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Exercise set 6
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MS-E1602 Large Random Systems, 2020-2021/IV

Exercise session: Wed 7.4. at 14-16   Solutions due: Mon 12.4. at 10

Exercise 1. Fill in the online course feedback questionnaire.

Hint: *You should receive the link to the questionnaire by email.*

Exercise 2. Consider the standard Brownian motion $B = (B_t)_{t\in[0,\infty)}$. The Brownian motion started from $x \in \mathbb{R}$ is the process defined by $B_t^{(x)} = x + B_t$. The generator G of the Brownian motion is the following operator. For a smooth and compactly supported function $f \colon \mathbb{R} \to \mathbb{R}$, set

$$Gf(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathsf{E}\Big[f(B_t^{(x)})\Big].$$

Show that

$$Gf(x) = \frac{1}{2} f''(x).$$

Hint: *Recall the distribution of B_t for a given $t > 0$. Perform a Taylor expansion of f at x to the second order. Control the error terms when this Taylor approximation is used in the defining formula of $Gf(x)$.*

Exercise 3. Let $\mathcal{G} = (\mathrm{V}, \mathrm{E})$ be a finite graph. Consider the Ising model on $\mathcal{G}$, i.e., the probability measure on $\Omega_{\mathcal{G}} = \{-1,+1\}^{\mathrm{V}}$ given by

$$\mathsf{P}\left[\{\sigma\}\right] = \frac{e^{-\beta H(\sigma)}}{Z(\beta, B)}, \qquad \text{where } H(\sigma) = -\sum_{\{x,y\}\in\mathrm{E}} \sigma_x \sigma_y - B \sum_{x\in\mathrm{V}} \sigma_x.$$

For $\tau \in \{-1,+1\}^{\mathrm{V}}$, $x \in \mathrm{V}$, and $\epsilon \in \{-1,+1\}$, denote

$$c_\epsilon^{(x)}(\tau) := \Big(1 + \exp\big(-2\epsilon\beta B - 2\epsilon\beta \sum_{y:\{x,y\}\in\mathrm{E}} \tau_y\big)\Big)^{-1}.$$

(a) Let $x \in \mathrm{V}$ and $\tau \in \{-1,+1\}^{\mathrm{V}}$. Show that the conditional distribution of σ_x given that σ coincides with τ outside x is given by

$$\mathsf{P}\Big[\sigma_x = \epsilon \;\Big|\; \sigma_y = \tau_y \;\forall y \neq x\Big] = c_\epsilon^{(x)}(\tau).$$

(b) Let $X = (X_t)_{t\in[0,\infty)}$ be a continuous time Markov process on the state space $\{-1,+1\}^{\mathrm{V}}$ with jump rates

$$\lambda(\sigma,\tau) = \begin{cases} c_{\tau_x}^{(x)}(\sigma) & \text{if } \exists x \in \mathrm{V} \text{ s.t. } \tau_x \neq \sigma_x \text{ and } \tau_y = \sigma_y \;\forall y \neq x \\ 0 & \text{if } \#\big\{x \in \mathrm{V} \;\big|\; \sigma_x \neq \tau_x\big\} \neq 1 \end{cases}$$

Show that the Ising model probability measure P is the unique stationary measure of the process X.

Exercise 4. Fix parameters $L, N \in \mathbb{N}$ with $N \leq L$. Let $\mathcal{G}_L = (\mathrm{V}_L, \vec{\mathrm{E}}_L)$ be the directed cycle graph with the set of sites $\mathrm{V}_L = \{1, \ldots, L\}$ and the set of directed links $\vec{\mathrm{E}}_L = \big\{(x,y) \,\big|\, x, y \in \mathrm{V}_L,\ y - x \equiv 1 \pmod{L}\big\}$. We denote $x \curvearrowright y$, if $(x,y) \in \vec{\mathrm{E}}_L$. The totally asymmetric simple exclusion process (TASEP) on $\mathcal{G}_L$ with N particles and activation rate $v > 0$ is a continuous time Markov process $X = (X_t)_{t\in[0,\infty)}$ with

$$\text{state space} \qquad \mathcal{S}_L^{(N)} = \Big\{ Y \subset \mathrm{V}_L \,\Big|\, \#Y = N \Big\}$$

$$\text{and jump rates} \qquad \lambda(Y, Y') = \begin{cases} v & \text{if } Y' = (Y \setminus \{x\}) \cup \{y\} \text{ for some } x \curvearrowright y \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the uniform distribution $\mu_{\text{unif.}}$ on $\mathcal{S}_L^{(N)}$ is the unique stationary distribution for the process $(X_t)_{t\geq 0}$.
(b) Define the average speed s in the stationary distribution as

$$s = \lim_{\varepsilon \searrow 0} \frac{\mu_{\text{unif.}}\big[X_\varepsilon \neq X_0\big]}{\varepsilon\, N}.$$

Calculate s.
(c) Calculate s in the limit as $L \to \infty$, $\frac{N}{L} \to \rho \in (0,1)$. What is the optimal value of the density ρ for maximum speed s? What is the optimal value of ρ for maximum traffic flow sN (optimality asymptotically as $L \to \infty$)?

The last two exercises concern the totally asymmetric simple exclusion process (TASEP) on the integer lattice $\mathbb{Z}$, a process $\xi = (\xi_t)_{t\geq 0}$, which is constructed as follows. The state space $\mathcal{S}$ consists of all subsets $Y \subset \mathbb{Z}$, which we identify with $\mathcal{S} = \{0,1\}^{\mathbb{Z}}$ in such a way that Y corresponds to

$$\xi = \big(\xi(x)\big)_{x\in\mathbb{Z}} \qquad \text{with } \xi(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y. \end{cases}$$

Choose an initial configuration $\xi_0 = \big(\xi_0(x)\big)_{x\in\mathbb{Z}} \in \mathcal{S}$. For each $x \in \mathbb{Z}$, take an independent Poisson process with intensity $v > 0$, and denote its arrival times $(T_n^x)_{n\in\mathbb{N}}$. The rules to define $\xi_t \in \mathcal{S}$ for $t \geq 0$ are the following: for any $x \in \mathbb{Z}$

- $t \mapsto \xi_t(x)$ is continuous from the right, and constant on any time interval that does not contain any T_n^x or T_n^{x-1}
- at times $t = T_n^x$, $\xi_t(x)$ and $\xi_t(x+1)$ are determined in terms of the left limits:
  * if $\xi_{t-}(x) = 1$ and $\xi_{t-}(x+1) = 0$, then $\xi_t(x) = 0$ and $\xi_t(x+1) = 1$
  * otherwise, $\xi_t(x) = \xi_{t-}(x)$ and $\xi_t(x+1) = \xi_{t-}(x+1)$.

Exercise 5. Show that the process $(\xi_t)_{t\geq 0}$ (the TASEP on $\mathbb{Z}$) becomes (almost surely) well defined by the rules given above.
Hint: *Show that (almost surely) for any $x \in \mathbb{Z}$ and $t \geq 0$ there are only finitely many Poisson process arrivals that could affect $\xi_t(x)$ according to the rules.*

Exercise 6. Let $\rho \in (0,1)$. Suppose that the initial state ξ_0 of the TASEP on $\mathbb{Z}$ is taken random and independent of the Poisson processes, so that its coordinates $\xi_0(x)$, $x \in \mathbb{Z}$, are independent and $\mathsf{P}\big[\xi_0(x) = 1\big] = \rho$ for each x. Show that for any $t \geq 0$ also the coordinates $\xi_t(x)$, $x \in \mathbb{Z}$, are independent and $\mathsf{P}\big[\xi_t(x) = 1\big] = \rho$.
Interpretation: *In other words, the product of Bernoulli measures $\mu = \bigotimes_{x\in\mathbb{Z}} \mathrm{Bernoulli}(\rho)$ is a stationary measure for the TASEP on $\mathbb{Z}$ — for any $\rho \in (0,1)$.*