Aalto University
Department of Mathematics and Systems Analysis

Exercise set 4
K Kytölä & O Abuzaid

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Exercise session: Wed 24.3. at 14-16   Solutions due: Mon 29.3. at 10

In Exercises 1 and 2, let $(\mathfrak{X}, \varrho)$ be a metric space, $\mathscr{B}$ the Borel sigma algebra on $\mathfrak{X}$, and $\mathscr{M}_1(\mathfrak{X})$ the set of Borel probability measures on $\mathfrak{X}$. For $\mu, \nu \in \mathscr{M}_1(\mathfrak{X})$ define

$$d_{\mathrm{LP}}(\mu,\nu) = \inf\left\{\delta > 0 \;\middle|\; \forall E \in \mathscr{B}:\ \mu[E] \leq \nu[E^\delta] + \delta \text{ and } \nu[E] \leq \mu[E^\delta] + \delta\right\},$$

where $E^\delta = \left\{x \in \mathfrak{X} \,\middle|\, \varrho(x, E) < \delta\right\}$ is the (open) δ-thickening of E.

Recall that a coupling of probability measures $\mu, \nu \in \mathscr{M}_1(\mathfrak{X})$ is a probability measure λ on $\mathfrak{X} \times \mathfrak{X}$ such that for all $E \in \mathscr{B}$ we have $\lambda[E \times \mathfrak{X}] = \mu[E]$ and $\lambda[\mathfrak{X} \times E] = \nu[E]$.

Exercise 1.

(a) Show that d_{LP} is a metric on $\mathscr{M}_1(\mathfrak{X})$.
(b) Assume that there exists a coupling λ of μ and ν in which

$$\lambda\Big[\left\{(x_1,x_2) \in \mathfrak{X} \times \mathfrak{X} \,\middle|\, \varrho(x_1,x_2) \geq \varepsilon\right\}\Big] \leq \varepsilon.$$

Show that then $d_{\mathrm{LP}}(\mu,\nu) \leq \varepsilon$.

Interpretation: *If ε is very small in part (b), then in the coupling λ the two components X_1 and X_2 of $(X_1, X_2) \sim \lambda$ are very close with very high probability. The conclusion then says that then the measures μ and ν are very close.*

Exercise 2. Assume that $\nu_n \in \mathscr{M}_1(\mathfrak{X})$, $n \in \mathbb{N}$, are such that for some $\nu \in \mathscr{M}_1(\mathfrak{X})$ we have $d_{\mathrm{LP}}(\nu_n, \nu) \to 0$ as $n \to \infty$. Show that we then have $\nu_n \xrightarrow{\mathrm{w}} \nu$ as $n \to \infty$.
Hint: *Use criterion (iii) for weak convergence. Recall that the closure of a set $E \subset \mathfrak{X}$ can be expressed in terms of the δ-thickenings E^δ.*

Exercise 3. Let $n \in \mathbb{N}$, and let $C \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and let $m \in \mathbb{R}^n$ be a vector. Define a function $p : \mathbb{R}^n \to \mathbb{R}$ by

$$p(x) = \frac{1}{Z} \exp\left(-\frac{1}{2}(x-m)^\top C^{-1} (x-m)\right),$$

where Z is a constant.

(a) Calculate $\int_{\mathbb{R}^n} p(x)\,\mathrm{d}^n x$, and show that p is a (correctly normalized) probability density on $\mathbb{R}^n$ if $Z = (2\pi)^{n/2}\sqrt{\det(C)}$.
Hint: *Do first a change of variables (translation) to reduce to the case $m = 0$. Then do an orthogonal change of variables to a basis in which C is diagonal.*

(b) Choose Z as in part (a), and suppose that $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is a random vector in $\mathbb{R}^n$, which has probability density $p : \mathbb{R}^n \to \mathbb{R}$ as above. Calculate the characteristic function $\varphi(\theta) = \mathsf{E}\left[e^{\mathrm{i}\,\theta\cdot\xi}\right]$, for $\theta \in \mathbb{R}^n$ where $\theta \cdot \xi = \sum_{j=1}^n \theta_j \xi_j$ denotes the inner product.

(c) Let ξ be the random vector as in (b), and let $a_1, \ldots, a_n \in \mathbb{R}$. Show that the linear combination $\sum_{j=1}^n a_j \xi_j$ is a random number with Gaussian distribution.

Exercise 4. Suppose that $B = (B_t)_{t\geq 0}$ is a standard Brownian motion and define three other stochastic processes $W^{(k)} = (W_t^{(k)})_{t\geq 0}$, $k = 1,2,3$, by setting

$$W_t^{(1)} = B_{s+t} - B_s$$
$$W_t^{(2)} = \lambda^{-1/2} B_{\lambda t}$$
$$W_t^{(3)} = \begin{cases} tB_{1/t} & , \text{ when } t > 0 \\ 0 & , \text{ when } t = 0 \end{cases}$$

where $s \geq 0$ and $\lambda > 0$ are constants. Show that all these three stochastic processes $W^{(k)} = (W_t^{(k)})_{t\geq 0}$, $k = 1,2,3$, are also standard Brownian motions.

Exercise 5. Suppose that $X = (X_t)_{t\in[0,1]}$ is a continuous and Gaussian process, for which $\mathsf{E}[X_t] = 0$ for all $t \in [0,1]$ and $\mathsf{Cov}[X_s, X_t] = s(1-t)$ for all $0 \leq s \leq t \leq 1$. Let $Y \sim N(0,1)$ be a random variable independent of X. Define a stochastic process $W = (W_t)_{t\in[0,1]}$ by setting $W_t = X_t + tY$.

(a) Show that W has the same finite dimensional distributions as a standard Brownian motion on the time interval $[0,1]$.

The event $\{W_1 = y\}$ has zero probability, but it is natural to define conditioning on this event by the following limiting procedure. For any $\varepsilon > 0$, the event $\{|W_1 - y| < \varepsilon\}$ has positive probability, so we can consider the conditional distribution of the process W given the event $\{|W_1 - y| < \varepsilon\}$, and then take the (weak) limit of this conditional distribution as $\varepsilon \searrow 0$.

(b) Show that the conditional distribution of the process W given $\{W_1 = y\}$ is Gaussian, and calculate their mean and covariance functions.

Note: *Interpret the distribution of the process as the collection of its finite dimensional distributions, and conditioning as defined by the limiting procedure above.*

Exercise 6. Let $B = (B_t)_{t\in[0,\infty)}$ be a standard Brownian motion. For $t \geq e$ define

$$\lambda(t) = \sqrt{t \log(\log(t))}.$$

(a) Show that almost surely $\limsup_{n\to\infty} \frac{|B_n|}{\lambda(n)} = \sqrt{2}$, where the lim sup is taken along $n \in \mathbb{N}$.

Hint: *Observe that the restriction to Brownian motion to integer times, $(B_n)_{n\in\mathbb{N}}$, is a random walk with Gaussian steps. Recall the law of iterated logarithm proven earlier.*

(b) Find a function $\tilde{\lambda}\colon \mathbb{N} \to [0,\infty)$ with the following properties: $\frac{\tilde{\lambda}(n)}{\lambda(n)} \to 0$ as $n \to \infty$, and almost surely $\max_{s\in[n,n+1)} |B_s - B_n| \leq \tilde{\lambda}(n)$ except for finitely many values of $n \in \mathbb{N}$.

Hint: *You may use the fact that* $\mathsf{P}\big[\max_{s\in[0,1]} B_s > r\big] = \frac{2}{\sqrt{2\pi}} \int_r^\infty e^{-\frac{1}{2}v^2}\, \mathrm{d}v$ *for* $r \geq 0$.

(c) Show that almost surely we have (with lim sup taken along $t \in [0,\infty)$)

$$\limsup_{t\to\infty} \frac{|B_t|}{\lambda(t)} = \sqrt{2}.$$

(d) Show that almost surely

$$\limsup_{t\to 0} \frac{|B_t|}{\sqrt{t \log\big(|\log(1/t)|\big)}} = \sqrt{2}.$$

Hint: *Use (c) and the last part of Exercise 4 above.*