Multirate Digital Signal Processing

Basic Sampling Rate Alteration Devices

- Up-sampler - Used to increase the sampling rate by an integer factor
- Down-sampler - Used to decrease the sampling rate by an integer factor

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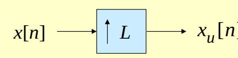
Up-Sampler

Time-Domain Characterization

- An up-sampler with an **up-sampling factor** L, where L is a positive integer, develops an output sequence $x_u[n]$ with a sampling rate that is L times larger than that of the input sequence $x[n]$
- Block-diagram representation



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Up-Sampler

- Up-sampling operation is implemented by inserting $L-1$ equidistant zero-valued samples between two consecutive samples of $x[n]$
- Input-output relation

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

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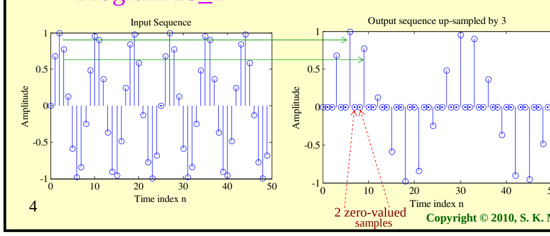
Up-Sampler

- Figure below shows the up-sampling by a factor of 3 of a sinusoidal sequence with a frequency of 0.12 Hz obtained using Program 13_1



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Up-Sampler

- In practice, the zero-valued samples inserted by the up-sampler are replaced with appropriate nonzero values using some type of filtering process
- Process is called **interpolation** and will be discussed later

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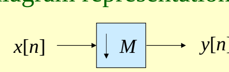
Down-Sampler

Time-Domain Characterization

- An down-sampler with a **down-sampling factor** M, where M is a positive integer, develops an output sequence $y[n]$ with a sampling rate that is $(1/M)$-th of that of the input sequence $x[n]$
- Block-diagram representation



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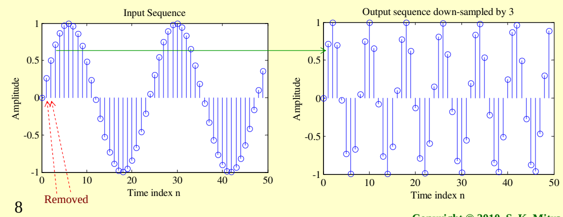
Down-Sampler

- Down-sampling operation is implemented by keeping every M-th sample of $x[n]$ and removing $M-1$ in-between samples to generate $y[n]$
- Input-output relation

$$y[n] = x[nM]$$

Down-Sampler

- Figure below shows the down-sampling by a factor of 3 of a sinusoidal sequence of frequency 0.042 Hz obtained using Program 13_2



Basic Sampling Rate Alteration Devices

- Sampling periods have not been explicitly shown in the block-diagram representations of the up-sampler and the down-sampler
- This is for simplicity and the fact that the mathematical theory of multirate systems can be understood without bringing the sampling period T or the sampling frequency F_T into the picture

Down-Sampler

- Figure below shows explicitly the input-output sampling rates of the down-sampler

$$x[n] = x_a(nT) \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] = x_a(nMT)$$

Input sampling frequency

$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = \frac{F_T}{M} = \frac{1}{T'}$$

Up-Sampler

- Figure below shows explicitly the input-output sampling rates of the up-sampler

$$x[n] = x_a(nT) \longrightarrow \boxed{\uparrow L} \longrightarrow y[n]$$

$$= \begin{cases} x_a(nT/L), & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Input sampling frequency

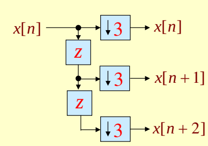
$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = LF_T = \frac{1}{T'}$$

Serial-to-Parallel Converter

- A bank of down-samplers can be used to convert a serial digital data into parallel form
- Consider the structure shown below



Serial-to-Parallel Converter

- The multirate structure shown on the previous slide converts an input sequence into a sequence vector of length 3
- The first 4 samples of the sequences $v_1[n]$, $v_2[n]$, and $v_3[n]$ are shown below:

$$\begin{array}{llllll} v_1[n]: & x[n] & x[n+3] & x[n+6] & x[n+9] & \cdots \\ v_2[n]: & x[n+1] & x[n+4] & x[n+7] & x[n+10] & \cdots \\ v_3[n]: & x[n+2] & x[n+5] & x[n+8] & x[n+11] & \cdots \end{array}$$

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Serial-to-Parallel Converter

- The multirate structure shown on the previous slide converts an input sequence into a sequence vector of length 3
- At time instant n, the vector generated is

$$\begin{bmatrix} x[n] \\ x[n+1] \\ x[n+2] \end{bmatrix}$$

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Serial-to-Parallel Converter

- The first output vector generated at the input time index n is

$$[x[n] \quad x[n+1] \quad x[n+2]]^T$$

- The next vector generated at the input time index n+3 is

$$[x[n+3] \quad x[n+4] \quad x[n+5]]^T$$

and so on

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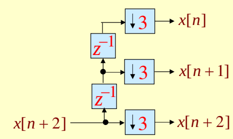
Serial-to-Parallel Converter

- A realizable form of the serial-to-parallel converter of slide 12 is shown below which has a system delay of 3 sample periods



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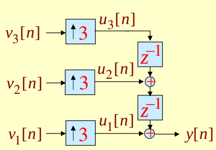
Parallel-to-Serial Converter

- A bank of up-samplers can be used to convert a vector of digital data into serial form
- Consider the structure shown below



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Parallel-to-Serial Converter

- The first 9 samples of the sequences $u_1[n]$, $u_2[n]$, $u_3[n]$, and $y[n]$ are shown below:

Sequence	n	n+1	n+2	n+3	n+4	n+5	n+6	n+7	n+8
$u_3[n]$:	$v_3[n]$	0	0	$v_3[n+1]$	0	0	$v_3[n+2]$	0	0
$u_2[n]$:	$v_2[n]$	0	0	$v_2[n+1]$	0	0	$v_2[n+2]$	0	0
$u_1[n]$:	$v_1[n]$	0	0	$v_1[n+1]$	0	0	$v_1[n+2]$	0	0
$u_3[n-2]$:	0	0	$v_3[n]$	0	0	$v_3[n+1]$	0	0	$v_3[n+2]$
$u_2[n-1]$:	0	$v_2[n]$	0	0	$v_2[n+1]$	0	0	$v_2[n+2]$	0
$y[n]$	$v_1[n]$	$v_2[n]$	$v_3[n]$	$v_1[n+1]$	$v_2[n+1]$	$v_3[n+1]$	$v_1[n+2]$	$v_2[n+2]$	$v_3[n+2]$

$$y[n] = v_1[n] + v_2[n-1] + v_3[n-2]$$

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Parallel-to-Serial Converter

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Parallel-to-Serial Converter

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Parallel-to-Serial Converter

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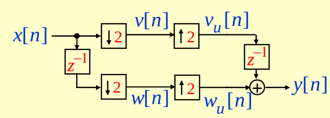
A Simple Multirate Structure



- The operation of the above multirate structure can be analyzed by writing down the relations between various signal variables, and the input as shown in the next slide

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A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$

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A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$

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A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$
$v_u[n]$:	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0	$x[8]$
$w_u[n]$:	$x[-1]$	0	$x[1]$	0	$x[3]$	0	$x[5]$	0	$x[7]$

A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$
$v_u[n]$:	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0	$x[8]$
$w_u[n]$:	$x[-1]$	0	$x[1]$	0	$x[3]$	0	$x[5]$	0	$x[7]$
$v_u[n-1]$:	0	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0
$y[n]$:	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$

$$y[n] = v_u[n-1] + w_u[n] = x[n-1]$$

Basic Sampling Rate Alteration Devices

- The up-sampler and the down-sampler are linear but time-varying discrete-time systems
- We illustrate the time-varying property of a down-sampler
- The time-varying property of an up-sampler can be proved in a similar manner

Basic Sampling Rate Alteration Devices

- Consider a factor-of-M down-sampler defined by $y[n] = x[nM]$
- Its output $y_1[n]$ for an input $x_1[n] = x[n - n_0]$ is then given by

$$y_1[n] = x_1[Mn] = x[Mn - n_0]$$

- From the input-output relation of the down-sampler we obtain

$$y[n - n_0] = x[M(n - n_0)]$$
$$= x[Mn - Mn_0] \neq y_1[n]$$

Up-Sampler

Frequency-Domain Characterization

- Consider first a factor-of-2 up-sampler whose input-output relation in the time-domain is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Up-Sampler

- In terms of the z-transform, the input-output relation is then given by

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-2m} = X(z^2)$$

Up-Sampler

- In a similar manner, we can show that for a factor-of-L up-sampler

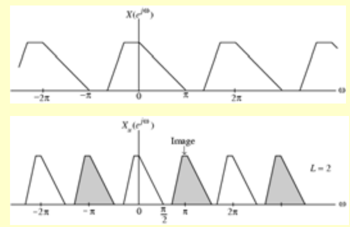
$$X_u(z) = X(z^L)$$

- On the unit circle, for $z = e^{j\omega}$, the input-output relation is given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

Up-Sampler

- Figure below shows the relation between $X(e^{j\omega})$ and $X_u(e^{j\omega})$ for $L = 2$ in the case of a typical sequence $x[n]$



Up-Sampler

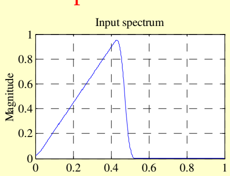
- As can be seen, a factor-of-2 sampling rate expansion leads to a compression of $X(e^{j\omega})$ by a factor of 2 and a 2-fold repetition in the baseband $[0, 2\pi]$
- This process is called imaging as we get an additional "**image**" of the input spectrum

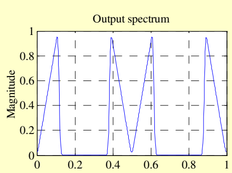
Up-Sampler

- Similarly in the case of a factor-of-L sampling rate expansion, there will be $L-1$ additional images of the input spectrum in the baseband
- Lowpass filtering of $x_u[n]$ removes the $L-1$ images and in effect "fills in" the zero-valued samples in $x_u[n]$ with interpolated sample values

Up-Sampler

- Program 13_3 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $L = 4$





Down-Sampler

Frequency-Domain Characterization

- Applying the z-transform to the input-output relation of a factor-of-M down-sampler

$$y[n] = x[Mn]$$

we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n}$$

- The expression on the right-hand side cannot be directly expressed in terms of $X(z)$

Down-Sampler

- To get around this problem, define a new sequence $x_{\text{int}}[n]$:

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Then

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn]z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_{\text{int}}[k]z^{-k/M} = X_{\text{int}}(z^{1/M})$$

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Down-Sampler

- Now, $x_{\text{int}}[n]$ can be formally related to $x[n]$ through

$$x_{\text{int}}[n] = c[n] \cdot x[n]$$

where

$$c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- A convenient representation of $c[n]$ is given by

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

where $W_M = e^{-j2\pi/M}$

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Down-Sampler

- Taking the *z*-transform of $x_{\text{int}}[n] = c[n] \cdot x[n]$ and making use of

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

we arrive at

$$X_{\text{int}}(z) = \sum_{n=-\infty}^{\infty} c[n]x[n]z^{-n} = \frac{1}{M} \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{M-1} W_M^{kn} \right) x[n]z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \left(\sum_{n=-\infty}^{\infty} x[n] W_M^{kn} z^{-n} \right) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z W_M^{-k}\right)$$

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Down-Sampler

- Hence,

$$Y(z) = X_{\text{int}}(z^{1/M})$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

- On the unit circle,

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

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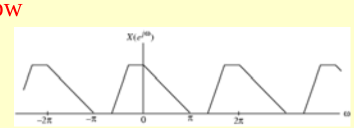
Down-Sampler

- Consider a factor-of-2 down-sampler with an input $x[n]$ whose spectrum is as shown below



- The DTFTs of the output and the input sequences of this down-sampler are then related as

$$Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\}$$

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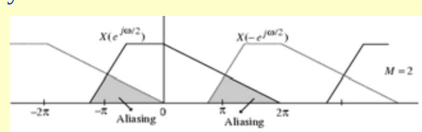
Down-Sampler

- Now $X(-e^{j\omega/2}) = X(e^{j(\omega - 2\pi)/2})$ implying that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below



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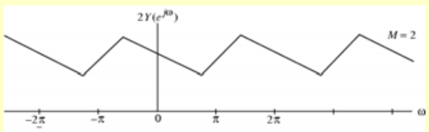
Down-Sampler

- The plots of the two terms have an overlap, and hence, in general, the original "shape" of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below



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Down-Sampler

- This overlap causes the **aliasing** that takes place due to under-sampling
- There is no overlap, i.e., no aliasing, only if

$$X(e^{j\omega}) = 0 \qquad \text{for } |\omega| \geq \pi/2$$

- Note: $Y(e^{j\omega})$ is indeed periodic with a period 2π, even though the stretched version of $X(e^{j\omega})$ is periodic with a period 4π

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Down-Sampler

- For the general case, the relation between the DTFTs of the output and the input of a factor-of-M down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

- ➡ $Y(e^{j\omega})$ is a sum of M uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of $1/M$

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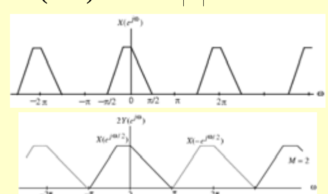
Down-Sampler

- Aliasing is absent if and only if

$$X(e^{j\omega}) = 0 \quad for\ |\omega| \geq \pi/M$$

as shown below for $M = 2$

$$X(e^{j\omega}) = 0 \ \text{ for } |\omega| \geq \pi/2$$



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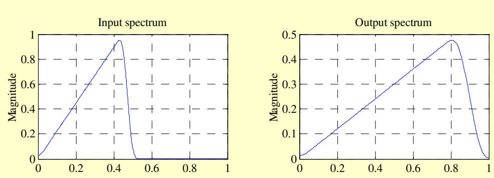
Down-Sampler

- Program 13_4 can be used to illustrate the frequency-domain properties of the down-sampler shown below for $M = 2$



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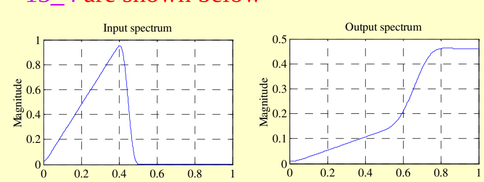
Down-Sampler

- The input and output spectra of a down-sampler with $M = 3$ obtained using Program 13_4 are shown below



- Effect of aliasing can be clearly seen

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Cascade Equivalences

- A complex multirate system is formed by an interconnection of the up-sampler, the down-sampler, and the components of an LTI digital filter
- In many applications these devices appear in a cascade form
- An interchange of the positions of the branches in a cascade often can lead to a computationally efficient realization

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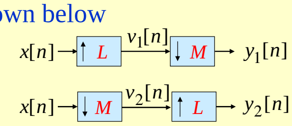
Up-Sampler and Down-sampler Cascade

- To implement a fractional change in the sampling rate we need to employ a cascade of an up-sampler and a down-sampler
- Consider the two cascade connections shown below



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Up-Sampler and Down-sampler Cascade

- Consider the top cascade shown in the previous slide
- Here, we have $V_1(z) = X(z^L)$

  and
  $$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(z^{1/M} W_M^{-k})$$
- Combining the last two equations we get
  $$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL})$$

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Up-Sampler and Down-sampler Cascade

- We next consider the bottom cascade shown in Slide 36
- Here, we have
  $$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

  and $Y_2(z) = V_2(z^L)$
- Combining the last two equations we get
  $$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k})$$

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Up-Sampler and Down-sampler Cascade

- It follows from the above that $Y_1(z) = Y_2(z)$ if
  $$\frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL}) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k})$$
- The above equality holds if and only if M and L are relatively prime, i.e. M and L do not have a common factor that is an integer $r > 1$, as then W_M^{-k} and W_M^{-kL} take the same set of values for $k = 0, 1, \ldots, M-1$

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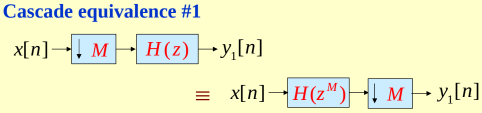
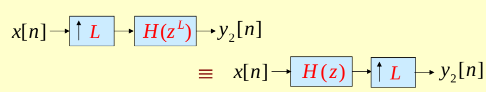
Noble Identities

- Two other cascade equivalences are shown below

Cascade equivalence #1



Cascade equivalence #2



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