Aalto University
School of Science
and Technology

Large Random Systems (MS-E1602)

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Course practicalities

- MyCourses web page:
  `https://mycourses.aalto.fi/course/view.php?id=29650`
- Zulip forum:
  `https://ms-e1602.zulip.cs.aalto.fi`
- Lectures: (Kalle Kytölä)
  - Mon 10-12, Wed 10-12
- Exercises: (Osama Abuzaid)
  - Wed 14-16
  - weekly exercise sets
  - turn in written solutions by following Monday at 10am
- Exam dates:
  - TBA
- Credits: 5 cr
- Prerequisites:
  - measure theoretic probability!
  - a little bit of calculus, linear algebra, combinatorics, topology, . . .
- Material:
  - lecture notes
- Grading:
  - exercises ($\approx$ 100%)
  - oral exam? ($\pm$15%)
- Your feedback:
  - any time!
  - online questionnaire

Course grading

~~Written exam~~ (still an option, but...)

~~3 hours, 4 problems à 6 points~~

Exercises: ~~up to 6 bonus points!~~

Turn in written solutions to weekly problem sets via MyCourses Assignments. (Mondays at 10:00 am)

Grading based on exercises + oral exam adjustment

The solutions to weekly problem sets are graded, and course grade is based primarily on these.

Honor code:

- You may discuss problems with others (incl. TA), but everyone must write their own solutions independently.
- During the week 12.-16.4. Zoom meetings to explain selected solutions + random topic to the lecturer and/or TA. $\rightsquigarrow$ $\pm 15\%$

Course contents / topics

Mathematical theory

- 0-1 laws, phase transitions
- Large deviations
- Couplings and monotonicity
- **Convergence in distribution (= weak convergence)**
- Tightness

Models and examples

- **Random walk, Brownian motion**
- Percolation
- Totally asymmetric simple exclusion process
- **Curie-Weiss and Ising models**
- Voter model and contact process

Course contents / approximate schedule

Week 1

I Introduction

II 0-1 laws: phase transition in percolation

Week 2

III 0-1 laws: law of iterated logarithm for random walk

IV Weak convergence on $\mathbb{R}$

Week 3

V Curie-Weiss model of ferromagnetism and large deviations

VI weak convergence on metric spaces

Week 4

VII Weak convergence: tightness and Prokhorov's theorem

VIII Donsker's theorem: random walk $\rightarrow$ Brownian motion

Week 5

IX Correlation inequalities for the Ising model

X Thermodynamic limit of the Ising model

Week 6

XI Interacting particle systems: existence of dynamics

XII Voter model (contact process, TASEP)

Objectives

Upon successful completion of the course you will…

- … have gained familiarity with a number of prominent probabilistic models of various phenomena;
- … be able to analyze the models mathematically rigorously, using probability theory in combination with a broad range of mathematics;
- … be able to select methods that are appropriate for the analysis of the models;
- … know and be able to utilize the fundamental results about weak convergence of probability measures.

Large random systems?

Large?

e.g. "size parameter" n

limit $n \to \infty$

Random?

Probability spaces

$(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$

indexed by "size parameter" n.

- Ω sample space (outcomes)
- $\mathcal{F}$ sigma-algebra (events)
- P probability measure (probabilities)

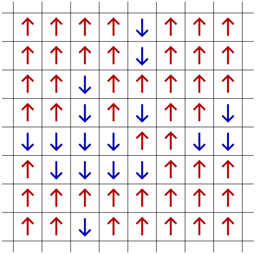
…and then…

$$\lim_{n\to\infty} \mathsf{P}_n[\cdots], \qquad \lim_{n\to\infty} \mathsf{E}_n[\cdots], \qquad \lim_{n\to\infty} \mathsf{P}_n?$$

Example 1: Ising model

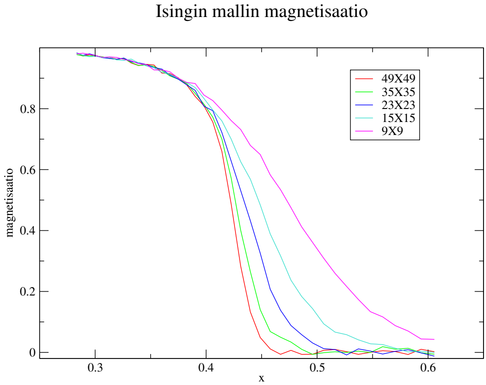
Ising model of a ferromagnet:

- $L \times L$ grid, each square magnetized up($\uparrow$) or down($\downarrow$)
- probability of configuration $\sim$ $x^{\text{number of disagreeing neighbors}}$
- low/high x $\leftrightarrow$ low/high temperature



Thermodynamic limit: $L \to \infty$

Example 1: Phase transition in a ferromagnet

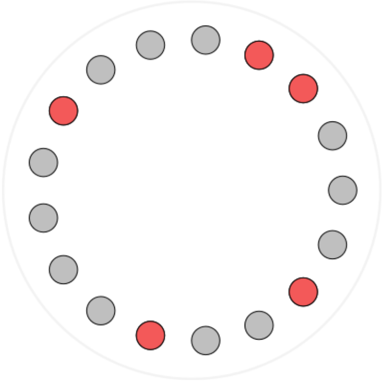


low temperature: $x < \sqrt{2} - 1$
ferromagnetic
$\lim_{L\to\infty}(\text{magn.}) \neq 0$

high temperature: $x > \sqrt{2} - 1$
paramagnetic
$\lim_{L\to\infty}(\text{magn.}) = 0$

Example 2: TASEP

Totally asymmetric simple exclusion process



- L sites, periodic chain
- N vehicles

- at Poissonian rate λ, vehicle attempts advance to next site
  - advance if next site empty
  - do nothing if next site occupied

Example 2: TASEP

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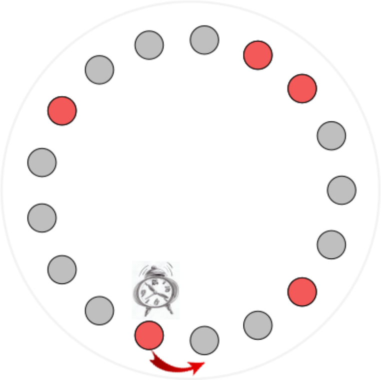

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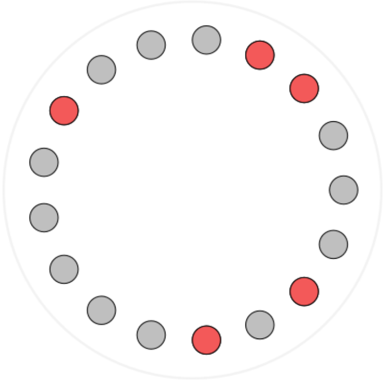

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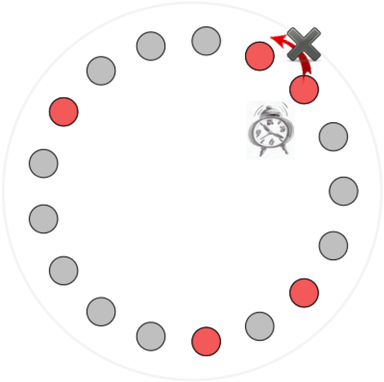

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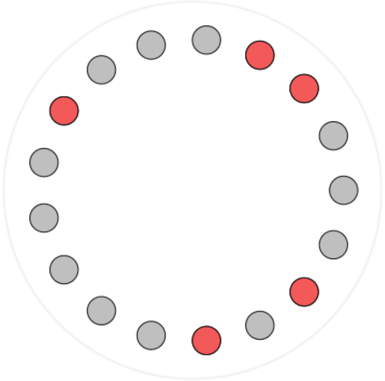

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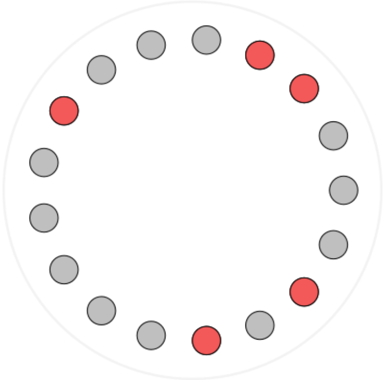

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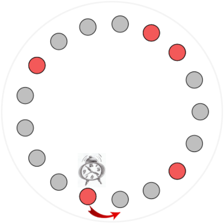
$$L \to \infty, N \to \infty, \frac{N}{L} \to \rho$$

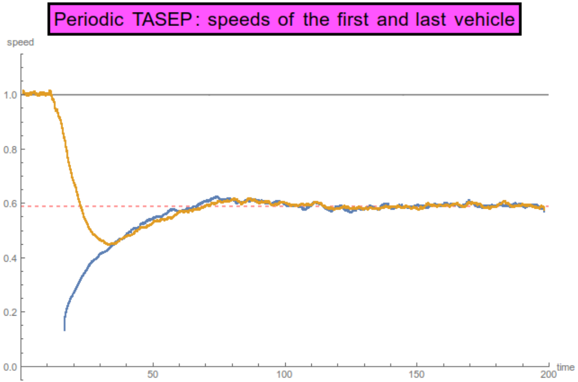
Example 2: behavior of TASEP

Totally asymmetric simple exclusion process

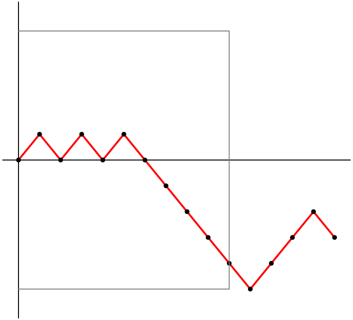


speed
$\to \frac{L-N}{L-1} \approx 1-\rho$

traffic
$\to \frac{L-N}{L-1}N \approx (1-\rho)\rho$



Example 3: Random walks and Brownian motion



Random walk $S_n = \sum_{j=1}^{n} \xi_j$

$(\xi_j)_{j=1}^{\infty}$ i.i.d., $\mathsf{P}\big[\xi_j = \pm 1\big] = \frac{1}{2}$

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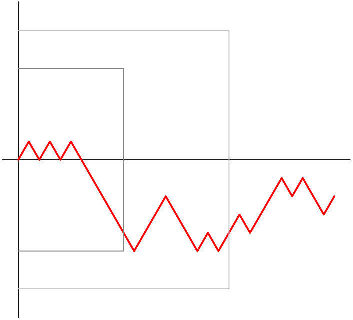

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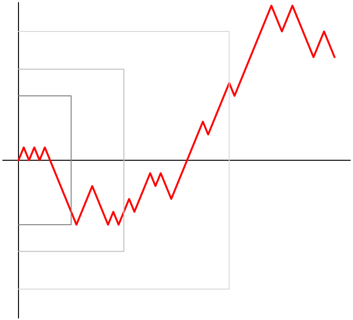

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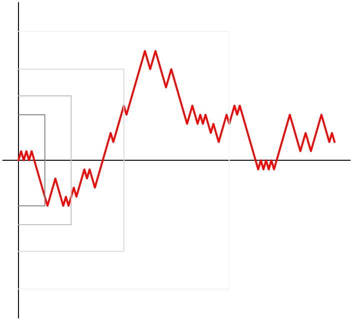

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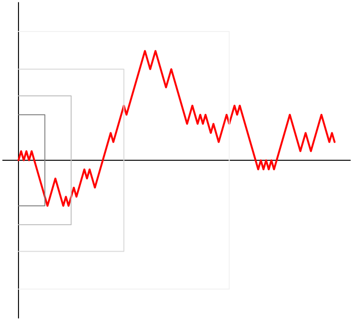

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Scaling $X_t^{(\delta)} \; = \; \sqrt{\delta}\, S_{t/\delta}$

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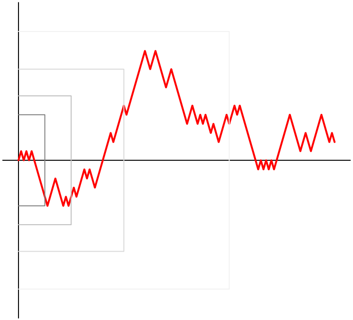

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Theorem [Donsker 1951]

As $\delta \to 0$, we have $\big(X_t^{(\delta)}\big)_{t\in[0,1]} \xrightarrow{\text{w}}$ Brownian motion $\big(B_t\big)_{t\in[0,1]}$.

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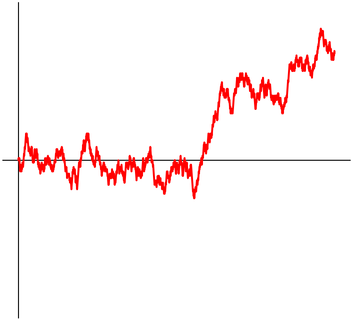

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Limit? Weak convergence of probability measures on $\Big(\mathcal{C}([0,1]), \|\cdot\|_\infty\Big)$

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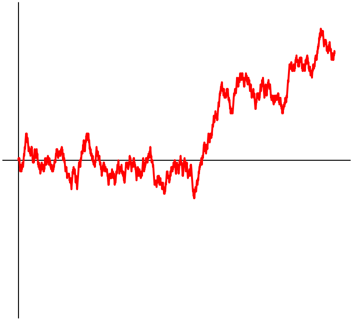

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With probability one:

$t \mapsto B_t$ nowhere differentiable,
$\dim\left\{(t, B_t) \;\middle|\; t \in [0,1]\right\} = \frac{3}{2}$,
$\left\{t \in [0,1] \;\middle|\; B_t = 0\right\}$ Cantor set, ...

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