Fast and accurate single frequency estimator

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A new fast and accurate method for estimating the frequency of a complex sinusoid in complex white Gaussian noise is proposed. The proposed estimator has the lowest computational complexity among the estimators. Its performance and estimation range are similar to the Fitz and L&R estimators, and its threshold is 20 log(m) less than the Kay estimator.

Introduction: The problem of estimating the frequency of a single complex sinusoid in additive white Gaussian noise (AWGN) arises in many applications. Signal interception and detection, carrier synchronisation and other digital communication measurements are examples of applications requiring rapid frequency estimation. The observed discrete-time signal of this problem can be described as

$$x_k = Ae^{j(\omega_0 k+\theta)} + n_k, k = 0, 1, \ldots, N-1 \tag{1}$$

where amplitude A, frequency ω_0 and phase θ are deterministic but unknown constants. The noise n_k is assumed to be a complex white Gaussian process with zero-mean and variance σ_n^2. The maximum likelihood (ML) solution is well known and involves locating the peak of the periodogram [1]. However, this approach requires far too many computations. Several fast and accurate frequency estimators have been reported [2–4]. Subsequently, Kim *et al.* [5] designed a fast estimator by signal averaging, which achieves lower threshold than the Kay estimator. Leung *et al.* [6] proposed the use of the differenced phase of autocorrelation functions that gives better frequency estimates than Kay's method. In [7], frequency estimation with wider estimation range is achieved using an iterative algorithm.

In this Letter, we propose a new frequency estimator for a single complex sinusoid. The proposed estimator is computationally efficient with performance comparable to the existing estimators. It has the same threshold and estimation range as, but much less computational complexity than, the Fitz and L&R estimators.

Derivation of proposed estimator: It has been shown in [2] that the signal could be approximated by the following expression:

$$s_k = Ae^{j(w_0 k+\theta+u_k)}, \quad 0 \le k \le N-1 \tag{2}$$

where $\{u_k\}$ is a sequence of statistically independent, identically distributed zero-mean real Gaussian with variance $\sigma_n^2/2A^2$. The argument of s_k is given by

$$\arg(s_k) = w_0 k + \theta + u_k, \quad 0 \le k \le N-1 \tag{3}$$

To avoid phase unwrapping, Kay [2] employed differenced phase data, and Fitz [3] and L&R [4] used the sample autocorrelation function to develop their frequency estimators.

In the proposed approach, we partition the observation sequence into three sections: $\{s_k\}_0^{2m-1}$, $\{s_k\}_{2m}^{N-2m-1}$ and $\{s_k\}_{N-2m}^{N-1}$, and develop the ML estimator based on signals in the first and third sections. Assume that $|w_0| < \pi/m$, m is a design parameter that satisfies $0 < 4m \le N$. A new observation vector $\mathbf{B}$ is defined as follows:

$$\mathbf{B} = (B_1, B_2, B_3)^T \tag{4}$$

where

$$B_1 = \arg\left\{\left(\sum_{k=0}^{m-1} s_{N-2m+k}\right)\left(\sum_{k=0}^{m-1} s_k\right)^*\right\} = (N-2m)w_0 + 2\pi K_1 + v_1 \tag{4a}$$

$$B_2 = \arg\left\{\left(\sum_{k=0}^{m-1} s_{N-m+k}\right)\left(\sum_{k=0}^{m-1} s_{m+k}\right)^*\right\} = (N-2m)w_0 + 2\pi K_1 + v_2 \tag{4b}$$

$$B_3 = \arg\left\{\left(\sum_{k=0}^{m-1} s_{N-m+k}\right)\left(\sum_{k=0}^{m-1} s_k\right)^*\right\} = (N-m)w_0 + 2\pi K_2 + v_3 \tag{4c}$$

It can be shown that, at high input SNR values, v_1, v_2 and v_3 are real zero-mean Gaussian with variance

$$\sigma_V^2 = \frac{m\sigma^2 \sin^2(w_0/2)}{A^2 \sin^2(mw_0/2)} \tag{5}$$

The parameters K_1 and K_2 are two unknown, deterministic integers. We can obtain the covariance matrix of the vector $\mathbf{B}$:

$$\textstyle\sum_{\mathbf{B}} = \sigma_V^2 \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \tag{6}$$

The frequency estimation problem is then equivalent to the estimation of the mean of the observation vector $\mathbf{B}$, and the ML estimator is obtained by minimising the following form:

$$Q(\omega_0) = (\mathbf{B} - 2\pi\mathbf{K} + \omega_0\mathbf{p})^T(\textstyle\sum_{\mathbf{B}})^{-1}(\mathbf{B} - 2\pi\mathbf{K} + \omega_0\mathbf{p}) \tag{7}$$

where

$$\mathbf{K} = (K_1, K_1, K_2)^T$$

and

$$\mathbf{p} = (N-2m, N-2m, N-m)^T$$

By setting the derivative of $Q(\omega_0)$ to zero, we obtain the ML estimator of ω_0 as

$$\hat{\omega}_0 = \frac{(\mathbf{B} - 2\pi\mathbf{K})(\sum_{\mathbf{B}})^{-1}\mathbf{p}^T}{\mathbf{p}^T(\sum_{\mathbf{B}})^{-1}\mathbf{p}} = \frac{(N-3m)(B_1 + B_2 - 4\pi K_1) + 2m(B_3 - 2\pi K_2)}{2(N^2 - 4mN + 5m^2)} \tag{8}$$

For correct frequency estimation and to avoid phase unwrapping, we consider $|c| < \pi$, $c = B_3 - (B_1 + B_2)/2$ and obtain the closed form expression

$$2\pi K_1 + k = \frac{B_1 + B_2 + B_3}{3} - \frac{3N - 5m}{3m}c \tag{9}$$

where k is a zero-mean real Gaussian with variance

$$\sigma_k^2 = \frac{N^2 - 4mN + 5m^2 + m - 1}{8\pi^2 m(m^2 + m - 1)A^2} \tag{10}$$

Then K_1 can be estimated by

$$\hat{K}_1 = round(K_1 + k/2\pi) \tag{11}$$

where $round(x)$ rounds x towards the nearest integer, and $\hat{K}_2 = \hat{K}_1$. Note that the estimation range is enlarged from $\pi/(N-m)$ to π/m.

Performance analysis: From (4) and (8), it can be shown that the variance of the estimated frequency $\hat{\omega}_0$ is

$$Var(\hat{\omega}_0) = \frac{1}{4(N^2 - 4mN + 5m^2)^2} Var\{(N-3m)(B_1 + B_2) + 2mB_2\} = \frac{m\sigma_V^2}{2(N^2 - 4mN + 5m^2)} \tag{12}$$

It is also known that the CRBs for unbiased joint frequency and phase estimation are given by the diagonal elements of the inverse of Fisher's information matrix.

The CRB for the unbiased frequency estimation is

$$\vartheta(N) = \frac{6\sigma_n^2}{N(N^2 - 1)A^2} \tag{13}$$

The SNR degradation compared to the Kay estimator is evaluated by the variance ratio given as

$$E = 10\log\left(\frac{Var(\hat{\omega}_0)}{\vartheta(N)}\right)\Big|_{\omega_0 \to 0} \simeq 10\log\left(\frac{1 - 1/N^2}{3\alpha(1 - \alpha + 0.3125\alpha^2)}\right) \tag{14}$$

where $\alpha = 4m/N$, $0 < \alpha \le 1$, stands for the data utilisation rate. At $\alpha = 1$, (14) yields

$$E|_{\omega_0\to 0,\alpha=1} \simeq 10\log\left(\frac{16}{15}-\frac{1}{m^2}\right) \simeq 0.28\,\text{dB} \quad (15)$$

This shows that the proposed estimator has 0.28 dB loss from CRB. In comparison, the Fitz and L&R estimators have more than 3 dB loss, while the loss is 0.51 dB by [6] and 0.1 dB by [5].

It can be seen from (14) that the smaller the α, the more computationally efficient the estimator is, but at the expense of performance degradation. Also note that the parameter m is important to the estimator performance. When N is fixed, larger m results in better estimation, but requires more computations. The threshold, estimation range and computational complexity for different single frequency estimators are shown in Table 1.

Table 1: Threshold, estimation range and computational requirements of various frequency estimators and proposed estimator

Estimator	Threshold (dB)	Estimation range	Multiplications	Additions	Angle operation
Kay	6	π	$2(N-1)$	$N-1$	$N-1$
Fitz	<0	$<\pi/M$	$M(2N-M+1)/2$	$M(N-(M-1)/2)$	M
L&R	<0	$<\pi/(M+1)$	$M(2N-M-1)/2+1$	$M(N-(M-1)/2)$	1
[5]	6–20 log(K)	$<\pi/K$	$2(N-K)$	$(N-K)(K+1)$	$N-K$
[6]	1	$<\pi$	$N(N+1)/2$	$N(N+1)/2$	1
[7]	−2	$<\pi$	$(N-1)\log_2 N-3$	$\log_2 N-1$	$\log_2 N-1$
Proposed	6–20 log(m)	$<\pi/m$	6	$4m+6$	3

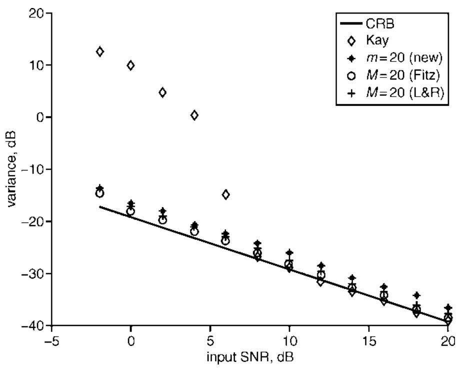

Fig. 1 *Performance comparison of different estimators for $w_0=0.04\pi$*

Computer simulations were carried out to compare the performance of the proposed estimators and the Kay, Fitz, and L&R estimators. Fig. 1 shows the frequency variance $Var(\hat{w}_0)$ against input SNR for $w_0=0.04\pi$ with data length $N=80$ and $m=20$. The threshold of the proposed estimator is 20 log(m) dB less than that of the Kay estimator; and its performance is comparable to the Fitz and L&R estimators. Since the proposed estimator has the least computational complexity among these estimators, it is more suitable for real-time applications.

Conclusion: We have presented a new single frequency estimator that can achieve a lower threshold than the Kay estimator, and is more computationally efficient than the existing estimators. The drawback of the proposed estimator, as well as the Fitz and L&R estimators, is a reduced estimation range.

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