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Exercise set 3
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MS-E1602 Large Random Systems, 2020-2021/IV

Exercise session: Wed 17.3. at 14-16 Solutions due: Mon 22.3. at 10

Note: *In the formulation of Exercises 2 and 5, we use the Landau notation o and $\mathcal{O}$ for asymptotic behavior. If g is a positive function, and f is any function defined on the same set, then we write*

$$f(x) = o\Big(g(x)\Big) \quad \text{as } x \to x_0, \qquad \text{if } \left|\frac{f(x)}{g(x)}\right| \to 0 \text{ as } x \to x_0$$

$$f(x) = \mathcal{O}\Big(g(x)\Big), \qquad \text{if for some constant } C > 0 \text{ we have } \left|\frac{f(x)}{g(x)}\right| \le C.$$

We also write $f_1(x) = f_2(x) + o(g(x))$ to mean $f_1(x) - f_2(x) = o(g(x))$, etc. As an example that you may use in the solution to Excercise 2 below, the Stirling approximation of n-factorial states that

$$n! = n^n e^{-n} \sqrt{2\pi n}\Big(1 + \mathcal{O}(n^{-1})\Big).$$

Exercise 1. Recall that $P \sim \text{Poisson}(\lambda)$ if $\mathsf{P}\big[P = k\big] = \frac{1}{k!} e^{-\lambda} \lambda^k$ for all $k \in \mathbb{Z}_{\ge 0}$.

(a) Let P_1, P_2 be two independent Poisson distributed random variables, $P_1 \sim \text{Poisson}(\lambda_1)$ and $P_2 \sim \text{Poisson}(\lambda_2)$. Show that $P_1 + P_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

(b) Let $(X_j)_{j \in \mathbb{N}}$ be independent, $X_j \sim \text{Poisson}(1)$, and $S_n = \sum_{j=1}^n X_j$. Show that, when $a > 1$, the following rate of large deviations holds

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathsf{P}\left[\frac{S_n}{n} \ge a\right] = -a + 1 + a \log a.$$

Interpretation: *For $a > 1$, the probability of $\{S_n \ge na\}$ is exponentially small in n, approximately given by $\mathsf{P}[S_n \ge na] \approx e^{-nJ(a)}$, where $J(a)$ is the limit in part (b).*

Exercise 2. This exercise concerns the Cramér entropy, and a related rate of large deviations for the simple random walk.

(a) Let $x \in (-1, 1)$, and suppose a sequence $(k_n)_{n \in \mathbb{N}}$ of integers k_n satisfies $k_n = \frac{n}{2}(1 + x) + \mathcal{O}(1)$. Using the Stirling approximation, show that

$$\log \binom{n}{k_n} = n\Big(\log(2) - I(x)\Big) + \mathcal{O}(\log(n)),$$

where $I(x)$ is the Cramér entropy

$$I(x) = \frac{1+x}{2} \log(1 + x) + \frac{1-x}{2} \log(1 - x).$$

(b) Consider the simple random walk, $X_n = \sum_{s=1}^n \xi_s$, where $(\xi_s)_{s \in \mathbb{N}}$ are i.i.d. steps with $\mathsf{P}\big[\xi_s = \pm 1\big] = \frac{1}{2}$. Show that for $0 < a < 1$ we have the following rate of large deviations

$$-\lim_{m\to\infty} \frac{1}{2m} \log \mathsf{P}\left[\frac{X_{2m}}{2m} \ge a\right] = I(a).$$

Exercise 3. For the Curie–Weiss model, the Helmholtz free energy reads

$$g(\beta, m) = \frac{1}{\beta}\Big(\frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m) - \log(2)\Big) - m^2.$$

(a) For a fixed $\beta > \frac{1}{2}$, let $\bar{m} = \bar{m}(\beta)$ be the unique positive solution of $\frac{\partial}{\partial m} g(\beta, m) = 0$. Calculate $\lim_{\beta \searrow \frac{1}{2}} \frac{\bar{m}(\beta)}{(\beta - \frac{1}{2})^{1/2}}$.

(b) For fixed $B > 0$ and $\beta > 0$, let $\tilde{m} = \tilde{m}(\beta, B)$ be the unique positive solution of $\frac{\partial}{\partial m}\big(g(\beta, m) - Bm\big) = 0$. Calculate $\lim_{B \searrow 0} \frac{\tilde{m}(\frac{1}{2}, B)}{B^{1/3}}$.

Note: *The results of (a) and (b) establish two critical exponents for the Curie–Weiss model*

Exercise 4. The total variation metric ϱ_{TV} on the set of probability measures on $\mathbb{R}$ is defined by $\varrho_{\mathrm{TV}}(\mu, \nu) = \sup_{E \in \mathscr{B}} \big|\mu[E] - \nu[E]\big|$, where $\mathscr{B}$ is the collection of Borel subsets of $\mathbb{R}$. Find a sequence $(\mu_n)_{n \in \mathbb{N}}$ of probability measures on $\mathbb{R}$ which converges weakly, but which is not a Cauchy sequence with respect to ϱ_{TV}.

Exercise 5. Let $(\mathfrak{X}, \varrho)$ be a metric space. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence of points $x_n \in \mathfrak{X}$, which satisfies two conditions:

1°) any subsequence $\big(x_{n_k}\big)_{k \in \mathbb{N}}$ has a further subsequence $\big(x_{n_{k_j}}\big)_{j \in \mathbb{N}}$ that converges

2°) the limits of any two convergent subsequences of $(x_n)_{n \in \mathbb{N}}$ are the same.

Prove that the sequence (x_n) converges.

Attention: *Recall arguments from the lectures, which used this logic. Pay attention to more such arguments in the rest of the course.*

Exercise 6. Let $(\mathfrak{X}, \varrho)$ be a metric space, and let ν_1, ν_2 be two Borel probability measures on $\mathfrak{X}$. Show that either of the following is a sufficient condition for $\nu_1 = \nu_2$:

(i) for all closed sets $F \subset \mathfrak{X}$ we have $\nu_1[F] = \nu_2[F]$

(ii) for all bounded continuous functions $f \colon \mathfrak{X} \to \mathbb{R}$ we have $\int_{\mathfrak{X}} f \, \mathrm{d}\nu_1 = \int_{\mathfrak{X}} f \, \mathrm{d}\nu_2$.

Note: *In particular, by (ii), the weak limit of a sequence of probability measures is unique if it exists.*