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Direct collocation.

Two discretization schemes for trajectory optimization

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Abstract

Two discretization schemes for optimal control methods, collocation and the method of differential inclusions, are described. They replace the original infinite dimensional problem with a finite-dimensional approximation and allow the use of ordinary nonlinear optimization. The schemes are applied to aircraft trajectory optimization problems. Both methods seem to produce rapidly results that are accurate enough for most purposes. In addition, it was noted that unlike other direct methods, the method of differential inclusions is not seriously disturbed by singular controls.

Introduction

Since the early 60's the basic methods to solve optimal control problems have been the calculus of variations and the Pontryagin Maximum Principle. They provide a set of necessary conditions that a trajectory satisfies if it is optimal. The conditions constitute a nonlinear multipoint boundary value problem that is solved by methods like multiple shooting or quasilinearization. The methods that solve an optimal control problem via satisfying the necessary conditions are commonly referred to as indirect methods.

An indirect method provides an accurate result but also requires a good initial guess. This stems mainly from the nonlinear and unstable nature of the boundary value problem and Newton-type solution methods. In many cases the state trajectories may be easy to guess, but the adjoint variables, whose physical meaning is often obscure, are not.

To overcome the difficulties associated with the initial guess, alternative methods have been developed. Direct methods for infinite-dimensional problems start from a feasible nominal solution and develop it towards optimality by exploiting the cost function gradient. Unfortunately these methods succeed only partly in their goals. First order gradient methods show poor converge in the neighborhood of the optimal solution, whereas second order gradient methods again suffer from the small convergence radius. Also the treatment of state inequality constraints must in practice be based on penalty functions. A major drawback of the methods is the need to integrate the state and the cost function gradient equations numerous times during the solution process.

In many cases the accuracy of the solution is not as important as is the robust convergence of the solution method. The convergence could be improved by replacing the original infinite-dimensional problem with a finite-dimensional approximation, in which the differential equation constraint is satisfied only pointwise and integration of the system is totally avoided. Furthermore, restricting to a finite dimension allows the use of ordinary nonlinear optimization and its developed tools. Even state constraints may be simply added to the problem constraints.

The discretization of the problem can be carried out in a number of ways. For example, Hargraves et al. [4] present the state trajectories with high-order patched polynomials that satisfy the state equations in the least-squares sense. Betts and Huffman [2] discuss ordinary trapetzoidal, Hermite-Simpson and Runge-Kutta discretization. In the following we describe two schemes, direct collocation and a recently proposed method of differential inclusions [11]. The first scheme, direct collocation, relies on implicit integration. The solution is sought among piecewise defined polynomials that have to satisfy the differential equations in a finite set of collocation points. The controls and the coefficients of the polynomials are selected through nonlinear optimization to satisfy the state equations and to minimize the cost function. The method has been applied to various trajectory optimization tasks (see. e.g.,[6]) but also to facilitate the solving of complex pursuitevasion games (see [7]).

The second approach, the method of differential inclusions, relates to set valued analysis and differential inclusion (see, e.g., [1]). Optimal control problems and especially minimum time problems can be treated on the basis of what is called 'the set of attainability' (see, e.g., [5]). The set describes the states that can be reached from the given state within a given time interval. In modern framework the set of attainability provides an efficient way to discretize the infinite-dimensional problem. The differential equation constraint is replaced by a requirement that each state must be attainable from the previous state. According to the approach, the role of the controls is merely to parameterize the set of attainable states. The controls may therefore be suppressed from the optimization process by finding another, control independent way to represent the set. The method was first applied to trajectory optimization problems in [11].

In the report we will first briefly describe both methods. We then describe two test problems that are often encountered in aircraft trajectory optimization and apply the methods to them. We compare the accuracy of the resulting trajectories and the computational effort required by the methods. In addition, we compare the capability of the methods to solve problems involving singular controls by including a dynamic pressure constraint to the third test problem.

Direct collocation

Consider an optimal control problem of the Mayer form, hereafter referred to as P1,

$$
\begin{aligned}
\min \quad & \psi(x(T),T) \\
\text{subject to} \quad & \\
\dot{x}(t) &= f(x(t),u(t)) \\
x(0) &= x_{init} \\
x(T) &= x_{final} \\
C(x(t),u(t)) &\le 0 \\
S(x(t)) &\le 0, \quad t \in [0,T]
\end{aligned}
$$

where $x(t) \in R^n$, $u(t) \in R^u$, $f : R^n \times R^u \mapsto R^n$, $C : R^n \times R^u \mapsto R^c$ and $S : R^n \mapsto R^s$. Possible explicit time dependence of $f(\cdot)$ may be suppressed with a new independent variable and problems of Bolza type, i.e. with integral cost functional, can be turned into Mayer form by adding a new state variable. The final time T may be fixed or free.

In the method of direct collocation, the finite dimensional solution subspace is the space of piecewise polynomials of time and given degree, defined in the interval $t \in [0,T]$. We use Hermite interpolation with 3rd degree polynomials for the state variables and linear polynomials for the control variables. The state equation must be satisfied in the middle of each interval.

For simplicity, consider an equidistant division of the solution interval

$$t_j = j\frac{T}{m} := j\Delta t, \quad j = 0,\dots,m.$$

In the jth subinterval we seek state component trajectories of the form

$$x_{ij}(t) = a + bt + ct^2 + dt^3, \quad t \in [t_{j-1}, t_j]. \tag{1}$$

We hereafter drop the subscripts i and j for clarity. Introducing a new transformed time variable

$$\tau := \frac{t - t_{j-1}}{\Delta t}$$

and differentiating expression (1) with respect to τ yields the following system of equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x(0) \\ \dot{x}(0) \\ x(1) \\ \dot{x}(1) \end{pmatrix}.$$

The independent variable is τ and ($\dot{}$) means differentiation with respect to τ. Evaluating (1) at $\tau = 1/2$ and substituting the coefficients solved from the above system of equations leads to

$$x(1/2) = \frac{x(0) + x(1)}{2} + \Delta t \frac{f(0) - f(1)}{8},$$

where $f(\tau)$ is an abbreviation of $f_i(x(\tau), u(\tau))$ and refers to the corresponding state equation component. Note that $\frac{dx}{d\tau} = \Delta t \frac{dx}{dt}$. In the same way we obtain the expression for $\dot{x}(1/2)$:

$$\dot{x}(1/2) = -3\frac{x(0) - x(1)}{2\Delta t} - \frac{f(0) + f(1)}{4}.$$

Using the expression for $x(1/2)$ and linear interpolation of the controls, $f(1/2)$ may be calculated. Define the defect at the center of the interval j as

$$\Delta_j := \dot{x}(1/2) - f(1/2).$$

When the values of the state variables at the ends of the interval are chosen such that the defect is driven to zero, the cubic provides an approximation of the state component trajectory without explicit integration. The controls at the time points may now be selected freely within their bounds to minimize the objective function, as far as the constraints $\Delta_j = 0$, initial and terminal constraints, and possible state constraints are satisfied. Thus the inifinite dimensional optimal control problem P1 may be approximated by an ordinary finite dimensional nonlinear optimization problem

$$\begin{aligned}
&\min_{(x_0, x_1, \ldots, x_m, u_0, u_1, \ldots, u_m; T)} \quad \psi(x_m, T)\\
&\text{subject to}\\
&\begin{aligned}
\Delta_j &= 0,\; j = 1, \ldots, m\\
x_0 &= x_{init}\\
x_m &= x_{final}\\
S(x_j) &\leq 0,\; j = 0, \ldots, m\\
C(x_j, u_j) &\leq 0,\; j = 0, \ldots, m\\
-T &\leq 0.
\end{aligned}
\end{aligned}$$

Here x_j refers to state vector x at the time instant t_j. The state constraints may be satisifed only pointwise, since the differential equations are satisfied only in the middle points of the segments. If violations occur, the time division should be made denser to suppress them.

Applying direct collocation leads to a nonlinear optimization problem where the number of the decision variables is $(n + u)(m + 1) + 1$ when the final time is free. The number of constraints amounts to $nm + (n_{cieq} + n_{sieq})(m + 2) + n_{init} + n_{final}$, where n_{cieq}, n_{sieq} refer to the number of control and state inequality constraints and n_{init} and n_{final} to the number of initial and final conditions, respectively. The nonlinearity of the collocation constraints

depends on the state equations. Some of the state and control variable constraints may be simple bounds.

Differential inclusion

Another way to discretize the problem is to require that each subsequent state can be attained from the preceding state. Given t_0, $x(t_0) = x_0$ and t_1, the *set of attainability* $K(x_0, t_0, t_1)$ is defined as the collection of the states that can be reached from x_0 in $[t_0, t_1]$ with admissible controls $u(t)$, $t \in [t_0, t_1]$ [5]. In general, this set cannot be expressed explicitly. To approximate it we use the *set of attainable state rates* at state x(t), which is defined as the set of all the state rates that can be produced in a given state by varying the controls within their allowable bounds. The set is defined as

$$\mathcal{H}(x(t)) = \{\dot{x}(t) \in R^n \mid \dot{x}(t) = f(x(t), u(t)),\ C(x(t), u(t)) \leq 0\}.$$

In the following, we drop the argument t for clarity. The set $\mathcal{H}(x)$ is sometimes called the ***hodograph*** of the system. The control u may be regarded as a parameter vector that describes the hodograph. We may therefore assume that u can be eliminated from the definition above. That is, there exist smooth functions $p : R^n \times R^n \mapsto R^p$ and $q : R^n \times R^n \mapsto R^q$ such that the hodograph can be expressed as

$$\mathcal{H}(x) = \{\dot{x} \in R^n \mid p(\dot{x}, x) = 0, q(\dot{x}, x) \leq 0\}.$$

The existence of such functions depends on the system under consideration. In practice, they are derived by eliminating the controls from the state equations and then using the control constraints.

The shape of the hodograph plays a significant role when examining the type of possible solutions. In terms of necessary conditions the pointwise maximization of the Hamiltonian with respect to controls is equal to maximizing the Hamiltonian with respect to state rates that are constrained to lie in the hodograph. In the space of the state rates the Hamiltonian is defined as

$$H = p^T \dot{x},$$

where p stands for the adjoint vector. Therefore we are faced with a series of linear optimization problems with nonlinear constraints. If $\mathcal{H}(x)$ is always strictly convex, the