An Improved Single Frequency Estimator

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***Abstract*— A new method for estimating the frequency of a complex sinusoid in complex white Gaussian noise is proposed. Its computational complexity is comparable to Kay's method [1], but it attains the Cramer–Rao bound (CRB) down to lower signal-to-noise ratio (SNR) values. Simulation results are included to demonstrate the performance of the proposed method.**

I. Introduction

The PROBLEM of estimating the frequency of a complex sinusoid in white Gaussian noise arises in many applications [2]. The observed discrete-time signal in this situation can be represented as

$$x_k = Ae^{j(\omega_0 k+\theta)} + n_k, \; k = 0, 1, \cdots, N-1 \tag{1}$$

where the amplitude A, the frequency ω_0, and the phase θ are deterministic but unknown constants. The noise n_k is assumed to be a zero-mean complex white Gaussian process (WGP) with variance σ_n^2. It is well known that the maximum likelihood estimate (MLE) in this case is given by the location of the peak of the periodogram [3]. However, this approach requires too many computations, even if fast Fourier transform (FFT) techniques are employed. Therefore, simpler methods have been reported by Kay [1], Fitz [4], and Luise and Reggiannini (L&R) [5]. All these methods achieve the Cramer–Rao bound (CRB) [2] under the following two conditions. First, the input signal-to-noise ratio (SNR) $(A^2/2)/\sigma_n^2$ should be greater than a certain *threshold.* Second, $|\omega_0|$ should be smaller than the *estimation range*.

The threshold, the estimation range, and the computational requirements for the three estimators mentioned above are compared with the proposed estimator in Table I.[1] The Kay estimator has better estimation range and requires fewer multiplications than the other estimators; however, its threshold is limited to 6 dB. The Fitz and L&R estimators can operate at lower SNR values, but at the expense of the estimation range and the computational complexity. The computational complexity of the proposed estimator is comparable to the Kay estimator. However, it has a design parameter K, which can be adjusted to achieve lower threshold values at the expense of the estimation range.

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[1] We assume that the arg() function is realized by a read only memory. M is a design parameter for the Fitz and L&R estimators [4], [5]. Fitz and L&R estimators achieve CRB when $M = N/2$ [4], [5]. We can decrease the number of multiplications by decreasing M; however, as M gets smaller, the estimator's performance gets worse [4], [5].

The reason for the 6 dB threshold in the Kay estimator is explained in Section II. Section III shows how to modify Kay's algorithm to achieve lower threshold values. Section IV provides simulation results to demonstrate the performance of the proposed estimator.

II. Threshold Effect in the Kay Estimator

We rederive the Kay estimator to explain why threshold occurs. Equation (1) can be rewritten as

$$x_k = Ae^{j(\omega_0 k+\theta)}(1+\hat{n}_k), \; k = 0, 1, \cdots, N-1 \tag{2}$$

where

$$\hat{n}_k = \frac{n_k}{A} e^{-j(\omega_0 k+\theta)}. \tag{3}$$

$\hat{n}_k$ is a complex WGP with variance σ_n^2/A^2. Equation (2) can be rewritten

$$x_k = \hat{A}e^{j(\omega_0 k+\theta+u_k)}, \; k = 0, 1, \cdots, N-1 \tag{4}$$

where u_k is the phase term of $(1+\hat{n}_k)$. At high-input SNR values (i.e., $A^2/\sigma_n^2 >> 1$), u_k is the same as the imaginary part of $\hat{n}_k$; therefore, it is a real WGP with variance

$$\sigma_u^2 = \sigma_{\hat{n}}^2/2 = \sigma_n^2/(2A^2). \tag{5}$$

The argument of x_k is given by

$$arg(x_k) = \omega_0 k + \theta + u_k, \; k = 0, 1, \cdots, N-1. \tag{6}$$

To avoid phase unwrapping [1], Kay considered the differenced phase data

$$\begin{aligned} arg(x_k x_{k-1}^*) &= arg(x_k) - arg(x_{k-1}) \\ &= \omega_0 + u_k - u_{k-1}, \; k = 1, \cdots, N-1. \end{aligned} \tag{7}$$

The minimum variance unbiased estimator [2] for the linear model of (7) is the Kay estimator [1], as follows:

$$\hat{\omega}_0 = \sum_{k=1}^{N-1} w_k arg(x_k x_{k-1}^*) \tag{8}$$

where w_k is a proper weighting constant.

Since u_k is a noise term after taking the nonlinear arg( ) function, (5) is not valid at lower SNR values. For example, when $\hat{n}_k = -1.01 + 0j$, though the imaginary part of $\hat{n}_k$ is zero, $|u_k| = \pi$ which is large and results in $\sigma_u^2 > \sigma_n^2/(2A^2)$. At low values of SNR, this situation occurs more often and eventually leads to the threshold effect in the Kay estimator. One way to reduce the threshold is to boost the SNR before taking the arg() function so that (5) is valid for lower values of the SNR. We will show how this can be achieved in the next section.

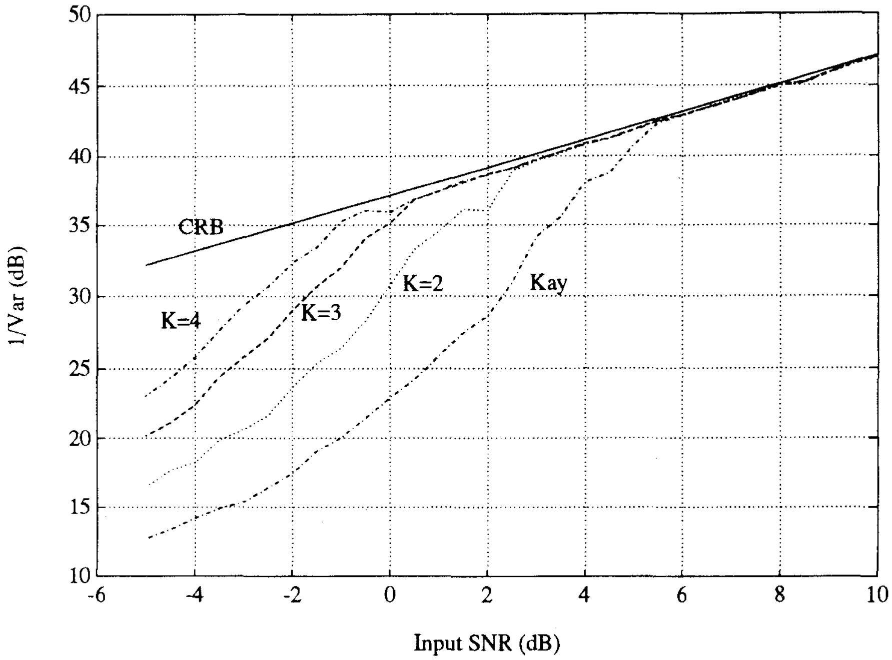

Fig. 1. Performance of the Kay estimator (Kay) and the proposed estimator for different values of the design parameter K. CRB is the Cramer–Rao bound.

III. Derivation of the Proposed Estimator

We first show how to achieve 3 dB less threshold than the Kay estimator and then generalize the method. We define the average value a_k as

$$a_k = (x_k + x_{k-1})/2,\ k = 1, \cdots, N-1. \tag{9}$$

After some manipulation, it can be shown that

$$a_k = Ae^{j(\omega_0(k-1/2)+\theta)}(cos(\frac{\omega_0}{2}) + \frac{\tilde{n}_k + \tilde{n}_{k-1}}{2}) \tag{10}$$

where $\tilde{n}_k$ is a complex WGP whose variance is σ_n^2/A^2. Note that the variance of $\frac{\tilde{n}_k+\tilde{n}_{k-1}}{2}$ is 3 dB less than $\hat{n}_k$ of (2). The argument of a_k is given by

$$arg(a_k) = (k-1/2)\omega_0 + \theta + arg(cos(\frac{\omega_0}{2}) + \frac{\tilde{n}_k + \tilde{n}_{k-l}}{2})$$
$$k = 1, \cdots, N-1. \tag{11}$$

At high input SNR, this can be written as

$$arg(a_k) = (k-1/2)\omega_0 + \theta + \frac{u_k + u_{k-1}}{2cos(\omega_0/2)},\ k = 1, \cdots, N-1 \tag{12}$$

where u_k is the imaginary part of $\tilde{n}_k$. This can be viewed as a real WGP with variance

$$\sigma_u^2 = \sigma_n^2/(2A^2). \tag{13}$$

To avoid phase unwrapping, we consider the differenced phase data y_k

$$y_k = arg(a_k a_{k-1}^*) = arg(a_k) - arg(a_{k-1})$$
$$= \omega_0 + \frac{u_k - u_{k-2}}{2cos(\omega_0/2)},\ k = 2, \cdots, N-1. \tag{14}$$

In matrix form

$$\mathbf{y} = \mathbf{1}\omega_0 + \frac{1}{2cos(\omega_0/2)}\mathbf{Pu} \tag{15}$$

where

$$\mathbf{y} = (y(N-1)\ y(N-1)\ \cdots\ y(2))^t \tag{16}$$
$$\mathbf{u} = (u(N-1)\ u(N-2)\ \cdots\ u(0))^t \tag{17}$$
$$\mathbf{1} = (1\ 1\ \cdots 1)^t \tag{18}$$
$$[\mathbf{P}]_{il} = \begin{cases} 1, & \text{if } l = i \\ -1, & \text{if } l = i+2 \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

The minimum variance unbiased estimator and its estimation variance for the linear model of (15) is well known [2], and is given by

$$\hat{\omega}_0 = \frac{\mathbf{1}^t\mathbf{C}^{-1}\mathbf{y}}{\mathbf{1}^t\mathbf{C}^{-1}\mathbf{1}} \tag{20}$$

$$var(\hat{\omega}_0) = \frac{1}{\mathbf{1}^t\mathbf{C}^{-1}\mathbf{1}} \tag{21}$$

where

$$\mathbf{C} = \frac{1}{4cos^2(\omega_0/2)}\frac{\sigma_n^2}{2A^2}\mathbf{PP}^t. \tag{22}$$

The final form of this estimator is

$$\hat{\omega}_0 = \sum_{k=2}^{N-1} w_k y_k = \sum_{k=2}^{N-1} w_k arg(a_k a_{k-1}^*) \tag{23}$$

where w_k is a weighting constant derived from (20) and a_k is defined in (9).

It can be seen from (2) and (10) that the SNR before taking the arg() function for the proposed estimator is 3 dB

TABLE I
THRESHOLD, ESTIMATION RANGE, NUMBER OF MULTIPLICATIONS REQUIRED FOR KAY, FITZ, L&R, AND THE PROPOSED ESTIMATOR

estimator	threshold	estimation range	multiplications	multiplications to achieve CRB
Kay	$6dB$	π	$2(N-1)$	$2(N-1)$
Fitz	$< 0dB$	$< \frac{\pi}{M}$	$M(N-(M-1)/2)$	$\frac{3}{8}N^2$
L&R	$< 0dB$	$< \frac{2\pi}{M+1}$	$M(N-(M-1)/2)$	$\frac{3}{8}N^2$
proposed	$(6-20logK)dB$	$< \frac{\pi}{K}$	$2(N-K)$	$2(N-K)$

higher than that of the Kay estimator for the same input SNR. Therefore, the proposed estimator experiences the threshold effect at 3 dB lower input SNR than the Kay estimator.

The problem with the proposed estimator is that its performance degrades as $|\omega_0|$ gets large due to the $cos(\omega_0/2)$ term in (22). However, this is true for other estimation methods (Fitz and L&R) that achieve better threshold than the Kay estimator. In fact, the proposed estimator has less degradation as $|\omega_0|$ gets large than the Fitz or L&R estimators. The proposed estimator has less than 0.1 dB loss from CRB until $\omega_0 = 0.3$, independent of N, while the Fitz and the L&R estimators have more than 3 dB loss at the same ω_0 when $N = 10$, and get worse at high values of N.

We can generalize the proposed estimator to achieve more threshold gain. We define the average value a_k as

$$a_k = \frac{1}{K}\sum_{m=0}^{K-1} x_{k-m}, \; k = K-1, \cdots, N-1. \tag{24}$$

We do not show the detailed derivation for general K, since it is a straightforward extension to the case $K = 2$. The final form of this estimator is

$$\hat{\omega}_0 = \sum_{k=K}^{N-1} w_k arg(a_k a_{k-1}^*) \tag{25}$$

where w_k is a weighting constant that can be derived as was done for the $K = 2$ case. As we have averaged K input samples before taking the arg() function, the SNR to the input of the arg() function is $20log(K)$ dB better than the Kay estimator. Therefore, its threshold is $20log(K)$ dB lower than the Kay estimator. One obvious question is: Can we increase K up to $N-1$ to achieve better threshold? The answer is no. As K gets large, the proposed estimator does not achieve CRB even at high SNR. This is due to the fact that some information has been lost through the averaging operation in (24). Using simulation, we have found that for $N > 24$, the proposed estimator with $K = 2$, 3, and 4 gives less than 0.2 dB degradation from the CRB. It should be noted that division by K in (24) is not needed, as we only use the argument of a_k.

IV. COMPUTER SIMULATIONS

A computer simulation was performed to compare the performance of the Kay estimator and the proposed estimator. A data record of $N = 25$ points was chosen. Fig. 1 shows the inverse of $var(\omega_0)$ versus input SNR for $\omega_0 = 0.04\pi$. As predicted by the derivation, the threshold of the proposed estimator is $20log_{10}K$ dB less than that of the Kay estimator. The Kay estimator can be thought as a special case of the proposed estimator with $K = 1$.

V. CONCLUSIONS

We have proposed a new single-frequency estimator that can achieve a lower threshold than the Kay estimator, and is as computationally efficient. The threshold is shown to be $20log_{10}K$ dB less than the Kay estimator, and can easily be used to achieve the threshold of 0 dB ($K = 4$).

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