Aalto University
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Utility theory: elicitation of utility functions, risk attitudes and stochastic dominance

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Presentation 3
25.9.2020

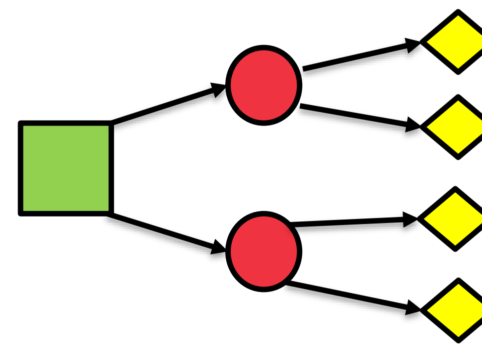
MS-E2191 Graduate Seminar on Operations Research
Fall 2020

Presentation content

- Recap of last week's topic.
- What is utility theory?
- Risk attitudes: risk prone, neutral, averse.
- Elicitation of utility functions(i.e. Getting data from DM to get utility function ”parameters”
- Stochastic dominance.
- References
- Home assignment

Decision trees and expected value

- Last weeks speakers Jussi Leppinen and Kalle Alaluusua presented **decision trees**.
- They solved the decision trees by calculating the expected value = EV for the tree's branches.
- The EV-method is viable if the decision maker = DM is **risk-neutral**.
- Example: St. Petersburg paradox.



Theory of expected utility

- The idea is to incorporate **DM's preference regarding risk** into the value of the possible outcomes.
- For this we introduce the utility function **u(a)**.
- Can be seen as a weight to the values of the outcomes, and can be used in decision trees instead of EV.

$$EU(a) = \sum_{i=1}^{n} p_i \cdot u(a_i)$$

Example: Expected value of utility

You have two lotteries to choose from:

A. 50-50, 0€ or 100 000€ (EV = 50 000€)

B. 50-50, 20 000€ or 50 000€ (EV = 35 000€)

Cash	0€	20 000€	50 000€	100 000€
Equivalent	Dirt	Good car	Awesome second-hand car!	Nice new car
Utility	0	0.4	0.8	1

E(u(A)) = 0.5, E(u(B)) = 0.6

How to define the DM's utility function?

- Unlike the previous slide's example, utility functions are rarely given.
- How to quantify the DM's preferences?

Basis for defining a utility function

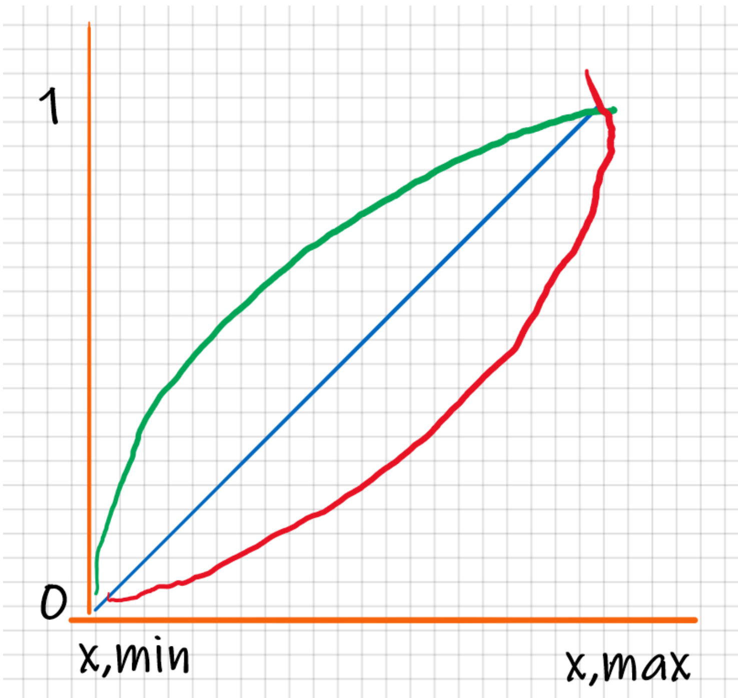
- According to the theory of utility: if DM's preference follows the axioms
  - Complete ordering
  - Continuity
  - Independence

  then there exists an utility function u(a), which describes this preference.

Further basic concepts

- **Certainty equivalent (CE)** = The safe consequence for which DM is indifferent of lottery or certain outcome. It follows that $u(CE(a)) = EU(a)$.
- **Risk attitude** = The DM’s relationship with risk.
- **Risk premium (RP)** = the difference $RP(a) = EV(a) – CE(a)$.

Risk attitude illustrated with u(x)



- Risk-neutral, risk-averse(concave), risk-prone(convex).
- **Arrow/Pratt measure**:

$$r(x) = -\frac{u''(x)}{u'(x)}$$

  - Quantifies DM's absolute risk attitude.
- Relative risk attitude:

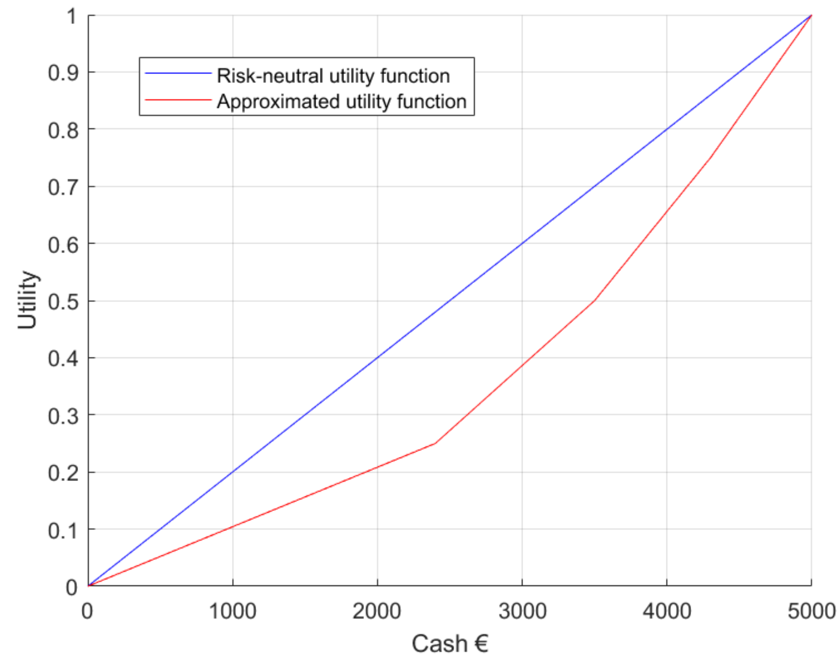
$$r^*(x) = -\frac{u''(x)}{u'(x)} \cdot x$$

Determining the utility function

- Determining the DM's utility function is done by evaluating simple alternatives.
- Most methods build upon the **basic-reference-lottery (BRL**), and the lottery's certainty equivalent (**CE***).
- The DM is presented with the BRL, for which the DM has to give a CE which he is willing to take instead of the lottery.
- Points are elicited from the DM after which an appropriate function is interpolated or a curve is fitted.

Example: Bisection version of CE-method

- The DM is asked to give CE's for simple 50-50 lotteries.
- Simple, generally easy and quick to process for DM, as situation is essentially a coin toss.
- Example slightly convex, so we could have a risk-prone DM.



Other methods for elicitation of utility function

- Quantile version of certain equivalent method
- Variable probability method
- Method of equal utility differences
- Trade-off method for utility functions

- Those interested in these methods may find more information in **Eisenführ et al.**, *Rational decision making (2010) p.264-268.*

Consistency checks

- It is essential to conduct consistency checks, as errors and distortions in the determination of utility functions is common.
- A good way is to use multiple methods for elicitation, and to look for inconsistencies.
- Eliciting can be done by computer programs with appropriate interfaces for consistency and ease of processing for the DM.

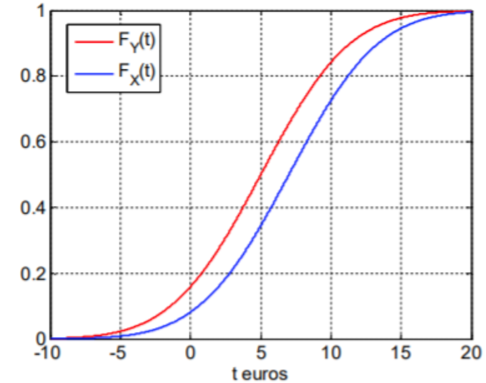
Stochastic dominance

- Stochastic dominance is in essence stochastic ordering: an option being “bigger” or “better” than another.
- Quantifying the “better” property can help in choosing the optimal alternative of many random variables.
- Useful when utility function is not known.

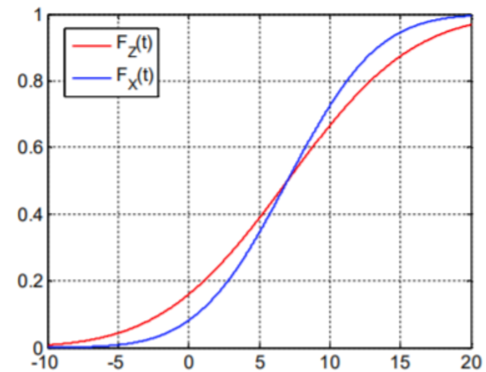
First degree stochastic dominance

$$P(A \geq x) \geq P(B \geq x)$$
for all x, and
$$P(A \geq x) > P(B \geq x)$$
for some x

- If alternative A first degree dominates B, the DM should choose A if he prefers more to less(since A always at least as good as B).



$X \succcurlyeq_{FSD} Y$



Neither alternative dominates the other in the sense of FSD

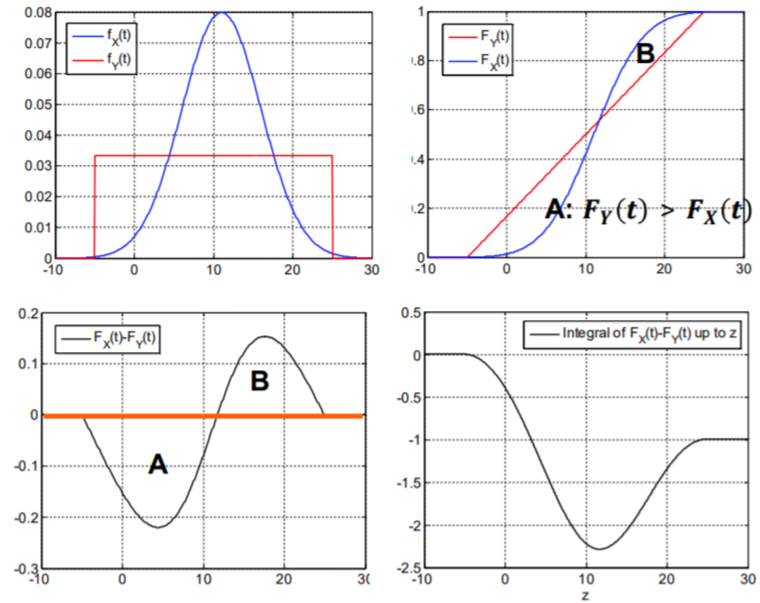
Liesiö et al., Decision making and problem solving, lecture 3

Second degree stochastic dominance

$$\int_{-\infty}^{x} \left(F_B(t) - F_A(t)\right)dt \geq 0$$

for all x

- "A has SSD over B if A is more predictable and has at least the same mean"
- Risk-averse DM should choose the second degree dominant alternative, since it is as "good" with less risk.



Liesiö et al., Decision making and problem solving, lecture 3

References

- Eisenführ, F., Weber, M., & Langer, T. (2010). *Rational Decision Making* (2010 ed.), p.235-275.
- https://en.wikipedia.org/wiki/Stochastic_dominance
- Liesiö et al, *Decision making and problem solving, lecture 3*

Homework assignment

Andy, Bea and Carlos are considering a group investment, and want to determine their preference on risk. With the bisection method, points on their utility functions have been determined in the table below.

- A: Plot the three investor's personal utility functions in the same figure. Fit an appropriate function for each investor. Determine for each investor whether they are risk-neutral, prone, or averse. (MatLab works well)

The group has many investment options to choose from, with varying risk.

- B: Assuming that the group chooses their investment according to their average preference, what kind of investment regarding risk will they choose? Who has the most influence on their choice (i.e. highest effect on average)? Which of the investors represents the group's average preference best?

Andy	0	710	1100	1630	2000
Bea	0	410	800	1300	2000
Carlos	0	1000	1400	1700	2000
u(x)	**0**	**0.25**	**0.5**	**0.75**	**1**