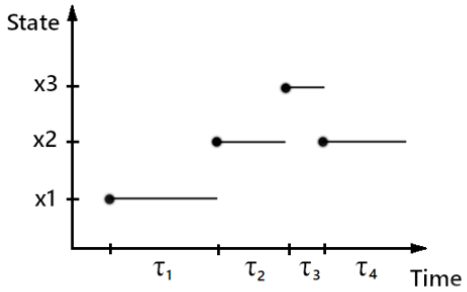
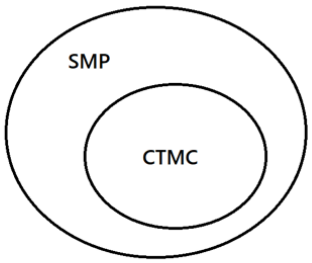
MS-E2191 - Graduate Seminar on Operations Research
Homework 23 model answer
Petri Määttä

1. What is the difference between a Semi-Markov Process and a Continuous-time Markov Chain?

A Continuous-time Markov Chain (CTMC) and a Semi-Markov Process (SMP) are both continuous-time stochastic processes with a discrete set of states, where the transition times between states are continuous random variables. For a CTMC, the transition times are exponentially distributed and as such the process is memoryless and satisfies the Markov property. (Memorylessness means that the further time spent in the state does not depend on the time already spent in the state.) For an SMP the transition times can be arbitrarily distributed and thus the process is not necessarily memoryless. A CTMC is therefore a kind of SMP. For both types of processes the sequence of states X_t (without considering time) is a Markov chain.





SMP: $\tau_i \sim$ any distribution

CTMC: $\tau_i \sim \exp(\lambda_i)$

2. Consider the (continuous-time) stochastic process of the above figure where the transition time T from state A to B (and B to A) has the probability density $f(t) = \frac{1}{\mu}$, $0 \leq t \leq \mu$ and 0 otherwise. Is the process memoryless? Prove your answer.

Let

- $T =$ the time spent in the state (random variable),
- $t =$ how much longer we will spend in the state,
- $s =$ how long we have already spent in the state.

Now

$$P(T > t) = \int_t^\mu \frac{1}{\mu} d\tau = 1 - \frac{t}{\mu}$$

and

$$\begin{aligned} P(T > s + t \mid T > s) &= \frac{P(T > s + t,\ T > s)}{P(T > s)} \\ &= \frac{P(T > s + t)}{P(T > s)} \\ &= \frac{1 - \frac{s+t}{\mu}}{1 - \frac{s}{\mu}} \\ &= 1 - \frac{t}{\mu - s} \\ &\neq P(T > t). \end{aligned}$$

Since the further time spent in the state depends on the time already spent in the state, the process is not memoryless.

Another approach

Since the expectation of a uniform distribution is the mean of the endpoints, we have that

- $\mathbf{E}[T \mid s = 0] = \frac{1}{2}\mu$
- $\mathbf{E}[T \mid s = \frac{\mu}{2}] = \frac{3}{4}\mu$

from which we see that the process has a memory.