Arbitrary-Rate Sampling Rate Converter

- The estimation of a discrete-time signal value at an arbitrary time instant between a consecutive pair of known samples can be solved by using some type of interpolation
- In this approach an approximating continuous-time signal is formed from a set of known consecutive samples of the given discrete-time signal

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Arbitrary-Rate Sampling Rate Converter

- The value of the approximating continuous-time signal is then evaluated at the desired time instant
- This interpolation process can be directly implemented by designing a digital interpolation filter

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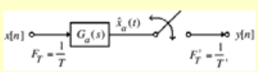
Ideal Sampling Rate Converter

- In principle, a sampling rate conversion by an arbitrary conversion factor can be implemented as follows
- The input digital signal is passed through an ideal analog reconstruction lowpass filter whose output is resampled at the desired output rate as indicated below



3

Ideal Sampling Rate Converter

- Let the impulse response of the analog lowpass filter is denoted by $g_a(t)$
- Then the output of the filter is given by

$$\hat{x}_a(t) = \sum_{\ell=-\infty}^{\infty} x[\ell] g_a(t - \ell T)$$

- If the analog filter is chosen to bandlimit its output to the frequency range $F_g < F'_T/2$, its output $\hat{x}_a(t)$ can then be resampled at the rate F'_T

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Ideal Sampling Rate Converter

- Since the impulse response $g_a(t)$ of an ideal lowpass analog filter is of infinite duration and the samples $g_a(nT' - \ell T)$ have to be computed at each sampling instant, implementation of the ideal bandlimited interpolation algorithm in exact form is not practical
- Thus, an approximation is employed in practice

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Ideal Sampling Rate Converter

- Problem statement: Given $N_2 + N_1 + 1$ input signal samples, $x[k]$, $k = -N_1, \ldots, N_2$, obtained by sampling an analog signal $x_a(t)$ at $t = t_k$ $= t_0 + kT_{in}$, determine the sample value $x_a(t_0 + kT_{in}) = y[\alpha]$ at time instant $t' = t_0 + kT_{in}$ where $-N_1 \le \alpha \le N_2$
- Figure on the next slide illustrates the interpolation process by an arbitrary factor

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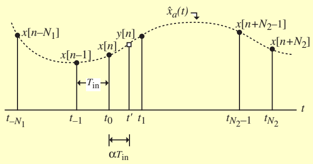
Ideal Sampling Rate Converter



- We describe next a commonly employed interpolation algorithm based on a finite weighted sum of input samples

Lagrange Interpolation Algorithm

- Here, a polynomial approximation $\hat{x}_a(t)$ to $x_a(t)$ is defined as

$$\hat{x}_a(t) = \sum_{k=-N_1}^{N_2} P_k(t)x[n+k]$$

where $P_k(t)$ are the Lagrange polynomials given by

$$P_k(t) = \prod_{\substack{\ell=-N_1 \\ \ell \neq k}}^{N_2} \left(\frac{t-t_\ell}{t_k - t_\ell} \right), \quad -N_1 \le k \le N_2$$

Lagrange Interpolation Algorithm

- Since

$$P_k(t_r) = \begin{cases} 1, & k = r, \\ 0, & k \neq r, \end{cases} \qquad -N_1 \le r \le N_2$$

it follows from the previous 3 equations that

$$\hat{x}_a(t_k) = x_a(t_k), \qquad -N_1 \le k \le N_2$$

Lagrange Interpolation Algorithm

- From $\hat{x}_a(t) = \sum_{k=-N_1}^{N_2} P_k(t)x[n+k]$

the value of $x_a(t)$ at an arbitrary value $t' = t_0 + \alpha T_{in}$ is given by

$$x_a(t') = \hat{x}_a(t_0 + \alpha T_{in}) = y[n] = \sum_{k=-N_1}^{N_2} P_k(\alpha)x[n+k]$$

where

$$P_k(\alpha) = P_k(t_0 + \alpha T_{in}) = \prod_{\substack{\ell=-N_1 \\ \ell \neq k}}^{N_2} \left(\frac{\alpha - \ell}{k - \ell} \right)$$

Lagrange Interpolation Algorithm

- Example - Design a fractional-rate interpolator with an interpolation factor of 3/2 using a 3rd-order polynomial approximation with $N_1 = 2$ and $N_2 = 1$
- The output $y[n]$ of the interpolator is thus computed using

$$y[n] = P_{-2}(\alpha)x[n-2] + P_{-1}(\alpha)x[n-1] + P_0(\alpha)x[n] + P_1(\alpha)x[n+1]$$

Lagrange Interpolation Algorithm

- Here, the Lagrange polynomials are given by

$$P_{-2}(\alpha) = \frac{(\alpha+1)\alpha(\alpha-1)}{-6} = \frac{1}{6}(-\alpha^3 + \alpha)$$

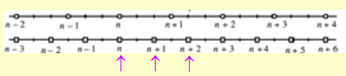
$$P_{-1}(\alpha) = \frac{(\alpha+2)\alpha(\alpha-1)}{2} = \frac{1}{2}(\alpha^3 + \alpha^2 - 2\alpha)$$

$$P_0(\alpha) = \frac{(\alpha+2)(\alpha+1)(\alpha-1)}{-2} = -\frac{1}{2}(\alpha^3 + 2\alpha^2 - \alpha - 2)$$

$$P_1(\alpha) = \frac{(\alpha+2)(\alpha+1)\alpha}{-6} = \frac{1}{6}(\alpha^3 + 3\alpha^2 + \alpha)$$

Lagrange Interpolation Algorithm

- Figure below shows the locations of the samples of the input and the output for an interpolator with a conversion factor of 3/2
- Locations of the output samples $y[n]$, $y[n+1]$, and $y[n+2]$ in the input sample domain are marked with an arrow



Input sample index
Output sample index

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Lagrange Interpolation Algorithm

- From the figure on the previous slide it can be seen that the value of α for computation of $y[n]$, to be labeled α_0, is 0
- Substituting this value of α in the expressions for the Lagrange polynomial coefficients derived earlier we get

$$P_{-2}(\alpha_0) = 0, \quad P_{-1}(\alpha_0) = 0$$
$$P_0(\alpha_0) = 1, \quad P_1(\alpha_0) = 0$$

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Lagrange Interpolation Algorithm

- The value of α for computation of $y[n+1]$, to be labeled α_1, is 2/3
- Substituting this value of α in the expressions for the Lagrange polynomial coefficients we get

$$P_{-2}(\alpha_1) = 0.0617, \quad P_{-1}(\alpha_1) = -0.2963$$
$$P_0(\alpha_1) = 0.7407, \quad P_1(\alpha_1) = 0.4938$$

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Lagrange Interpolation Algorithm

- The value of α for computation of $y[n+2]$, to be labeled α_2, is 4/3
- Substituting this value of α in the expressions for the Lagrange polynomial coefficients we get

$$P_{-2}(\alpha_2) = -0.1728, \quad P_{-1}(\alpha_2) = 0.7407$$
$$P_0(\alpha_2) = -1.2963, \quad P_1(\alpha_2) = 1.7284$$

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Lagrange Interpolation Algorithm

- Substituting the values of the Lagrange polynomial coefficients in the interpolator output equation for n, $n+1$, and $n+2$, and combining the three equations into a matrix form we arrive at

$$\begin{bmatrix} y[n] \\ y[n+1] \\ y[n+2] \end{bmatrix} = \begin{bmatrix} P_{-2}(\alpha_0) & P_{-1}(\alpha_0) & P_0(\alpha_0) & P_1(\alpha_0) \\ P_{-2}(\alpha_1) & P_{-1}(\alpha_1) & P_0(\alpha_1) & P_1(\alpha_1) \\ P_{-2}(\alpha_2) & P_{-1}(\alpha_2) & P_0(\alpha_2) & P_1(\alpha_2) \end{bmatrix} \begin{bmatrix} x[n-2] \\ x[n-1] \\ x[n] \\ x[n+1] \end{bmatrix}$$

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Lagrange Interpolation Algorithm

- The input-output relation of the interpolation filter can be compactly written as

$$\begin{bmatrix} y[n] \\ y[n+1] \\ y[n+2] \end{bmatrix} = \mathbf{H} \begin{bmatrix} x[n-2] \\ x[n-1] \\ x[n] \\ x[n+1] \end{bmatrix}$$

where $\mathbf{H}$ is the block coefficient matrix

- For the factor-of-3/2 interpolator, we have

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.0617 & -0.2963 & 0.7407 & 0.4938 \\ -0.1728 & 0.7407 & -1.2963 & 1.7284 \end{bmatrix}$$

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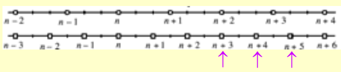
Lagrange Interpolation Algorithm

- It should be evident from an examination of



Input sample index

Output sample index

that the filter coefficients to compute $y[n+3]$, $y[n+4]$, and $y[n+5]$ are again given by the same block matrix **H**

$$\begin{bmatrix} y[n+3] \\ y[n+4] \\ y[n+5] \end{bmatrix} = \mathbf{H} \begin{bmatrix} x[n] \\ x[n+1] \\ x[n+2] \\ x[n+3] \end{bmatrix}$$

Lagrange Interpolation Algorithm

- In practice, the overall system delay of the fractional rate interpolator will be 3 sample periods
- Hence, the input-output relation of a practical interpolator will be

$$\begin{bmatrix} y[n] \\ y[n+1] \\ y[n+2] \end{bmatrix} = \mathbf{H} \begin{bmatrix} x[n] \\ x[n+1] \\ x[n+2] \\ x[n+3] \end{bmatrix}$$

Lagrange Interpolation Algorithm

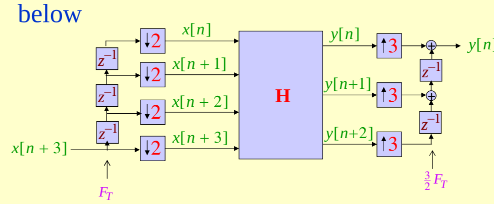
- The next set of length-3 output vector is then computed using

$$\begin{bmatrix} y[n+3] \\ y[n+4] \\ y[n+5] \end{bmatrix} = \mathbf{H} \begin{bmatrix} x[n+2] \\ x[n+3] \\ x[n+4] \\ x[n+5] \end{bmatrix}$$

and so on

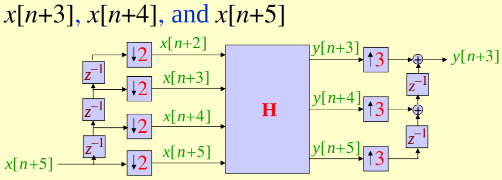
Lagrange Interpolation Algorithm

- ➡ The desired interpolation filter is a time-varying filter
- A realization of the interpolator is given below



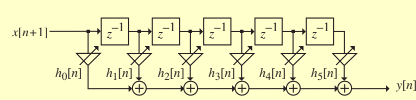
Lagrange Interpolation Algorithm

- Because of the factor-of-2 down-sampling, the next set of input samples appearing at the input of the block filter **H** is $x[n+2]$, $x[n+3]$, $x[n+4]$, and $x[n+5]$



Lagrange Interpolation Algorithm

- A realization of the factor-of-3 interpolator in the form of a time-varying filter is shown below



Lagrange Interpolation Algorithm

- The coefficients of the 5-th order time-varying FIR filter have a period of 3 and are assigned the values indicated below

Time	$h_0[n]$	$h_1[n]$	$h_2[n]$	$h_3[n]$	$h_4[n]$	$h_5[n]$
n	$P_1(\alpha_0)$	$P_0(\alpha_0)$	$P_{-1}(\alpha_0)$	$P_{-2}(\alpha_0)$	0	0
$n+1$	0	$P_1(\alpha_1)$	$P_0(\alpha_1)$	$P_{-1}(\alpha_1)$	$P_{-2}(\alpha_1)$	0
$n+2$	0	0	$P_1(\alpha_2)$	$P_0(\alpha_2)$	$P_{-1}(\alpha_2)$	$P_{-2}(\alpha_2)$

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Lagrange Interpolation Algorithm

- Substituting the expressions for the Lagrange polynomials in the output equation we arrive at

$$y[n]=\alpha^3(-\tfrac{1}{6}x[n-2]+\tfrac{1}{2}x[n-1]-\tfrac{1}{2}x[n]+\tfrac{1}{6}x[n+1])$$
$$+\alpha^2(\tfrac{1}{2}x[n-1]-x[n]+\tfrac{1}{2}x[n+1])$$
$$+\alpha(\tfrac{1}{6}x[n-2]-x[n-1]+\tfrac{1}{2}x[n]+\tfrac{1}{3}x[n+1])$$
$$+x[n]$$

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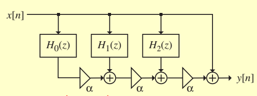
Lagrange Interpolation Algorithm

- A digital filter realization of the equation on the previous slide leads to the Farrow structure shown below



- In the above structure

$$H_0(z)=-\tfrac{1}{6}z^{-2}+\tfrac{1}{2}z^{-1}-\tfrac{1}{2}+\tfrac{1}{6}z$$
$$H_1(z)=\tfrac{1}{2}z^{-1}-1+\tfrac{1}{2}z$$
$$H_2(z)=\tfrac{1}{6}z^{-2}-z^{-1}+\tfrac{1}{2}+\tfrac{1}{3}z$$

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Lagrange Interpolation Algorithm

- In the Farrow structure only the value of α is changed periodically with the remaining digital filter structure kept unchanged
- Figures on the next slide show the input and the output of the above interpolator for a sinusoidal input of frequency of 0.05 Hz sampled at a 1-Hz rate

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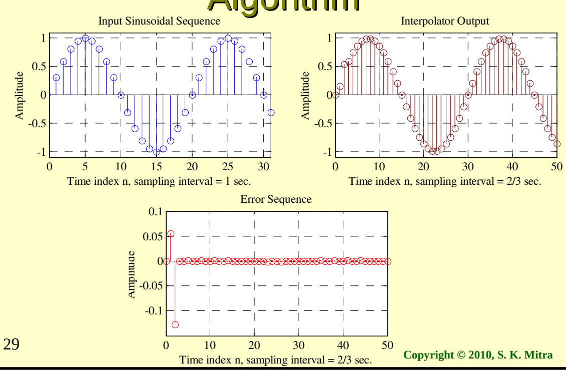
Lagrange Interpolation Algorithm



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Spline Interpolation

- Here, a polynomial approximation $\hat{x}_a(t)$ to $x_a(t)$ is made using the B-spline functions as the basis
- The time instants t_k , $m \le k \le N+m$, at which the samples $x_a(t_k)$ of the signal $x_a(t)$ are known, are called knots

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Spline Interpolation

- The Lth order B-spline $B_m^{(L)}(t)$ defined in the interval $[t_m,\ldots,t_{N+m}]$ is given by
$$B_m^{(L)}(t) = \sum_{i=m}^{N+m} a_i \phi_i(t)$$
where $\phi_i(t)$, called truncated power functions, are polynomials of degree L:
$$\phi_i(t) = (t-t_i)_+^L = \begin{cases} 0, & t < t_i \\ (t-t_i)^L, & t \geq t_i \end{cases}$$

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Spline Interpolation

- The polynomial approximation $\hat{x}_a(t)$ to $x_a(t)$ is given by
$$\hat{x}_a(t) = \sum_{k=m}^{N+m} B_k^{(L)}(t) x_a(t_k)$$
- The coefficients a_i in $B_m^{(L)}(t) = \sum_{i=m}^{N+m} a_i \phi_i(t)$ are determined by imposing specific conditions at the knots t_m and t_{N+m}

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Spline Interpolation

- It follows from the definition of the truncated power functions that $B_m^{(L)}(t) = 0$ for $t \leq t_m$
- An additional condition, $B_m^{(L)}(t) = 0$ for $t \geq t_{N+m}$ is also imposed
- Hence, for $t \geq t_{N+m}$ we have
$$\sum_{i=m}^{N+m} a_i (t-t_i)^L = 0$$

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Spline Interpolation

- The set of equations at the bottom of the previous slide has nontrivial solutions for $N > L$
- An elegant and simple solution exists for $N = L + 1$
- We develop the solution for the cubic B-spline next

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Cubic B-Spline

- Here $L = 3$ and therefore $N = 4$
- For notational convenience, we choose $m = 0$
- In this case, $\sum_{i=0}^{4} a_i (t-t_i)^3 = 0$ in matrix form becomes
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ t_0 & t_1 & t_2 & t_3 & t_4 \\ t_0^2 & t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_0^3 & t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Cubic B-Spline

- Since the matrix equation given in the previous slide is an underdetermined system and all rows are linearly independent, assuming $t_i \neq t_j$ if $i \neq j$, we can choose any one coefficient as the free parameter and solve for the other 4 coefficients in terms of the free parameter

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Cubic B-Spline

- Considering a_4 to be the free parameter, we rewrite the matrix equation in Slide 59 as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ t_0 & t_1 & t_2 & t_3 \\ t_0^2 & t_1^2 & t_2^2 & t_3^2 \\ t_0^3 & t_1^3 & t_2^3 & t_3^3 \end{bmatrix} = -a_4 \begin{bmatrix} 1 \\ t_4 \\ t_4^2 \\ t_4^3 \end{bmatrix}$$

- We can solve the above matrix equation for a_i using Cramer's rule

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Cubic B-Spline

- For example,

$$a_0 = -a_4 \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_4 & t_1 & t_2 & t_3 \\ t_4^2 & t_1^2 & t_2^2 & t_3^2 \\ t_4^3 & t_1^3 & t_2^3 & t_3^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_0 & t_1 & t_2 & t_3 \\ t_0^2 & t_1^2 & t_2^2 & t_3^2 \\ t_0^3 & t_1^3 & t_2^3 & t_3^3 \end{vmatrix}}$$

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Cubic B-Spline

- The numerator and the denominator of the previous equation are determinants of Vandermonde matrices and have nonzero values if the knots t_i are distinct
- It can be shown that

$$a_0 = -a_4 \frac{(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)}{(t_0 - t_1)(t_0 - t_2)(t_0 - t_3)}$$

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Cubic B-Spline

- Choosing the free parameter a_4 to be

$$a_4 = \frac{1}{(t_4 - t_0)(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)}$$

we arrive at

$$a_0 = \frac{1}{(t_0 - t_1)(t_0 - t_2)(t_0 - t_3)(t_0 - t_4)}$$

- In a similar manner the expressions for the remaining 3 coefficients can be derived and are given in the next slide

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Cubic B-Spline

$$a_1 = \frac{1}{(t_1 - t_0)(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)}$$

$$a_2 = \frac{1}{(t_2 - t_0)(t_2 - t_1)(t_2 - t_3)(t_2 - t_4)}$$

$$a_3 = \frac{1}{(t_3 - t_0)(t_3 - t_1)(t_3 - t_2)(t_3 - t_4)}$$

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B-Spline

- In the general case, the coefficients are given by

$$a_i = \frac{(-1)^{L+1}}{\prod_{k=m, i \neq k}^{N+m} (t_i - t_k)}, \qquad m \le i \le N + m$$

and the Lth order B-spline function is given by

$$B_m^{(L)}(t) = (-1)^{L+1} \sum_{i=m}^{N+m} \frac{(t - t_i)_+^L}{\prod_{k=m, i \neq k}^{N+m} (t_i - t_k)}$$

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Normalized B-Spline

- Since the maximum value of the B-spline decreases with increasing L, it is a common practice to use instead, a normalized form given by
$$\beta_m^{(L)}(t) = (t_{N+m} - t_m) B_m^{(L)}(t)$$
for interpolation
- In digital signal processing applications, the knots are uniformly spaced at sampling instants

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Second-Order Normalized B-Spline

- Here $L = 2$ and hence, $N = 3$
- The knots are at $t_i = i$, $m \le i \le m+3$
- As a result

$$a_m = -\frac{1}{(m-m-1)(m-m-2)(m-m-3)} = \frac{1}{6}$$
$$a_{m+1} = -\frac{1}{(m+1-m-1)(m+1-m-2)(m+1-m-3)} = -\frac{1}{2}$$
$$a_{m+2} = -\frac{1}{(m+2-m-1)(m+2-m-2)(m+2-m-3)} = \frac{1}{2}$$
$$a_{m+3} = -\frac{1}{(m+3-m-1)(m+3-m-2)(m+3-m-3)} = -\frac{1}{6}$$

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Second-Order Normalized B-Spline

- The expression for the second-order B-spline is given below:

$$B_m^{(2)}(t) = \begin{cases} 0, & t < m \\ a_m(t-m)^2, & m \le t < m+1 \\ a_m(t-m)^2 + a_{m+1}(t-m-1)^2, & m+1 \le t < m+2 \\ a_m(t-m)^2 + a_{m+1}(t-m-1)^2 + a_{m+1}(t-m-1)^2, & m+2 \le t < m+3 \\ 0, & t \ge m+3 \end{cases}$$

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Second-Order Normalized B-Spline

- The corresponding normalized second-order B-spline is given by

$$\beta_m^{(2)}(t) = (m+3-m) B_m^{(2)}(t) = \begin{cases} 0, & t < m \\ \frac{t^2}{2} - mt + \frac{m^2}{2}, & m \le t < m+1 \\ -t^2 + 3t + 2mt - 3m - m^2 - \frac{3}{2}, & m+1 \le t < m+2 \\ \frac{t^2}{2} - 3t - mt + \frac{m^2}{2} + 3m + \frac{9}{2}, & m+2 \le t < m+3 \\ 0, & t \ge m+3 \end{cases}$$

- A plot of $\beta_m^{(2)}(t)$ and the corresponding power functions for several values of m are shown in the next slide

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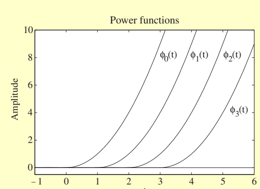
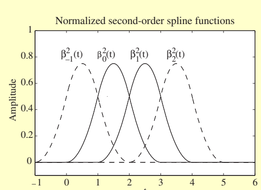
Second-Order Normalized B-Spline





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Spline Interpolation

- The interpolation formula is obtained by forming a linear combination of the normalized B-splines weighted by the known values of the function $x_a(t)$ at the knots $t_k = n + k$
- The interpolated value at the time instant $t' = t_0 + \alpha T_{in}$ is given by

$$\hat{x}_a(t') = \hat{x}_a(t_0 + \alpha T_{in})$$
$$= y[n] = \sum_{k=m}^{L+m+1} \beta_k^{(L)}(t_0 + \alpha T_{in}) x[n+k]$$

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Spline Interpolation

- It should be noted that, unlike the Lagrange interpolation algorithm, in the case of spline interpolation, $\hat{x}_a(t_k) \neq x_a(t_k)$
- We illustrate next the development of the interpolation formula using the normalized second-order B-spline

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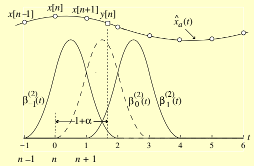
Interpolation Using Second-Order B- Spline

- The interpolation process is illustrated below



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Interpolation Using Second-Order B- Spline

- As can be seen from the above figure, the position $t' = 1 + \alpha$ of the desired value $y[n]$ of $x_a(t)$ is between the knots $t = 1$ and $t = 2$
- Here we thus have

$$y[n] = \sum_{k=-1}^{2} \beta_k^{(2)}(\alpha) x[n+k]$$

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Interpolation Using Second-Order B- Spline

where

$$\beta_{-1}^{(2)}(\alpha) = \frac{\alpha^2}{2} - \alpha + \frac{1}{2}$$

$$\beta_0^{(2)}(\alpha) = -\alpha^2 + \alpha + \frac{1}{2}$$

$$\beta_1^{(2)}(\alpha) = \frac{\alpha^2}{2}$$

$$\beta_2^{(2)}(\alpha) = 0$$

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Interpolation Using Second-Order B- Spline

- The interpolation formula is then given by

$$y[n] = \sum_{k=-1}^{1} \beta_k^{(2)}(\alpha) x[n+k]$$

$$= \left(\frac{\alpha^2}{2} - \alpha + \frac{1}{2}\right) x[n-1] + \left(-\alpha^2 + \alpha + \frac{1}{2}\right) x[n] + \frac{\alpha^2}{2} x[n+1]$$

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Interpolation Using Second-Order B- Spline

- The equation in the previous slide can be rewritten as

$$y[n] = \left(\tfrac{1}{2} x[n-1] + \tfrac{1}{2} x[n]\right) + \alpha\left(-x[n-1] + x[n]\right) + \alpha^2\left(\tfrac{1}{2} x[n-1] - x[n] + \tfrac{1}{2} x[n+1]\right)$$

leading to the Farrow structure shown on the next slide

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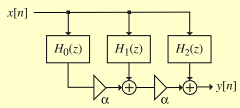
Interpolation Using Second-Order B- Spline



$$H_0(z) = \tfrac{1}{2} z^{-1} - 1 + \tfrac{1}{2} z$$

$$H_1(z) = -z^{-1} + 1$$

$$H_2(z) = \tfrac{1}{2} z^{-1} + \tfrac{1}{2}$$

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Arbitrary-Rate Sampling Rate Converter

Practical Considerations

- A direct design of a fractional-rate sampling rate converter in most applications is not practical
- This is due to two main reasons:
  – length of the time-varying filter needed is usually very large
  – real-time computation of the corresponding filter coefficients is nearly impossible

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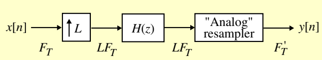
Arbitrary-Rate Sampling Rate Converter

- As a result, the fractional-rate sampling rate converter is almost realized in a hybrid form as indicated below for the case of an interpolator



- The digital sampling rate converter can be implemented in a multistage form to reduce the computational complexity

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