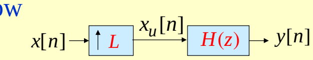
Nyquist Filters

- Under certain conditions, a lowpass filter can be designed to have a number of zero-valued coefficients
- When used as interpolation filters these filters preserve the nonzero samples of the up-sampler output at the interpolator output
- Moreover, due to the presence of these zero-valued coefficients, these filters are computationally more efficient than other lowpass filters of same order

*L*th-Band Filters

- These filters, called the Nyquist filters or *L*th-band filters, are often used in single-rate and multi-rate signal processing
- Consider the factor-of-*L* interpolator shown below



- The input-output relation of the interpolator in the *z*-domain is given by

$$Y(z) = H(z)X(z^L)$$

*L*th-Band Filters

- If $H(z)$ is realized in the *L*-band polyphase form, then we have

$$H(z) = \sum_{i=0}^{L-1} z^{-i} E_i(z^L)$$

- Assume that the *k*-th polyphase component of $H(z)$ is a constant, i.e., $E_k(z) = \alpha$:

$$H(z) = E_0(z^L) + z^{-1}E_1(z^L) + \ldots + z^{-(k-1)}E_{k-1}(z^L)$$
$$+ \alpha z^{-k} + z^{-(k+1)}E_{k+1}(z^L) + \ldots + z^{-(L-1)}E_{L-1}(z^L)$$

*L*th-Band Filters

- Then we can express $Y(z)$ as

$$Y(z) = \alpha z^{-k} X(z^L) + \sum_{\substack{\ell=0 \\ \ell \neq k}}^{L-1} z^{-\ell} E_\ell(z^L) X(z^L)$$

- As a result,

$$y[Ln+k] = \alpha\, x[n]$$

- Thus, the input samples appear at the output without any distortion for all values of *n*, whereas, in-between $(L-1)$ output samples are determined by interpolation

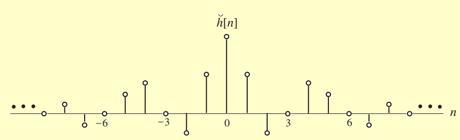
*L*th-Band Filters

- A filter with the above property is called a Nyquist filter or an *L*th-band filter
- Its impulse response has many zero-valued samples, making it computationally attractive
- For example, the impulse response of an *L*th-band filter for $k = 0$ satisfies the following condition

$$h[Ln] = \begin{cases} \alpha, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

*L*th-Band Filters

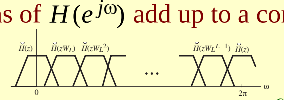
- Figure below shows a typical impulse response of a third-band filter ($L = 3$)



- *L*th-band filters can be either FIR or IIR filters

*L*th-Band Filters

- If the 0-th polyphase component of $H(z)$ is a constant, i.e., $E_0(z) = \alpha$ then it can be shown that
  $\sum_{k=0}^{L-1} H(zW_L^k) = L\alpha = 1$ (assuming $\alpha = 1/L$)
- Since the frequency response of $H(zW_L^k)$ is the shifted version $H(e^{j(\omega - 2\pi k/L)})$ of $H(e^{j\omega})$, the sum of all of these L uniformly shifted versions of $H(e^{j\omega})$ add up to a constant



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Half-Band Filters

- An Lth-band filter for $L = 2$ is called a half-band filter
- The transfer function of a half-band filter is thus given by
  $$H(z) = \alpha + z^{-1}E_1(z^2)$$
  with its impulse response satisfying
  $$h[2n] = \begin{cases} \alpha, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

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Half-Band Filters

- The condition
  $$H(z) = \alpha + z^{-1}E_1(z^2)$$
  reduces to
  $$H(z) + H(-z) = 1 \quad \text{(assuming } \alpha = 0.5\text{)}$$
- If $H(z)$ has real coefficients, then
  $$H(-e^{j\omega}) = H(e^{j(\pi-\omega)})$$
- Hence
  $$H(e^{j\omega}) + H(e^{j(\pi-\omega)}) = 1$$

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Half-Band Filters

- $\Rightarrow$ $H(e^{j(\pi/2-\theta)})$ and $H(e^{j(\pi/2+\theta)})$ add up to 1 for all θ
- Or, in other words, $H(e^{j\omega})$ exhibits a symmetry with respect to the half-band frequency $\pi/2$, hence the name "half-band filter"

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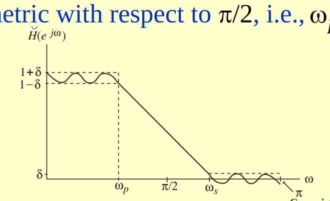
Half-Band Filters

- Figure below illustrates this symmetry for a half-band lowpass filter for which passband and stopband ripples are equal, i.e., $\delta_p = \delta_s$ and passband and stopband edges are symmetric with respect to $\pi/2$, i.e., $\omega_p + \omega_s = \pi$



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Half-Band Filters

- Attractive property: About 50% of the coefficients of $h[n]$ are zero
- This reduces the number of multiplications required in its implementation significantly
- For example, if $N = 101$, an arbitrary Type 1 FIR transfer function requires about 50 multipliers, whereas, a Type 1 half-band filter requires only about 25 multipliers

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Half-Band Filters

- An FIR half-band filter can be designed with linear phase
- However, there is a constraint on its length
- Consider a zero-phase half-band FIR filter for which $h[n] = \alpha * h[-n]$, with $|\alpha| = 1$
- Let the highest nonzero coefficient be $h[R]$

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Half-Band Filters

- Then R is odd as a result of the condition

$$h[2n] = \begin{cases} \alpha, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Therefore $R = 2K+1$ for some integer K
- Thus the length of $h[n]$ is restricted to be of the form $2R+1 = 4K+3$ [unless $H(z)$ is a constant]

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Design of Linear-Phase *L*th-Band Filters

- A lowpass linear-phase Lth-band FIR filter can be readily designed via the windowed Fourier series approach
- In this approach, the impulse response coefficients of the lowpass filter are chosen as $h[n] = h_{LP}[n] \cdot w[n]$ where $h_{LP}[n]$ is the impulse response of an ideal lowpass filter with a cutoff at π/L and $w[n]$ is a suitable window function

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Design of Linear-Phase *L*th-Band Filters

- Now, the impulse response of an ideal Lth-band lowpass filter with a cutoff at $\omega_c = \pi / L$ is given by

$$h_{LP}[n] = \frac{\sin(\pi n / L)}{\pi n}, \quad -\infty \le n \le \infty$$

- It can be seen from the above that

$$h_{LP}[n] = 0 \quad \text{for } n = \pm L, \pm 2L, \ldots$$

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Design of Linear-Phase *L*th-Band Filters

- Hence, the coefficient condition of the Lth-band filter

$$h[Ln] = \begin{cases} \alpha, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

is indeed satisfied

- Hence, an Lth-band FIR filter can be designed by applying a suitable window $w[n]$ to $h_{LP}[n]$

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Design of Linear-Phase *L*th-Band Filters

- There are many other candidates for Lth-band FIR filters
- Program 13_8 can be used to design an Lth-band FIR filter using the windowed Fourier series approach
- The program employs the Hamming window
- However, other windows can also be used

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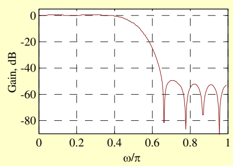
Design of Linear-Phase *L*th-Band Filters

- Figure below shows the gain response of a half-band filter of length-23 designed using Program 13_8



Design of Linear-Phase *L*th-Band Filters

- The filter coefficients are given by

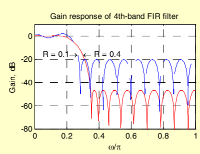
$$h[-11]=h[11]=-0.002315;\quad h[-10]=h[10]=0;$$
$$h[-9]=h[9]=0.005412;\quad h[-8]=h[8]=0;$$
$$h[-7]=h[7]=-0.001586;\quad h[-6]=h[6]=0;$$
$$h[-5]=h[5]=0.003584;\quad h[-4]=h[4]=0;$$
$$h[-3]=h[3]=-0.089258;\quad h[-2]=h[2]=0;$$
$$h[-1]=h[1]=0.3122379;\quad h[0]=0.5;$$

- As expected, $h[n] = 0$ for

$$n=\pm 2,\pm 4,\pm 6,\pm 8,\pm 10$$

Design of Linear-Phase *L*th-Band Filters

- We show below the gain response of a length 23 4th-band lowpass filter designed using the M-file `firnyquist` with 2 different roll-off factors



Design of Linear-Phase Half-Band Filters

- The problem of designing a real-coefficient half-band FIR filter can be transformed into the design of a single passband FIR filter with no stopband which can be easily designed using the Parks-McClellan algorithm
- An inverse transformation of the wideband filter then yields the half-band FIR filter

Design of Linear-Phase Half-Band Filters

- Let the specifications of the real-coefficient half-band filter $G(z)$ of order N be as follows:
- Passband edge at ω_p, stopband edge at ω_s, passband ripple of δ_p, and stopband ripple of δ_s
- Now for a half-band filter $\delta_p = \delta_s = \delta$, $\omega_p + \omega_s = \pi$ and the order N is even with $N/2$ odd

Design of Linear-Phase Half-Band Filters

- Now, consider the design of a wide-band linear-phase FIR filter $F(z)$ of degree $N/2$ with a passband from 0 to $2\omega_p$, a transition band from $2\omega_p$ to π, and a passband ripple of 2δ
- Since $N/2$ is odd, $F(z)$ has a zero at $z=-1$
- Let $f[n]$ denote the impulse response of $F(z)$

Design of Linear-Phase Half-Band Filters

- Define

$$G(z) = \tfrac{1}{2}[z^{-N/2} + F(z^2)]$$

- $G(z)$ can be seen to be the transfer function of a causal half-band lowpass FIR filter with an impulse response

$$g[n] = \begin{cases} \tfrac{1}{2} f[n/2], & n \text{ even} \\ 0, & n \text{ odd}, n \neq \frac{N}{2} \\ \frac{1}{2}, & n = \frac{N}{2} \end{cases}$$

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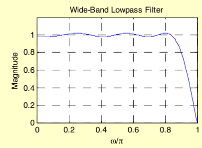
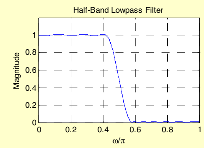
Design of Linear-Phase Half-Band Filters

- The plots below the magnitude response of an wide-band lowpass filter of degree 13 with a passband from 0 to 0.85π and a transition band from 0.9π to π and the magnitude response of the derived half-band filter





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Design of Half-Band IIR Filters

- Recall that an odd-order bounded real (BR) lowpass IIR transfer function $G(z) = P(z)/D(z)$ with a symmetric numerator and satisfying the power-symmetry condition

$$G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 1$$

can be decomposed in the form

$$G(z) = \tfrac{1}{2}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)]$$

where $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ are stable allpass functions

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Design of Half-Band IIR Filters

- It follows from the power-symmetry condition that $G(z)$ is a half-band lowpass transfer function
- A Butterworth half-band lowpass IIR filter $G(z)$ can be designed by first designing an odd-order analog Butterworth lowpass filter with a 3-dB cutoff frequency at $\Omega_c = 1$ and then applying a bilinear transformation
- We next consider the design of an elliptic IIR half-band filter

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Design of Half-Band IIR Filters

- It can be shown that any odd-order elliptic lowpass half-band filter $G(z)$ with a frequency response specification given by

$$1 - 2\delta_p \leq |G(e^{j\omega})| \leq 1, \quad \text{for } 0 \leq \omega \leq \omega_p$$
$$|G(e^{j\omega})| \leq \delta_s, \quad \text{for } \omega_s \leq \omega \leq \pi$$

and satisfying the conditions

$$\omega_p + \omega_s = \pi, \quad \delta_s^2 = 4\delta_p(1 - \delta_p)$$

is a power-symmetric transfer function

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Design of Half-Band IIR Filters

- It can be shown that the poles of the elliptic lowpass half-band filter lie on the imaginary axis
- Using the pole-interlacing property, we can readily identify the expressions for $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$

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Design of Half-Band IIR Filters

- **Design Steps:**
- Since $\omega_p + \omega_s = \pi$, $\delta_s^2 = 4\delta_p(1-\delta_p)$, only one of the bandedges and one of the ripples can be specified
- Let the specified stopband edge and stopband ripple be ω_s and δ_s, respectively
- Then ω_p and δ_p are determined using the equations at the top of the slide

Design of Half-Band IIR Filters

- Define $r = \dfrac{\tan(\omega_p/2)}{\tan(\omega_s/2)}$

$$r' = \sqrt{1-r^2}$$

$$q_0 = \frac{(1-\sqrt{r'})}{2(1+\sqrt{r'})}$$

and compute

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}$$

$$D = \left(\frac{1-\delta_s^2}{\delta_s^2}\right)^2$$

Design of Half-Band IIR Filters

- Next, the estimate of the order of $G(z)$ is determined by choosing the smallest odd integer satisfying

$$N \ge \frac{\log_{10}(16D)}{\log_{10}(1/q)}$$

- Now the integer value of N is almost always higher than the quantity on the RHS of the above equation

Design of Half-Band IIR Filters

- As a result, the corresponding value of δ_s will be smaller than the original specified value
- To determine the actual value of δ_s, the actual value of the parameter D is first computed from

$$D = \frac{10^{N\log_{10}(1/q)}}{16}$$

Design of Half-Band IIR Filters

- From the actual value of D, the actual value of δ_s is computed by solving

$$D = \left(\frac{1-\delta_s^2}{\delta_s^2}\right)^2$$

- From the new value of δ_s, the actual value of δ_p is obtained from

$$\delta_s^2 = 4\delta_p(1-\delta_p)$$

Design of Half-Band IIR Filters

- Next, the poles of the two allpass filters are computed as follows:

$$\lambda_k = \frac{2q^{1/4}\sum_{i=0}^{\infty}(-1)^i q^{i(i+1)} \sin((2i+1)k\pi/N)}{1+2\sum_{i=0}^{\infty}(-1)^i q^{i^2} \cos(2\pi ki/N)}$$

$$b_k = \sqrt{(1-r\lambda_k^2)\left(1-\frac{\lambda_k^2}{r}\right)}$$

$$c_k = 2b_k/(1+\lambda_k^2)$$

$$\alpha_{k-1} = (2-c_k)/(2+c_k)$$

Design of Half-Band IIR Filters

- In general, the two infinite sums in the expression for converge after the addition of 5 or 6 terms
- The poles of the two allpass filters are on the imaginary axis at $z = \pm j\sqrt{\alpha_k}$ and are inside the unit circle, as the parameters α_k are distinct with magnitudes less than 1

Design of Half-Band IIR Filters

- Using the pole-interlacing property, then poles of $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ are selected
- Their corresponding zeros are at the mirror-image locations
- **Example**
- We consider the design of an elliptic half-band lowpass filter

Design of Half-Band IIR Filters

- The specifications are:

$$\omega_s = 0.6\pi \qquad \delta_s = 0.016$$

- The transfer functions of the two allpass sections of the half-band filter designed using Program 13_9 are given by

$$\mathcal{A}_0(z) = \frac{0.23647 + z^{-1}}{1 + 0.23647 z^{-1}}, \quad \mathcal{A}_1(z) = \frac{0.71454 + z^{-1}}{1 + 0.71454 z^{-1}}$$

Design of Half-Band IIR Filters

- The pole-zero plot and the magnitude response of the designed elliptic lowpass half-band filter are shown below: