	Correct
Argentina	World Cup 2010	quarter-finals	3 July 2010	Germany[23]	4–0	Correct
Spain	World Cup 2010	semi-finals	7 July 2010	Spain[33]	0–1	Correct
Uruguay	World Cup 2010	3rd place play-off	10 July 2010	Germany	3–2	Correct

Is this something that might easily happen by chance? Or does it indicate a good prediction skill?

https://en.wikipedia.org/wiki/Paul_the_Octopus

Hypothesis testing, contrasted to posterior inference

On previous lectures, we learned how we can infer a full distribution for an unknown parameter θ, if we have two ingredients:

- prior $f(\theta)$ — which values of θ are probable in the first place
- likelihood $f(\vec{x} \mid \theta)$ — the stochastic model of how the data are generated, if θ has a particular value

What if we are not able to formulate any prior $f(\theta)$? Can we do any inference **only from the data and the likelihood function?**

We can still do **something**. We can consider a **particular value of** θ, and choose to **reject** it, if that θ makes the observed data seem "too unlikely". [We'll make this more precise.]

This leads to the classical hypothesis testing, which is the topic of this lecture. (This is an alternative to Bayesian inference.)

Hypothesis testing — first idea (not good)

Suppose we know the general stochastic model: $X \sim \text{Bin}(1000, \theta)$ (one thousand coin tosses), but don't know the parameter θ. We are considering if $\theta = 0.5$ seems plausible — or if the data seems too surprising (unlikely) for this parameter value.

Example 1. Observe $x = 510$ heads. If $\theta = 0.5$ is true,

$$\mathbb{P}(X = 510 \,|\, \theta = 0.5) = \binom{1000}{510} 0.5^{510}\, 0.5^{490} \approx 2.1\%.$$

Is this surprising? Should we reject $\theta = 0.5$?

Example 2. Observe $x = 500$ heads. If $\theta = 0.5$ is true,

$$\mathbb{P}(X = 500 \,|\, \theta = 0.5) = \binom{1000}{500} 0.5^{500}\, 0.5^{500} \approx 2.5\%.$$

Is this surprising? Should we reject $\theta = 0.5$? Probably not!

Hypothesis testing — classical method

Step	Example
Formulate a hypothesis H_0 about how data are generated.	$\vec{X} = 30$ coffee cups, each from $\mathrm{N}(10, 3^2)$
Formulate a test statistic $t = t(\vec{X})$, calculated from data	Sample mean $m(\vec{X})$
Work out the distribution of t (if H_0 is true).	$m(\vec{X}) \sim \mathrm{N}(\ldots)$
Reject H_0 if the *observed* value $t(\vec{x})$ is in the tails of the distribution; choose tails to have probability α	$\alpha = 0.05$

Idea: t in the 5% tails is *surprising* if H_0 is true. We reject H_0 in that case. The tails are called critical region (rejection region).

Even if H_0 is true, this procedure may cause H_0 to be rejected — but only $\alpha = 5\%$ of the time. This is called the significance level of the test.

$\longrightarrow$ Illustration on blackboard

Hypothesis testing — another view, with *p*-value

Step	Example
Formulate a hypothesis H_0 about how data are generated.	$\vec{X}$ = 30 coffee cups, each from $\mathrm{N}(10, 3^2)$
Formulate a test statistic $t = t(\vec{X})$, calculated from data	Sample mean $m(\vec{X})$
Work out the distribution of t (if H_0 is true).	$m(\vec{X}) \sim \mathrm{N}(\ldots)$
Calculate both tail probabilities corresponding to $t(\vec{x})$. This is the *p*-value	$p = 0.018$
Reject H_0 if $p < \alpha$	reject

Here we **first** calculated a *p*-value, and **then** applied the significance level $\alpha = 0.05$. A *p*-value $0.018 < 0.05$ was considered "surprising enough" that H_0 should be rejected.

Contents

Coffee machine — Normal model

A coffee machine is meant to give 10.0 cl coffee in each cup, at least on average. We assume the coffee volumes are normally distributed, but we don't know the mean. To test the hypothesis ($\mu = 10.0$), 30 cups were taken and measured:

11.05 9.65 10.93 9.46 10.27 10.02 10.07 10.74 11.15 10.40 10.12 11.20 10.07 10.27 9.99 9.80 10.83 10.21 11.26 10.11 10.49 10.10 10.15 11.02 10.00 11.68 10.51 11.20 11.29 10.15

Is the machine correctly calibrated (on average)?

Sample mean $m(\vec{x}) = 10.473$, which differs from the intended $\mu_0 = 10.0$.

But since the data are random, it is quite expected that the sample mean is not exactly 10.0!

Is the observed difference statistically significant?

Coffee machine — Normal model

11.05 9.65 10.93 9.46 10.27 10.02 10.07 10.74 11.15 10.40 10.12 11.20 10.07 10.27 9.99 9.80 10.83 10.21 11.26 10.11 10.49 10.10 10.15 11.02 10.00 11.68 10.51 11.20 11.29 10.15

Sample mean $m(\vec{x}) = 10.473$, sample standard deviation $\mathrm{sd}(\vec{x}) = 0.563$

H_0: $\mu = \mu_0 = 10.0$
H_1: $\mu \neq \mu_0 = 10.0$

Test statistic of the observed data:

$$t(\vec{x}) \;=\; \frac{m(\vec{x}) - \mu_0}{\mathrm{sd}(\vec{x})/\sqrt{n}} \;=\; \frac{10.473 - 10.0}{0.563/\sqrt{30}} \;=\; 4.60$$

Because sample size $n = 30$ fairly large, we work as if $\sigma = \mathrm{sd}(\vec{x}) = 0.563$ exactly ("known variance"). Then $t(\vec{X})$ has standard normal distribution.
[We could be more exact and use t distribution.]

$$\text{p value} \;\approx\; \mathbb{P}(|t(\vec{X})| \geq |t(\vec{x})| \mid H_0) \;\approx\; \mathbb{P}(|Z| \geq 4.60) \;\approx\; 4.2 \times 10^{-6}$$

Result: p-value very small. If H_0 were true, it would be very unlikely to obtain a sample mean so far (or further) from the hypothesized $\mu = 10.0$.

Hypothesis testing vs. confidence interval

Often, hypothesis testing at significance level α can be alternatively framed as the question:

If we calculate a $1 - \alpha$ confidence interval for the unknown parameter θ, does the interval contain the value θ_0 claimed by the null hypothesis?

If the interval contains θ_0, then the data is compatible with the possibility that $\theta = \theta_0$, as claimed.

If the interval is fully below or fully above θ_0, then the data speaks against the possibility that $\theta = \theta_0$.

(Possibly illustration on blackboard)

Coffee machine — testing vs. confidence interval

11.05 9.65 10.93 9.46 10.27 10.02 10.07 10.74 11.15 10.40 10.12 11.20
10.07 10.27 9.99 9.80 10.83 10.21 11.26 10.11 10.49 10.10 10.15 11.02
10.00 11.68 10.51 11.20 11.29 10.15

Sample mean $m(\vec{x}) = 10.473$, sample standard deviation $\mathrm{sd}(\vec{x}) = 0.563$

H_0: $\mu = \mu_0 = 10.0$
H_1: $\mu \neq \mu_0 = 10.0$

Again, work as if $\sigma = \mathrm{sd}(\vec{x}) = 0.563$ exactly ("known variance").

Computing e.g. 99% confidence interval, we obtain

$$10.473 \pm 2.58 \cdot \frac{0.563}{\sqrt{30}} \approx 10.473 \pm 0.265,$$

so the interval is completely above 10.0. Thus we reject the null hypothesis (that $\mu = 10.0$) at 1% significance level.

Caveat: In some situations, there are subtle differences between hypothesis testing and confidence intervals, but in the most common situations, this connection is probably helpful for understanding.

Null hypothesis H_0

The starting point of a hypothesis test is the null hypothesis H_0, which generally indicates that nothing new or surprising is needed to explain the observations. Often this is of the form "parameter=value" (and the most common parameter is *mean*).

Example

H_0: Paul's predictions are correct with probability $\theta = 0.5$

H_0: Coffee machine gives $\mu = 10.0$ cl on average, as intended

H_0: A proposed new medicine is no better than placebo

H_0: A portfolio manager performs no better than market average

The alternative hypothesis H_1 is usually the complement of the null hypothesis. So if H_0 says $\mu = 10$, then H_1 says $\mu \neq 10$. Note that such an alternative hypothesis does not claim any single value!

Test statistic and p-value

The "surprisingness" of an observed data $\vec{x} = (x_1, \ldots, x_n)$ is measured by first calculating a test statistic,

$$t(\vec{x}) = t(x_1, \ldots, x_n),$$

which *condenses* the n-dimensional data vector into one real number.

Then the p-value (related to the test statistic) is the *probability* that the test statistic would have the observed value $t(\vec{x})$, or something even further away from the expected value.

The probability and the expected value are calculated by assuming that the H_0 is *true*. Some typical interpretations

p-value	**Interpretation**
> 0.10	Data quite compatible with H_0
≈ 0.05	Data suggests against H_0
< 0.01	Data suggests strongly against H_0

Some more examples

Example (Coin tossing — Discrete data)

A coin that was claimed to be fair, was tossed 50 times, with 42 heads.
H_0: Heads probability $\theta = 1/2$
H_1: Heads probability $\theta \neq 1/2$

Example (Noisy observation — Little data)

Star brightness measurements claimed to be normal, with $\mu = 5$ and $\sigma = 3$. Measured once, with result $x_1 = 9.8$.
H_0: $\mu = 5$
H_1: $\mu \neq 5$

Example (Quality control — Composite hypothesis)

Shopkeeper claims that *at most* 5% of their tomatoes are bad. 50 tomatoes were tested, 4 were bad.
H_0: Proportion of bad $\theta \leq 0.05$
H_1: Proportion of bad $\theta > 0.05$
This is an example where H_0 is *composite* (allows many values).

Example. Coin tossing

Coin claimed to be fair, results 42 heads on 50 tosses.
H_0: Heads probability $\theta = 1/2$
H_1: Heads probability $\theta \neq 1/2$
Test statistic = heads count: $t(x) = 42$
$T = t(X) =$ "heads count according to H_0"

$$f(x) \;=\; \mathbb{P}(T = x \,|\, H_0) \;=\; \binom{50}{x} \left(\frac{1}{2}\right)^{x} \left(1 - \frac{1}{2}\right)^{50-x}$$

Test statistic has mean $t_0 = \mathbb{E}(T \mid H_0) = 25$.

$$\begin{aligned}
\text{p-value} \;&=\; \mathbb{P}(|T - t_0| \geq |t(x) - t_0| \,|\, H_0) \\
&=\; \mathbb{P}(|T - 25| \geq 17 \,|\, H_0) \\
&=\; \sum_{x=0}^{8} f(x) + \sum_{x=42}^{50} f(x) \;\approx\; 1.2 \times 10^{-6}.
\end{aligned}$$

Data is strongly against H_0.

Example. Noisy observation

Star brightness measurements claimed to be normal $\mu = 5$ and $\sigma = 3$. Single observation: $x_1 = 9.8$.
H_0: Mean $\mu = 5$
H_1: Mean $\mu \neq 5$
Test statistic = normalized difference from the hypothesized mean:
$z(\vec{x}) = \frac{x_1 - 2}{3} = 1.6$

$$\text{p-value} \;=\; \mathbb{P}(|Z| \geq 1.6 \mid H_0) \;=\; 2\mathbb{P}(Z \geq 1.6 \mid H_0) \;\approx\; 11\%,$$

Observation compatible with regular random chance.
Observation does not lead to rejection of H_0

Contents

Variant: Testing for μ, large non-normal data

Suppose the data source generates independent, identically distributed numbers $X_1, X_2, \ldots, X_n$ from some distribution with unknown mean μ. We study whether the mean could be μ_0.

H_0: $\mu = \mu_0$
H_1: $\mu \neq \mu_0$

Distribution unknown $\implies$ impossible to test?
No; if sample is big, and independent, then CLT says the *sample mean* is normal, even if the individual observations are not.

Test statistic just like in the normal model:

$$t(\vec{x}) \;=\; \frac{m(\vec{x}) - \mu_0}{\mathrm{sd}(\vec{x})/\sqrt{n}}.$$

Variant: Unknown variance

Often, the standard deviation σ **of the data source** is not known, but is estimated by the *sample* standard deviation $\mathrm{sd}(\vec{x})$.

If the sample is large (e.g. $n > 30$), the estimate is decent, but ...

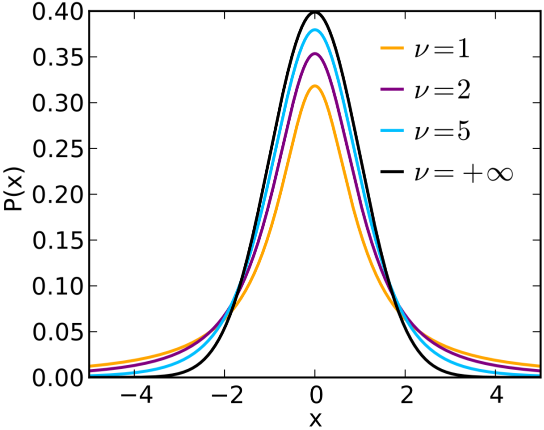
For small samples, we must note that the test statistic

$$t(\vec{X}) = \frac{m(\vec{X}) - \mu_0}{\mathrm{sd}(\vec{X})\,\sqrt{n}}$$

is the quotient of two random variables, and there is no reason to believe its distribution would be normal. It is not!

The real distribution of $t(\vec{X})$ is the Student's t-distribution with parameter $n-1$. The parameter is called "degrees of freedom". All is still fine — you simply do all computations with this t-distribution instead of the normal distribution. Again, you can use tables, or a computer. In R, `pt` is the CDF, and `qt` is the quantile function. (Compate to `pnorm` and `qnorm`.)

Student's t-distribution



Picture credit: Skbkekas, CC BY 3.0,
https://commons.wikimedia.org/w/index.php?curid=9546828

Interlude: Computing with distributions in R

distribution	density	CDF	quantile function	generate random
uniform	`dunif`	`punif`	`qunif`	`runif`
beta	`dbeta`	`pbeta`	`qbeta`	`rbeta`
normal	`dnorm`	`pnorm`	`qnorm`	`rnorm`
Student	`dt`	`pt`	`qt`	`rt`
exponential	`dexp`	`pexp`	`qexp`	`rexp`
...	`d...`	`p...`	`q...`	`r...`

Compare the 0.975-quantiles of standard normal, and Student with $n = 50$ and $n = 10$.

```
> qnorm(.975)
[1] 1.959964
> qt(.975, 49)
[1] 2.009575      => Slightly wider confidence intervals.
> qt(.975, 9)
[1] 2.262157      => Clearly wider confidence intervals.
```

Interlude: Computing with distributions in Matlab/Octave

distribution	density	CDF	quantile function	generate randor
uniform	`unifpdf`	`unifcdf`	`unifinv`	`unifrnd`
beta	`betapdf`	`betacdf`	`betainv`	`betarnd`
normal	`normpdf`	`normcdf`	`norminv`	`normrnd`
Student	`tpdf`	`tcdf`	`tinv`	`trnd`
exponential	`exppdf`	`expcdf`	`expinv`	`exprnd`
...	`...pdf`	`...cdf`	`...inv`	`...rnd`

Compare the 0.975-quantiles of standard normal, and Student with $n = 50$ and $n = 10$.

```
>> norminv(.975)
ans =
   1.959963984540054
>> tinv(.975, 49)
ans =
   2.009575237129235
>> tinv(.975, 9)
ans =
   2.262157162798204
```

Variant: Composite hypothesis

Shopkeeper claims that *at most* 5% of their tomatoes are bad.
50 tomatoes were tested, 4 were bad.
H_0: Proportion of bad $\theta \leq 0.05$
H_1: Proportion of bad $\theta > 0.05$
This is an example where H_0 is composite (allows many values).

Test statistic: Count of bads: $t(\vec{x}) = 4$
If the real proportion is θ (in the data source), then

$$\mathbb{P}_\theta(T = t) \; = \; f_\theta(t) \; = \; \binom{50}{t}\theta^t(1-\theta)^{50-t}$$

Because H_0 claims proportion is *small*, we apply a one-sided test: only *high* values *above* claimed mean are significant. We would like to find

$$\mathbb{P}_\theta\Big(T - \mathbb{E}_\theta(T) \geq t(\vec{x}) - \mathbb{E}_\theta(T) \Big) \; = \; \mathbb{P}_\theta(T \geq t(\vec{x})) \; = \; \sum_{t=4}^{50} f_\theta(t).$$

Trouble: the probability depends on θ. So let us choose the the *highest* possible p-value, from any θ that H_0 allows:

$$\text{p-value} \; = \; \max_{\theta \leq 0.05} \mathbb{P}_\theta(T \geq t(\vec{x})) \; = \; \mathbb{P}_{0.05}(T \geq t(\vec{x})) \; = \; \sum_{t=4}^{50} f_{0.05}(t) \; \approx \; 24\%$$

Contents

Accepting or rejecting

You could compute a *p*-value and just report it, refraining of making further decisions like "accept" or "reject".

But often you *need* to make a decision. Based on the test, you either accept or reject H_0. This may affect e.g. further studies (performed or not), taking a medicine for use, ...

To make a decision, you choose (either before or after computing *p*) a significance level α $(0 < \alpha < 1)$.

- If p-value $\geq \alpha$, the null hypothesis is *accepted*
- If p-value $< \alpha$, the null hypothesis is *rejected*

Typical, conventional significance levels are $\alpha = 1\%$ and $\alpha = 5\%$.

(This is a very crude way of making decisions. More advanced methods would explicitly consider the *consequences* of the decisions $\longrightarrow$ decision theory, but outside the scope of this course.)

Type I and II errors

Whichever decision we make (accept or reject), it may be correct or incorrect.

		Decision	
		H_0 accepted	H_0 rejected
Reality	H_0 true	Correct	Type I error
	H_0 false	Type II error	Correct

If rejection of H_0 is considered *discovering* an interesting phenomenon (deviation from the null hypothesis), then

- type I error is a *false positive* (false discovery)
- type II error is a *false negative* (failure to discover)

In statistical inference, it is not possible to avoid both errors completely. But by probability calculus, we may try to calculate the *probabilities* of making type I and II errors.

Probabilities of the errors

$p(\vec{x})$ = the p-value computed from data $\vec{x}$
$p(\vec{X})$ = random variable: what p-values *can* be obtained (when $\vec{X}$ follows a distribution)

If H_0 is true, then the probability of rejecting it (Type I error) is

$$\mathbb{P}(H_0 \text{ rejected} \,|\, H_0) \;=\; \mathbb{P}(p(\vec{X}) < \alpha \,|\, H_0) \approx \alpha$$

If H_0 is false, then the probability of accepting it (Type II error) is

$$\mathbb{P}(H_0 \text{ accepted} \,|\, H_1) \;=\; \mathbb{P}(p(\vec{X}) \geq \alpha \,|\, H_1)$$

By changing α, we can change both probabilities ... with a tradeoff

α	Type I error rate	Type II error rate
Small	Small	Large
Large	Large	Small

Two caricatures

Eve Eager

- Applies significance level $\alpha = 5\%$
- Is eager to reject null hypotheses, so makes many discoveries
- Has approx 5% rate of Type I errors (rejecting a true null hypothesis)
- Has lower type II rate than Cathy

Cathy Cautious

- Applies significance level $\alpha = 1\%$
- Is cautious of rejecting a null hypothesis, so makes fewer discoveries
- Has approx 1% rate of type I errors (rejecting a true null hypothesis)
- Has higher type II error rate than Ann (failure to make a discovery)

Example. Coin tossing

A coin is tossed 10 times and $\vec{x} = (0,0,1,0,0,0,0,0,0,0)$ is observed.
Test the fairness at significance 5%.

> H_0: *Heads probability* $\theta = 0.5$,
> H_1: *Heads probability* $\theta \neq 0.5$.

Test statistic: $t(\vec{x}) = \#$ heads
Stochastic model of the test statistic: $T = t(\vec{X})$

$$f_{H_0}(t) \;=\; \mathbb{P}(T = t \mid H_0) \;=\; \binom{10}{t}\left(\frac{1}{2}\right)^{10}$$

From this observed data $\vec{x}$, we compute

$$p(\vec{x}) \;=\; \mathbb{P}(\,|t(\vec{X}) - 5| \geq 4 \mid H_0) \;=\; \sum_{t=0}^{1} f_{H_0}(t) + \sum_{t=9}^{10} f_{H_0}(t) \;\approx\; 2.1\%.$$

Decision: Null hypothesis rejected at 5% level.
But what do we know about the error probabilities?

Coin tossing — Type I error rate

Possible p-values, as a function of the test statistic $t(\vec{x}) = \#\text{heads}$:

# heads	0	1	2	3	4	5	6	7	8	9	10
$f_{H_0}(t)$ [%]	0.1	1.0	4.4	11.7	20.5	24.6	20.5	11.7	4.4	1.0	0.1
p-value [%]	0.2	2.1	10.9	34.4	75.4	100	75.4	34.4	10.9	2.1	0.2

At 5% level, we reject the null at the critical region $\{0, 1, 9, 10\}$.
If H_0 is true, we land there with probability

$$\mathbb{P}(t(\vec{X}) \in \{0, 1, 9, 10\} \mid H_0) \;=\; \sum_{t=0}^{1} f_{H_0}(t) + \sum_{t=9}^{10} f_{H_0}(t) \;\approx\; 2.1\%.$$

So the type I error rate is 2.1% $\leq$ 5%.

It is not exactly 5% because in the discrete distribution of the test statistic, we do not have a point where the tail probabilities would be exactly 5%. Values 2 and 8 are in the acceptance region because their p-values are $> 5\%$.

Coin tossing — Type II error rate??

Possible p-values, as a function of the test statistic:

# heads	0	1	2	3	4	5	6	7	8	9	10
$f_{H_0}(t)$ [%]	0.1	1.0	4.4	11.7	20.5	24.6	20.5	11.7	4.4	1.0	0.1
p-value [%]	0.2	2.1	10.9	34.4	75.4	100	75.4	34.4	10.9	2.1	0.2

At 5% level, we accept the null in the complement of the critical region, that is $\{2, 3, \ldots, 7, 8\}$.
If H_1 is true, how probably do we land there ($\Rightarrow$ type II error)?
This is more difficult to calculate, because it depends on the true value of θ, and H_1 allows many values.
For example, if $\theta = 0.5001$, we have

$$\begin{aligned} \mathbb{P}(t(\vec{X}) \in \{2, 3, \ldots, 8\} \,|\, \theta = 0.5001) \;&\approx\; \mathbb{P}(t(\vec{X}) \in \{2, 3, \ldots, 8\} \,|\, H_0) \\ &=\; \sum_{t=2}^{8} f_{H_0}(t) \;\approx\; 97.9\%, \end{aligned}$$

so we have a huge type II error rate.

Type II error rate, if single alternative known

A coin tossed 10 times and $\vec{x} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$ is observed. **Extra assumption:** We know that either $\theta = 0.5$ or $\theta = 0.9$. Test the fairness hypothesis at $\alpha = 0.05$.

> H_0: *Heads probability* $\theta = 0.5$,
> H_1: *Heads probability* $\theta = 0.9$.

Our computations are as before (same H_0, same test statistic, same decisions). Now if H_1 is true, then the test statistic has distribution

$$f_{H_1}(t) \;=\; \mathbb{P}(T = t \mid H_1) \;=\; \binom{10}{t} 0.9^t (1 - 0.9)^{10-t}$$

and the type II error rate is

$$\mathbb{P}(\, t(\vec{X}) \in \{2, 3, \ldots, 8\} \mid H_1) \;=\; \sum_{t=2}^{8} f_{H_1}(t) \;\approx\; 26\%.$$

Contents

Further topics in hypothesis testing

The previous statistical tests concerned hypotheses *about the mean*, for example $p = 1/2$ or $\mu = 10.0$, and were based on *strong simplifying assumptions*, for example "data are normal" or "lots of data, so test statistic is normal".

Classical statistics offers more tests for advanced questions, e.g.

- Hypotheses of other parameters. E.g. is the standard deviation of our star measurements $\sigma = 3$ or not?
  $\longrightarrow \chi^2$ test etc.
- Weaker assumptions. E.g. data not normal and sample small, so sample mean not normal.
  $\longrightarrow$ distribution-specific tests; or nonparametric tests
- Tests for distribution shape. E.g. we would like to test whether the data are normal. $\longrightarrow$ more tests . . .

Further topics in hypothesis testing

For many specific yes/no questions about the unknown distribution (that generates the data), one can still apply the **same generic framework** of hypothesis testing:

1. Formulate a hypothesis H_0 about how the data are generated.
2. Formulate a test statistic $t(\vec{X})$ and work out its distribution, if H_0 is true.
3. Study how well the observed $t(\vec{x})$ fits into that distribution (is it in the tails or not).

Details of the test statistics and their distributions are different in each case.

More about such advanced tests e.g. on MS-C1620 Statistical inference.

Last lecture on Friday, Feb 19. We will try to wrap up what we have learned during the course, see how it fits together, and perhaps fill in some gaps.

For the last lecture, you are encouraged to **bring your questions** about any topics related to the course. You can also send such questions in advance by e-mail, or in the chat now.

Course exam on Wednesday, Feb 24. Remote exam due to circumstances. Problems in MyCourses, you work out your solutions on paper, take a photo, and submit. (Detailed instructions later)