Aalto University
Department of Mathematics and Systems Analysis

Exercise set 5
K Kytölä & O Abuzaid

MS-E1602 Large Random Systems, 2020-2021/IV

Exercise session: Wed 31.3. at 14-16 **Solutions due: Mon 5.4. at 10**

In Exercise 1, the notations are as in Exercises 1-2 of problem set 4.

Exercise 1. Let $(\mathfrak{X}, \varrho)$ be a separable metric space, and d_{LP} the Lévy-Prokhorov metric on the set $\mathscr{M}_1(\mathfrak{X})$ of Borel probability measures on $\mathfrak{X}$. Assume that $(\nu_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{M}_1(\mathfrak{X})$ such that $\nu_n \xrightarrow{\mathrm{w}} \nu \in \mathscr{M}_1(\mathfrak{X})$ as $n \to \infty$. Let $\varepsilon > 0$.

(a) Show that there exists a countable dense set $\{x_i \mid i \in \mathbb{Z}_{>0}\} \subset \mathfrak{X}$ and a radius $r \in (\frac{\varepsilon}{4}, \frac{\varepsilon}{2})$ such that $\nu[\partial B_r(x_i)] = 0$ for all $i \in \mathbb{Z}_{>0}$.
(b) Show that there exists some $k \in \mathbb{Z}_{>0}$ such that $\nu\big[\bigcup_{i=1}^k B_r(x_i)\big] \geq 1 - \varepsilon$.
(c) Show that there exist finitely many disjoint sets $A_1, \ldots, A_k \subset \mathfrak{X}$ such that $\nu[\partial A_i] = 0$ and[1] $\mathrm{diam}(A_i) < \varepsilon$ for all $i = 1, \ldots, k$, and $\nu\big[\mathfrak{X} \setminus \bigcup_{i=1}^k A_i\big] \leq \varepsilon$.
(d) Define the collection $\mathscr{A} = \big\{\bigcup_{i\in I} A_i \;\big|\; I \subset \{1, \ldots, k\}\big\}$ of subsets of $\mathfrak{X}$. Show that for any $A \in \mathscr{A}$ we have $\nu_n[A] \to \nu[A]$ as $n \to \infty$. Conclude that there exists $N > 0$ such that $\big|\nu_n[A] - \nu[A]\big| \leq \varepsilon$ for all $A \in \mathscr{A}$ and $n \geq N$.
(e) Let $E \subset \mathfrak{X}$ be a Borel set. Choose $A \in \mathscr{A}$ as $A = \bigcup_{i\in I_E} A_i$, where $I_E = \big\{i \;\big|\; A_i \cap E \neq \emptyset\big\}$. Show that $A \subset E^\varepsilon$, and that $E \subset A \cup E'$, where $\nu[E'] \leq \varepsilon$ and $\nu_n[E'] \leq 2\varepsilon$ for $n \geq N$.
(f) Show that for $n \geq N$ we have $\nu_n[E] \leq \nu[E^\varepsilon] + 3\varepsilon$ and $\nu[E] \leq \nu_n[E^\varepsilon] + 2\varepsilon$.
(g) Conclude that $d_{\mathrm{LP}}(\nu_n, \nu) \to 0$ as $n \to \infty$.

Exercise 2. Let S be a finite or countably infinite set (equipped with the discrete topology). Define $\varrho \colon S^{\mathbb{N}} \times S^{\mathbb{N}} \to [0, \infty)$ by the formula[2]

$$\varrho(\omega, \omega') = \sum_{\substack{i\in\mathbb{N} \\ \omega_i \neq \omega'_i}} 2^{-i} \qquad \text{for } \omega, \omega' \in S^{\mathbb{N}},\ \omega = (\omega_i)_{i\in\mathbb{N}},\ \omega' = (\omega'_i)_{i\in\mathbb{N}}.$$

(a) Show that ϱ is a metric on the set $S^{\mathbb{N}}$.
(b) Show that the projections $\pi_i \colon S^{\mathbb{N}} \to S$ defined by $\pi_i(\omega) = \omega_i$, are continuous.
(c) Show that $(S^{\mathbb{N}}, \varrho)$ is complete.
(d) Show that $(S^{\mathbb{N}}, \varrho)$ is separable.

Exercise 3. Let $\beta > 0$ and $B \in \mathbb{R}$. Consider the Ising model on a finite graph $\mathscr{G}$, i.e., the measure given by $\mathsf{P}_{\beta,B}[\{\sigma\}] = \frac{1}{Z(\beta,B)} e^{-\beta H_B(\sigma)}$ for all $\sigma \in \{-1, +1\}^{\mathscr{G}}$, where $H_B(\sigma) = -\sum \sigma_x \sigma_y - B \sum \sigma_x$ (first sum over bonds $\{x, y\}$ of the graph, second over sites x) and $Z(\beta, B) = \sum e^{-\beta H_B(\sigma)}$ (sum over spin configurations σ). Define the average magnetization as

$$M = \mathsf{E}_{\beta,B}\Big[\frac{1}{\#\mathscr{G}} \sum_{x\in\mathscr{G}} \sigma_x\Big].$$

Show that $M = \frac{\partial}{\partial B}\Big(\frac{1}{\beta\,\#\mathscr{G}} \log Z(\beta, B)\Big)$.

[1] The diameter of a set $E \subset \mathfrak{X}$ is $\mathrm{diam}(E) = \sup\big\{\varrho(x, y) \;\big|\; x, y \in E\big\}$.

[2] The notation $S^{\mathbb{N}}$ means the set of all functions $\mathbb{N} \to S$, i.e., of all sequences $\omega = (\omega_i)_{i\in\mathbb{N}}$ such that $\omega_i \in S \ \forall i \in \mathbb{N}$.

Exercise 4. Let F, F_1 and F_2 be three cumulative distribution functions.

(a) Define $G\colon (0,1) \to \mathbb{R}$ by
$$G(u) = \inf \left\{ x \in \mathbb{R} \;\middle|\; F(x) \geq u \right\}.$$
Let $U \sim \mathrm{Unif}([0,1])$ be a uniform random variable on the unit interval. Show that the cumulative distribution function of the random variable $G(U)$ is F.

(b) Assume that $F_1(x) \geq F_2(x)$ for all $x \in \mathbb{R}$. Show that there exists a distribution ν on the plane $\mathbb{R}^2$, such that if $(X_1, X_2) \sim \nu$, then we have $X_1 \leq X_2$ almost surely, and the c.d.f.'s of the components X_1 and X_2 are F_1 and F_2, respectively.

Exercise 5. Consider a finite graph with set of sites $\mathscr{G}$ and set of bonds $\mathrm{E}(\mathscr{G})$. Consider percolation with parameter p on the graph $\mathscr{G}$, i.e., the probability measure P_p on $\{0,1\}^{\mathrm{E}(\mathscr{G})}$ in which the components are independent and Bernoulli(p) distributed. We define as usual for two sites $x, y \in \mathscr{G}$ the event $\{x \leftrightsquigarrow y\}$ that there exists an open path from x to y in the graph (see Lecture 2 for details). We also say that a function $f\colon \{0,1\}^{\mathrm{E}(\mathscr{G})} \to \mathbb{R}$ is increasing if $\omega \preceq \omega'$ implies $f(\omega') \leq f(\omega)$ (see Lectures 9-10 for details).

(a) Show that the FKG inequality
$$\mathsf{E}_p[f\, g] \;\geq\; \mathsf{E}_p[f]\, \mathsf{E}_p[g]$$
holds for all increasing functions $f, g\colon \{0,1\}^{\mathrm{E}(\mathscr{G})} \to \mathbb{R}$.

(b) Let $x, y, z, w \in \mathscr{G}$ be sites. Show that the events $\{x \leftrightsquigarrow y\}$ and $\{z \leftrightsquigarrow w\}$ are positively correlated:
$$\mathsf{P}_p\big[x \leftrightsquigarrow y \text{ and } z \leftrightsquigarrow w\big] \;\geq\; \mathsf{P}_p\big[x \leftrightsquigarrow y\big]\, \mathsf{P}_p\big[z \leftrightsquigarrow w\big].$$

Exercise 6. Consider the Ising model on subgraphs of the square grid $\mathbb{Z}^2$ at zero external magnetic field $B = 0$ and at inverse temperature $\beta > 0$. Denote also $\alpha = e^{-2\beta} \in (0,1)$. The goal of this exercise is to show that in low temperatures (large $\beta > 0$, small $\alpha > 0$) there is a ferromagnetic phase for the model.

Let $(\mathscr{G}_n)_{n \in \mathbb{N}}$ be a sequence of finite induced subgraphs of $\mathbb{Z}^2$ such that $\mathscr{G}_n \uparrow \mathbb{Z}^2$ as $n \to \infty$ (i.e. $\bigcup_{n\in\mathbb{N}} \mathscr{G}_n = \mathbb{Z}^2$) — the precise choice does not matter, you may take $\mathscr{G}_n$ to be e.g. the box $\{-n, -n+1, \ldots, n-1, n\} \times \{-n, -n+1, \ldots, n-1, n\}$. Let P_n^+ denote the Ising probability measure on $\mathscr{G}_n$ with plus boundary conditions.

The dual graph of $\mathbb{Z}^2$ is the graph with vertex set $\mathrm{V}^* = (\mathbb{Z}+\frac{1}{2}) \times (\mathbb{Z}+\frac{1}{2})$ (interpreted as the centers of the plaquettes of the square grid) and nearest neighbor bonds E^*. Note that for each dual bond $\{p,q\} \in \mathrm{E}^*$, there is a unique bond $\{x,y\}$ of the square grid with the same midpoint $\frac{p+q}{2} = \frac{x+y}{2}$ (the bonds "cross each other"). To a spin configuration $\sigma \in \Omega = \{-1,+1\}^{\mathbb{Z}^2}$ associate a subset $\Gamma(\sigma) \subset \mathrm{E}^*$ of dual bonds by the rule that $\{p,q\} \in \Gamma(\sigma)$ if and only if $\sigma_x \neq \sigma_y$ for the corresponding $\{x,y\}$.

(a) Show that $\mathsf{P}_n^+[\{\sigma\}] = \frac{1}{z(\mathscr{G}_n)}\, \alpha^{\#\Gamma(\sigma)}$ for $\sigma \in \Omega_{\mathscr{G}_n} \subset \Omega$, where $z(\mathscr{G}_n) \geq 1$ is a normalization constant and $\#\Gamma$ denotes the number of dual bonds in $\Gamma \subset \mathrm{E}^*$.

(b) Prove that $\mathsf{P}_n^+[\sigma_0 \neq +1] \;\leq\; \sum_{\ell=4}^{\infty} \ell\, (4\alpha)^{\ell}$.

Hint: *Consider dual graph circuits surrounding the origin. Use (a) with careful rearrangements.*

(c) Prove that there exists a $\beta' > 0$ such that for $\beta \geq \beta'$ there exists a $m(\beta) > 0$ such that $\mathsf{E}_n^+[\sigma_0] \geq m(\beta)$ for all $n \in \mathbb{N}$. Conclude that for $\beta \geq \beta'$ the weak limit P of the P_n^+ has positive magnetization $\mathsf{E}[\sigma_0] > 0$.