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Exercise set 2
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Exercise session: Wed 10.3. at 14-16 Solutions due: Mon 15.3. at 10

Exercise 1. Let $(X_j)_{j\in\mathbb{N}}$ be independent, $X_j \sim \mathrm{N}(\mu,\sigma^2)$, and $S_n = \sum_{j=1}^n X_j$.

(a) Show that for all $x > 0$ we have

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \exp\left(-\frac{x^2}{2}\right) \leq \int_x^\infty \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y \leq \frac{1}{x} \exp\left(-\frac{x^2}{2}\right).$$

(b) For $a > \mu$, calculate the following rate of large deviations

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathsf{P}[S_n \geq na].$$

Interpretation: *For $a > \mu$, the probability of $\{S_n \geq na\}$ is exponentially small in n, approximately given by $\mathsf{P}[S_n \geq na] \approx e^{-nJ(a)}$, where $J(a)$ is the limit in part (b).*

Exercise 2. Let $X_1, X_2, \ldots$ be independent and identically distributed real valued random variables with a continuous cumulative distribution function $F \colon \mathbb{R} \to [0,1]$.

(a) Show that for any $n = 1, 2, \ldots$, almost surely (i.e., with probability one) there exists a unique permutation $\sigma_n \in \mathfrak{S}_n$ such that

$$X_{\sigma_n(1)} > X_{\sigma_n(2)} > \cdots > X_{\sigma_n(n)}.$$

Show moreover, that the random permutation σ_n follows the uniform distribution on $\mathfrak{S}_n$, i.e. $\mathsf{P}\big[\sigma_n = \pi\big] = \frac{1}{n!}$ for all $\pi \in \mathfrak{S}_n$.

Hint: *Show that the probabilities that $\sigma_n = \pi$ and $\sigma_n = \pi'$ are the same for any $\pi, \pi' \in \mathfrak{S}_n$.*

(b) Let $\sigma_n \in \mathfrak{S}_n$ and $\sigma_{n+1} \in \mathfrak{S}_{n+1}$ be the random permutations defined in part (a). Show that for any $\pi \in \mathfrak{S}_n$ we have

$$\mathsf{P}\Big[\sigma_{n+1}(1) = n+1 \;\Big|\; \sigma_n = \pi\Big] = \frac{1}{n+1}.$$

For all $n = 1, 2, \ldots$, define the event $R_n = \{X_n > \max(X_1, \ldots, X_{n-1})\}$.

(c) Show that the events $R_1, R_2, \ldots$ are independent.

Hint: *Observe that $R_n = \{\sigma_n(1) = n\}$ and use previous results.*

(d) Show that almost surely R_n occurs for infinitely many n.

(e) Show that almost surely $R_n \cap R_{n+1}$ occurs only for finitely many n.

Interpretation: *Think of an annual sports contest, where X_n is the score (e.g., javelin throw distance) of the winner in the contest of year n. Then R_n is the event that the earlier record score is broken by the winner of year n. The conclusions of parts (d) and (e) say that there are infinitely many years when a new record is set, but there are only finitely times when new records are set in two consecutive years.*

Exercise 3. Define a function λ by

$$\lambda(n) = \sqrt{n\,\log\big(\log(n)\big)}.$$

Let $\alpha > 1$ and for $k \in \mathbb{N}$ let[1] $n_k = \lfloor \alpha^k \rfloor$. Calculate the limits

$$\lim_{k\to\infty} \frac{n_k}{\alpha^k}, \qquad \lim_{k\to\infty} \frac{\lambda(n_k)}{\lambda(n_{k+1})}, \qquad \lim_{k\to\infty} \frac{\lambda(n_{k+1}-n_k)}{\lambda(n_{k+1})}.$$

Exercise 4. Let X_j, $j \in \mathbb{Z}_{>0}$, be independent identically distributed random variables with $X_j \sim \mathrm{Exp}(\lambda)$. Denote the maximum of the first n of them by $M_n = \max_{1\leq j\leq n} X_j$, and consider the shifted maxima $R_n = M_n - \frac{1}{\lambda}\log(n)$. Calculate the cumulative distribution functions $F_n(x) = \mathsf{P}[R_n \leq x]$, $n \in \mathbb{Z}_{>0}$, and show that they converge pointwise as $n \to \infty$. Calculate the limit, and show that it is a cumulative distribution function.[2]

Exercise 5.

(a) Let $P \sim \mathrm{Poisson}(\lambda)$, i.e., $\mathsf{P}\big[P = k\big] = \frac{\lambda^k}{k!}e^{-\lambda}$ for $k \in \mathbb{Z}_{\geq 0}$. Calculate the characteristic function $\mathsf{E}\big[e^{\mathrm{i}\theta P}\big]$.

(b) Let $B \sim \mathrm{Bin}(n,p)$, i.e., $\mathsf{P}\big[B = k\big] = \binom{n}{k}\,p^k\,(1-p)^{n-k}$ for $k = 0,1,\ldots,n$. Calculate the characteristic function $\mathsf{E}\big[e^{\mathrm{i}\theta B}\big]$.

(c) For $n \in \mathbb{N}$, let $B_n \sim \mathrm{Bin}(n,p_n)$, and assume $np_n \to \lambda$ as $n \to \infty$. Calculate the limit $\lim_{n\to\infty} \mathsf{E}\big[e^{\mathrm{i}\theta B_n}\big]$.

Exercise 6. Let $F_n \colon \mathbb{R} \to [0,1]$ be cumulative distribution functions, for $n \in \mathbb{N}$.

(a) Prove that the sequence $(F_n)_{n\in\mathbb{N}}$ has a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ which converges pointwise at all rational points, i.e., there exists a function $G\colon \mathbb{Q} \to [0,1]$ s.t.

$$\forall q \in \mathbb{Q} \qquad \lim_{k\to\infty} F_{n_k}(q) = G(q).$$

Prove also that if $q, q' \in \mathbb{Q}$ and $q < q'$, then we have $G(q) \leq G(q')$.

Assume now that the collection of c.d.f.'s is *tight*, i.e., for all $\varepsilon > 0$ there exists $R > 0$ such that for all n we have $F_n(-R) \leq \varepsilon$ and $F_n(R) \geq 1-\varepsilon$.

(b) Let $G\colon \mathbb{Q} \to [0,1]$ be as in part (a). Prove that

$$\lim_{q\to-\infty} G(q) = 0 \qquad \text{and} \qquad \lim_{q\to+\infty} G(q) = 1.$$

(c) Let $G\colon \mathbb{Q} \to [0,1]$ be as before. Define $F\colon \mathbb{R} \to [0,1]$ by

$$F(x) := \inf\Big\{ G(q) \;\Big|\; q \in \mathbb{Q} \cap (x,+\infty) \Big\}.$$

Prove that F is a cumulative distribution function.

(d) Let $F\colon \mathbb{R} \to [0,1]$ be as in part (c). Show that we have

$$\lim_{k\to\infty} F_{n_k}(x) = F(x) \qquad \text{for all continuity points } x \text{ of } F.$$

[1] The notation $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$, i.e., the largest $n \in \mathbb{Z}$ satisfying $n \leq x$.

[2] A function $F\colon \mathbb{R} \to [0,1]$ is a cumulative distribution function iff it is non-decreasing, right-continuous, and satisfies $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$.