Further Results in the Fast Estimation of a Single Frequency

Michael P. Fitz, *Member*

***Abstract*—This correspondence proposes a new frequency estimator for a single complex sinusoid in complex white Gaussian noise. The estimator is applicable to problems in communications requiring high speed, recursive frequency estimation. The estimator is computationally efficient yet obtains near optimum performance at moderate signal-to-noise ratios.**

I. Introduction

Rapid estimation of the frequency of a single complex sinusoid in AWGN is an important problem in a wide variety of applications [1]. Array signal processing, spectral estimation [2], carrier and clock synchronization for digital communications, FSK demodulation, Doppler rate estimation [3], and signal interception and detection are good examples of applications requiring rapid frequency estimation. Several fast, accurate frequency estimators have been previously proposed . Reference [4] examines the asymptotic performance of several estimation structures and presents a comparison by Monte Carlo simulation. The signal model for this letter is

$$x_n = \sqrt{E_s}\exp[j(\Omega n+\theta)]+v_n \qquad n=1,2,\ldots,N \tag{1}$$

where v_n is a discrete time, delta-correlated complex Gaussian noise process with a variance of N_0, θ is the carrier phase, and E_s is the energy per sample (note E_s/N_0 = SNR). It is assumed that the carrier phase is an unknown random variable that is uniformly distributed in $[-\pi,\pi]$.

In [4], Kay demonstrated that an asymptotically (high SNR) optimum estimate of frequency is obtained by an estimator of the form (equations (16) and (18) of [4])

$$\hat{\Omega}=\sum_{n=2}^{N} w_n \arg\{x_n x_{n-1}^*\} \quad \text{or} \quad \hat{\Omega}=\arg\left\{\sum_{n=2}^{N} w_n x_n x_{n-1}^*\right\} \tag{2}$$

where w_n is an estimator window function. Often in recursive, high speed applications this windowing is difficult to implement since the window changes with each new sample. Uniform time weighting in frequency estimation results in estimators having the form (equations (17) and (19) of [4])

$$\hat{\Omega}=\frac{1}{N-1}\sum_{n=2}^{N}\arg\{x_n x_{n-1}^*\} \quad \text{or} \quad \hat{\Omega}=\arg\left\{\sum_{n=2}^{N} x_n x_{n-1}^*\right\}. \tag{3}$$

While estimators of the form in (3) are significantly simpler to implement, their performance is significantly lower than the frequency estimators in (2). The work presented in this correspondence provides improved performance over estimators of the form in (3) in an architecture compatible with recursive, high speed, real-time applications.

The estimators in (3) are closely related to a digitally implemented balanced quadricorrelator frequency-tracking loop [5]. The equation governing this loop is

$$\hat{\Omega}_{N+1}=\hat{\Omega}_N+\mu\,\mathrm{Im}\left[x_N x_{N-1} e^{-j\hat{\Omega}_N}\right] \tag{4}$$

where μ is the loop gain (a first-order loop). From (4) it is seen that the quadricorrelator uses the last term in the summation in (3) as the innovation to the loop. This relationship is not really unexpected since both the estimation structure in (2) and the structure in (4) can be viewed as approximations to the maximum likelihood (ML) estimator of frequency [6]. This work investigates planar filtered[1] or open loop methods of frequency estimation instead of the traditional loop based architectures.

II. Derivation of the Planar Filtered Estimator

The proposed frequency estimation algorithm is an approximation of the ML estimator. The ML estimator of the frequency of a sinusoid in the model in (1) is the maximum of the periodogram [6], given as

$$\hat{\Omega}_{ML}=\arg\max\left|\sum_{n=1}^{N} x_n e^{-j\Omega n}\right|^2. \tag{5}$$

The frequency estimator presented in this letter results from examining the likelihood equation. The likelihood equation is derived by taking the derivative of the periodogram w.r.t. Ω and setting it equal to zero. The likelihood equation is, after grouping terms appropriately,

$$L_\Omega(\mathbf{x})=\mathrm{Im}\left[\sum_{m=1}^{N} m\hat{R}_N(m)e^{-j\Omega m}\right]=0 \tag{6}$$

where

$$\hat{R}_N(m)=\sum_{k=m+1}^{N} x_k x_{k-m}^* \tag{7}$$

is the unnormalized sample autocorrelation function. Since for the unmodulated sinusoid the sample autocorrelation function has the form

$$\hat{R}_N(m)=Ae^{j\Omega m}+noise, \tag{8}$$

the ML estimate is chosen to make the weighted sum of the quadrature components of the derotated sample autocorrelation functions zero.

[1] A planar filtered structure is one that uses both the I and Q components of the input signal and not just the input signal phase or phase difference like a phase or frequency tracking loop.

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The author is with the School of Electrical Engineering, Purdue University, West Lafayette, IN 47907-1285.

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Examination of (6) provides some insight into the design of a suboptimal frequency estimator. Note that higher lags for the sample autocorrelation function in (6) are given a greater weight. Intuitively this results since the higher lag sample autocorrelation functions are less affected by noise and give a better estimate of Ω. The reason for the improved performance can be seen by examining (7) when each autocorrelation lag has the same number of terms in the summation. In this case the noise in (8) has the same statistical description regardless of the lag, while the argument of the signal component is linearly proportional to m. Consequently, the estimates of the frequency derived from higher lags are more accurate (for N-m >> 1 in (7))[2]. The likelihood equation for the ML estimator given in (6) appropriately weights the higher lag sample autocorrelation functions to a greater degree. It should be noted that the work in [4] presents algorithms that use only $\hat{R}_N(1)$.

A practical estimator is produced by approximating the likelihood equation given in (6). Since

$$\arg\left\{E\left(\hat{R}_N(m)\right)\right\} = \Omega m, \tag{9}$$

and for large N (i.e., small noise in (7))

$$\begin{aligned} \operatorname{Im}\left(\hat{R}_N(m)e^{-j\Omega m}\right) &\approx A\sin\left(\arg\left\{\hat{R}_N(m)\right\} - \Omega m\right) \\ &\approx A\left(\arg\left\{\hat{R}_N(m)\right\} - \Omega m\right), \end{aligned} \tag{10}$$

then (6) is heuristically approximated for large N as

$$\sum_{m=1}^{N} m\left[\arg\left\{\hat{R}(m)\right\} - m\Omega\right] \approx 0. \tag{11}$$

Algebra reduces (11) to

$$\hat{\Omega}_{ML} \approx \sum_{m=1}^{N} \frac{6m}{N(N+1)(2N+1)} \arg\left\{\hat{R}(m)\right\}. \tag{12}$$

If the summation in (12) is truncated at indices less than N a practical estimator has the form

$$\hat{\Omega} = \frac{\sum_{m=1}^{J} m\arg\left\{\hat{R}_N(m)\right\}}{\sum_{m=1}^{J} m^2} = \sum_{m=1}^{J} C(J,m)\arg\left\{\hat{R}_N(m)\right\}. \tag{13}$$

This frequency estimator weights the argument of the sample autocorrelation function by the lag, m, as suggested in (6). Fig. 1 is a block diagram of this estimator. With the use of read only memory (ROM), a high speed implementation of this estimator is possible with only J+2 pipeline delays. A reviewer pointed out that (13) can be alternately viewed as a least squares fit of the phase of the estimated autocorrelation function phase versus lag to the true autocorrelation phase.

This algorithm is practical and performs well. The performance of this estimator is also quite near the CRLB given as [1]

$$\operatorname{var}(\Omega - \hat{\Omega}) \geq \frac{6}{N(N^2-1)} \frac{N_0}{E_s} \tag{14}$$

[2] It should be noted that the optimum lag for estimating the frequency is m_{opt}=2N/3 [3], but since this work only uses relatively small values of m compared to N, the performance actually improves monotonically with m.

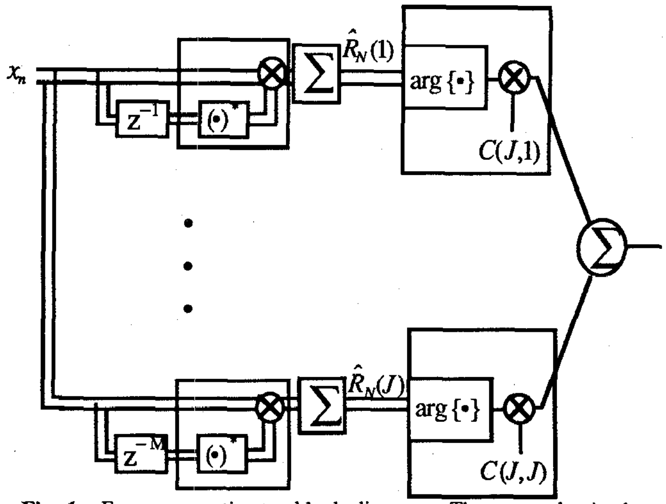

Fig. 1. Frequency estimator block diagram. The rectangles in the block diagram contain operations that potentially could be implemented in ROM.

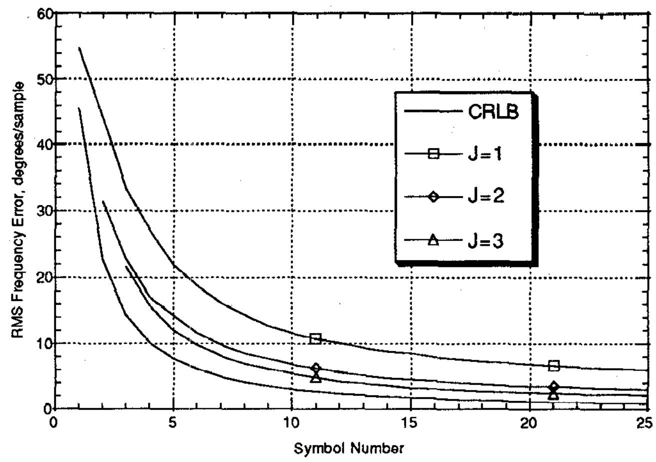

Fig. 2. Frequency estimator learning curves. E_s/N_0=2dB, Ω=0.5 radians/symbol, 10000 trials.

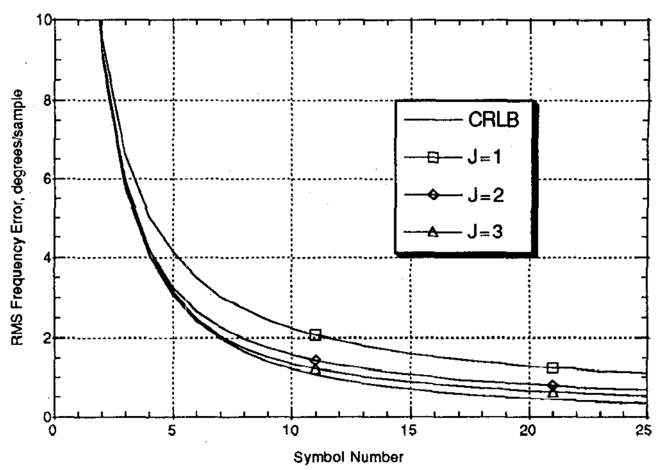

Fig. 3. Frequency estimator learning curves. E_s/N_0=10dB, Ω=0.5 radians/symbol, 10000 trials.

for relatively small indices, J, and is independent of Ω. Figs. 2-4 show plots of the estimator learning curves for different summation indices J at SNR=2dB, 10dB, and 30dB obtained by Monte Carlo simulation. The estimator acquires

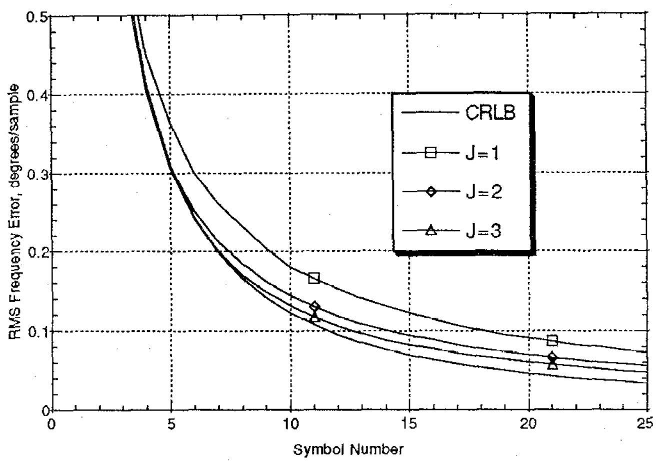

Fig. 4. Frequency estimator learning curves. E_s/N_0=30dB, Ω=0.5 radians/symbol, 10000 trials.

very rapidly so if the accumulators in Fig. 1 are given a fading factor the architecture is also capable of tracking a slowly varying frequency source. The planar filtered structure produces good performance at low SNR. It should be noted that (12) with J=1 is equivalent to the estimator of equation (19) in [4] and the frequency estimator presented in [7]. In implementing this algorithm one needs to ensure that $|J\Omega| << \pi$ so that phase unwrapping of the arguments in the summation of (9) is not necessary. This condition is met if the unknown frequency is a small fraction of the sampling rate (note that $\Omega/2\pi \approx 0.08/T$ in Figures 2-4), as is typical in many communications applications. This algorithm can be integrated with the planar filtered phase estimator proposed in [7] for a simple and rapidly acquiring carrier synchronization architecture.

References

[1] D.C. Rife and R.R. Boorstyn, "Single-Tone Parameter Estimation from Discrete-Time Observations," *IEEE Trans. Info. Theory*, vol. IT-20, September 1974, pp. 591-598.

[2] S.A. Kay, *Modern Spectral Estimation*, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[3] G.W. Lank, I.S. Reed, and G.E. Pollon, "A Semicoherent Detection and Doppler Estimation Statistic," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-9, March 1973, pp. 151-165.

[4] S. Kay, "A Fast and Accurate Single Frequency Estimator," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, December 1989, pp. 1987-1990.

[5] F.M. Gardner, "Characteristics of Frequency-Tracking Loops," *IEEE Trans. Commun.*, vol. COM-33, February 1985, pp. 131-139.

[6] A.J. Viterbi, *Principles of Coherent Communication*, McGraw-Hill, New York, 1966.

[7] M.P. Fitz and W.C. Lindsey, "Decision-Directed Burst Mode Digital Carrier Synchronization Techniques," *IEEE Trans. Commun.*, vol. COM-40, October 1992, pp. 1644-1653.