Aalto University
Department of Mathematics and Systems Analysis

Exercise set 1
K Kytölä & O Abuzaid

MS-E1602 Large Random Systems, 2020-2021/IV

Exercise session: Wed 3.3. at 14-16 (?) Solutions due: Mon 8.3. at 10

Exercise 1. If $\mathfrak{X}$ is a finite set, and μ, ν are two probability measures on $\mathfrak{X}$, we define the total variation distance between μ and ν as

$$\varrho_{\mathrm{TV}}(\mu, \nu) = \max_{E \subset \mathfrak{X}} \big| \mu[E] - \nu[E] \big|.$$

(a) Show that the total variation distance can be alternatively expressed as

$$\varrho_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{x \in \mathfrak{X}} \big| \mu[\{x\}] - \nu[\{x\}] \big|.$$

(b) Prove that the total variation distance is a metric[1] on the set of probability measures on $\mathfrak{X}$.

Exercise 2. Let σ be a uniformly distributed random permutation of the set $\{1, 2, \ldots, n\}$. Compute the following quantities about its cycle decomposition.[2]

(a) Let L be the length of the cycle that contains the element 1. What is the distribution of L, i.e. probabilities $\mathsf{P}[L = \ell]$? Calculate also $\mathsf{E}[L]$.
(b) Let S be the number of cycles in the cycle decomposition. Calculate $\mathsf{E}[S]$.
(c) What is the probability that elements 1 and 2 belong to the same cycle?

Exercise 3.

(a) Let $E_1, \ldots, E_n \subset \Omega$ be events. Prove the inclusion-exclusion formula:

$$\mathsf{P}\Big[\bigcup_{j=1}^{n} E_j\Big] = - \sum_{\substack{J \subset \{1,\ldots,n\} \\ J \neq \emptyset}} (-1)^{\# J} \, \mathsf{P}\Big[\bigcap_{j \in J} E_j\Big]$$

$$= \sum_{1 \le j_1 \le n} \mathsf{P}[E_{j_1}] - \sum_{1 \le j_1 < j_2 \le n} \mathsf{P}[E_{j_1} \cap E_{j_2}] + \sum_{1 \le j_1 < j_2 < j_3 \le n} \mathsf{P}[E_{j_1} \cap E_{j_2} \cap E_{j_3}] - \cdots .$$

(b) What is the probability that a uniformly distributed random permutation of the set $\{1, 2, \ldots, n\}$ has a fixed point, i.e., a cycle of length 1? Compute the limit of this probability as $n \to \infty$.

Hint: *In part (a), you may want to use indicator random variables and consider the complementary event. In part (b), set $E_j = \{$the point j is a fixed point$\}$.*

[1] *Recall:* A metric on a set A is a function $\varrho \colon A \times A \to [0, \infty)$ such that for any $a, b, c \in A$ we have (1): $\varrho(a, b) = 0 \iff a = b$, (2): $\varrho(a, b) = \varrho(b, a)$, and (3): $\varrho(a, c) \le \varrho(a, b) + \varrho(b, c)$.

[2] *Recall:* A permutation can be written as a composition of disjoint cycles so that each element appears in exactly one cycle, and up to the order of cycles this cycle decomposition is unique.

Exercise 4. Let $(X_n)_{n\in\mathbb{Z}_{\geq 0}}$ be the symmetric simple random walk with ± 1 steps, i.e. $X_n = \sum_{k=1}^{n} \xi_k$ where $(\xi_k)_{k\in\mathbb{N}}$ are i.i.d. and $\mathsf{P}\big[\xi_k = +1\big] = \frac{1}{2}$, $\mathsf{P}\big[\xi_k = -1\big] = \frac{1}{2}$. Prove that for any n and for any $x > 0$ we have

$$\mathsf{P}\big[X_n > x\big] \leq e^{-\frac{1}{2n}x^2}.$$

Hint: *Compute the exponential moments* $\mathsf{E}\big[e^{\theta X_n}\big]$*, and use Markov inequality. Choose the parameter* θ *judiciously.*

Exercise 5. Consider bond percolation on $\mathbb{Z}^d$, i.e., the probability measure on $\Omega = \{0,1\}^{\mathrm{E}(\mathbb{Z}^d)}$ obtained as the product of Bernoulli distributions on $\{0,1\}$ for each bond $e \in \mathrm{E}(\mathbb{Z}^d)$. Recall that for a given configuration $\omega = (\omega_e)_{e\in\mathrm{E}(\mathbb{Z}^d)} \in \Omega$, we interpret $\omega_e = 1$ as "the bond e is open", and we consider connectivity along paths of open bonds. The sigma algebra $\mathscr{F}$ on $\Omega = \{0,1\}^{\mathrm{E}(\mathbb{Z}^d)}$ is the cylinder sigma algebra (product sigma algebra), i.e., the smallest sigma algebra with respect to which the projections $\omega \mapsto \omega_e$ are measurable for all $e \in \mathrm{E}(\mathbb{Z}^d)$. Prove the following measurability properties.

(a) For any $x, y \in \mathbb{Z}^d$, the event $\{x \leftrightsquigarrow y\}$ that x is connected to y is measurable.
(b) For any $x \in \mathbb{Z}^d$, the size $\#\mathcal{C}_x$ of the connected component $\mathcal{C}_x$ of x is a measurable random variable.
(c) The event that there exists an infinite connected component is measurable.

Exercise 6. Consider the regular infinite tree $\mathcal{T}$ of degree $B \geq 3$ (see Figure). Declare each nearest neighbor bond of the tree open or closed, with respective probabilities p and $1-p$, independently. Denote by $\mathcal{C}$ the set of sites of the tree which are connected to the root site $\emptyset$ by a path of open bonds. Let $S = \#\mathcal{C}$ be the number of such sites. Calculate $\mathsf{E}[S]$ as a function of p. For which values of p is $\mathsf{E}[S]$ finite?

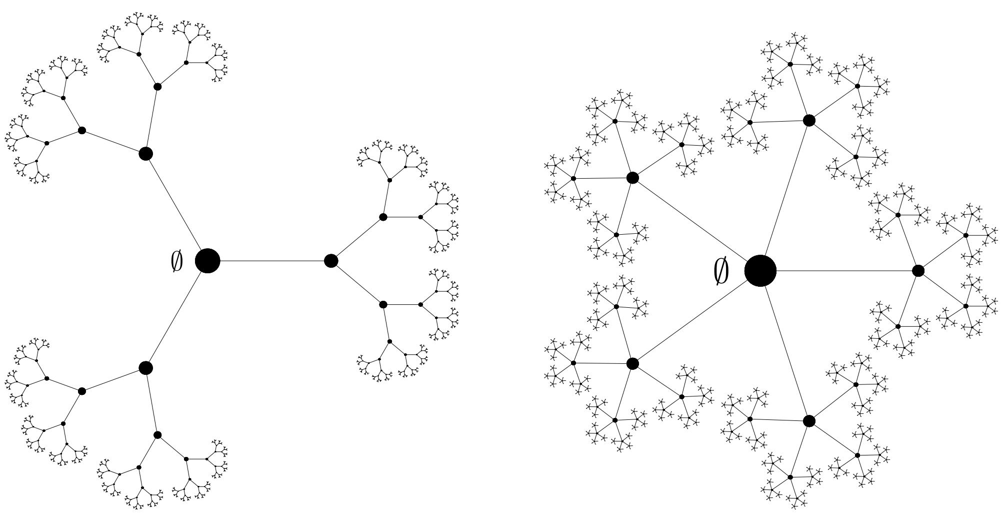

FIGURE: Regular infinite trees of degree 3 and 5.