Problem 1: Design of a PI current controller

The parameters of a DC motor are $R_a = 0.87\ \Omega$ and $L_a = 16$ mH. An ordinary PI current controller is used. The current-control bandwidth is required to be $\alpha_c = 2\pi \cdot 300$ rad/s.

(a) Derive the expressions for the controller gains according to the principle of internal model control.

(b) Derive the expressions for the controller gains by cancelling the pole of the open-loop system and by requiring that the 0-dB crossover angular frequency of the loop transfer function is α_c.

Solution

(a) The block diagram of the current-controlled system is shown in the figure below. The converter is assumed to be ideal: $u_a = u_{a,\text{ref}}$. The back-emf e_a is assumed to be quasi-constant. The PI controller and the admittance of the DC motor are

$$K(s) = k_p + \frac{k_i}{s} \qquad Y_a(s) = \frac{1}{sL_a + R_a}$$

respectively.
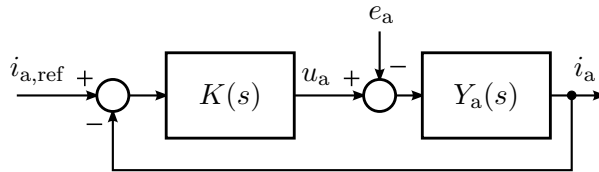
The closed-loop transfer function is

$$\frac{i_a(s)}{i_{a,\text{ref}}(s)} = H(s) = \frac{K(s)Y_a(s)}{1 + K(s)Y_a(s)} \tag{1}$$

and the desired closed-loop transfer function is

$$H(s) = \frac{\alpha_c}{s + \alpha_c} = \frac{\alpha_c/s}{1 + \alpha_c/s} \tag{2}$$

where $\alpha_c = 2\pi \cdot 300$ rad/s $= 1885$ rad/s. Equalling (1) and (2) gives the condition

$$K(s)Y_a(s) = \frac{\alpha_c}{s}$$

from which the controller transfer function can be solved:

$$K(s) = \frac{\alpha_c}{sY_a(s)} = \frac{\alpha_c}{s}(sL_a + R_a) = \alpha_c L_a + \frac{\alpha_c R_a}{s}$$

Hence, the PI controller gains are

$$k_p = \alpha_c L_a = 1885\ \text{rad/s} \cdot 0.016\ \text{H} = 30.2\ \text{V/A}$$
$$k_i = \alpha_c R_a = 1885\ \text{rad/s} \cdot 0.87\ \Omega = 1640\ \text{V/(As)}$$

(b) The loop transfer function is

$$L(s) = K(s)Y_a(s) = \frac{k_p s + k_i}{s} \frac{1}{sL_a + R_a} = \frac{s + k_i/k_p}{s/k_p} \frac{1/L_a}{s + R_a/L_a} \tag{3}$$

In order to cancel the pole of the open-loop system with the controller zero, the condition

$$s + \frac{k_i}{k_p} = s + \frac{R_a}{L_a} \qquad \Rightarrow \qquad \frac{k_i}{k_p} = \frac{R_a}{L_a} \tag{4}$$

should hold. Using this condition in (3) gives the loop transfer function

$$L(s) = \frac{k_p}{sL_a}$$

The loop-transfer function is required to have the gain of 0 dB (the unity gain) at the crossover angular frequency α_c:

$$|L(j\alpha_c)| = \left| \frac{k_p}{j\alpha_c L_a} \right| = 1$$

The proportional gain can now be solved:

$$k_p = \alpha_c L_a = 1885 \text{ rad/s} \cdot 0.016 \text{ H} = 30.2 \text{ V/A}$$

Based on (4), the integral gain is

$$k_i = \frac{R_a}{L_a} k_p = \alpha_c R_a = 1885 \text{ rad/s} \cdot 0.87 \ \Omega = 1640 \text{ V/(As)}$$

The result is equal to the result obtained in Part (a).

Remark: If needed, the gain and phase margins can be determined from $L(j\omega)$. In this design example, the controller gains led to the loop-transfer function $L(s) = \alpha_c/s$. As an example, the phase margin ϕ is

$$\phi = 180° + \underline{/L(j\alpha_c)} = 180° + \underline{/\frac{\alpha_c}{j\alpha_c}} = 90°$$

Problem 2: Waveforms in a balanced three-phase system

Sketch the waveforms of balanced three-phase voltages on the squared paper (or the grid below).
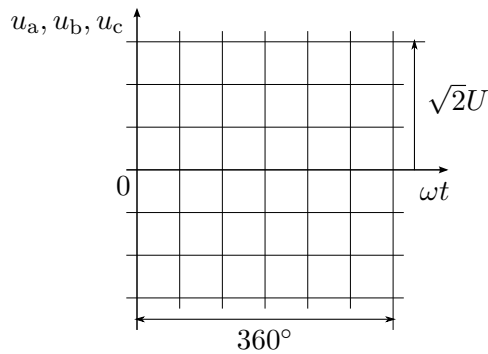

Hint: It is convenient to use a 6×6 grid to draw these waveforms. You can fist mark the points corresponding to zero crossings and peak values. It is also worth noticing that $\sin(\pi/6) = 1/2$ and mark these points on the grid.

Solution

A balanced set of three-phase sinusoidal AC voltages is

$$u_\mathrm{a} = \hat{U}\sin(\omega t) \qquad u_\mathrm{b} = \hat{U}\sin(\omega t - 2\pi/3) \qquad u_\mathrm{c} = \hat{U}\sin(\omega t - 4\pi/3)$$

The peak value of the voltage $\hat{U} = \sqrt{2}U = \sqrt{2/3}U_\mathrm{LL}$, where U is the rms line-to-neutral voltage and U_LL is rms line-to-line voltage. The waveforms are shown in the figure.
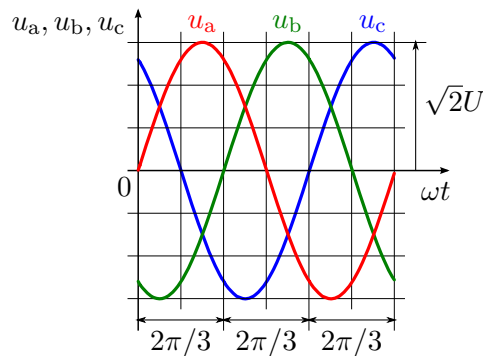
$u_\mathrm{a} = \sqrt{2}U\sin(\omega t)$
$u_\mathrm{b} = \sqrt{2}U\sin(\omega t - 2\pi/3)$
$u_\mathrm{c} = \sqrt{2}U\sin(\omega t - 4\pi/3)$

## Problem 3: Power in single-phase and three-phase systems

The purpose of this problem is to demonstrate that the instantaneous power in a balanced three-phase system is constant in steady state.

(a) A single-phase load is fed with the voltage $u(t) = \sqrt{2}U \sin(\omega t)$ and it draws the current $i(t) = \sqrt{2}I \sin(\omega t - \varphi)$. Derive the expressions for the instantaneous power and the average power.
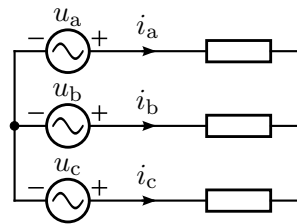
Hint: The trigonometric product-to-sum identity may be useful:

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

(b) A balanced three-phase load is fed with balanced three-phase voltages. Derive the expressions for the instantaneous power and the average power.
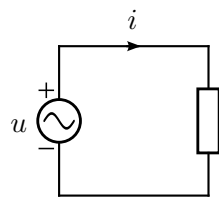
(a)   (b)

Solution

(a) The voltage and the current are

$$u = \sqrt{2}U \sin(\omega t) \qquad i = \sqrt{2}I \sin(\omega t - \varphi)$$

Using the given trigonometric identity, the instantaneous power becomes

$$\begin{aligned} p &= ui \\ &= 2UI \sin(\omega t)\sin(\omega t - \varphi) \\ &= UI \cos\varphi - UI \cos(2\omega t - \varphi) \end{aligned}$$

The second term is a second-harmonic component, i.e., the instantaneous power oscillates at twice the supply frequency. The first term is the average power

$$P = \frac{1}{T}\int_0^T p(t)\mathrm{d}t = UI \cos\varphi$$

where $T = 2\pi/\omega$ is the fundamental period.

(b) The phase voltages are

$$u_\mathrm{a} = \sqrt{2}U \sin(\omega t) \qquad u_\mathrm{b} = \sqrt{2}U \sin(\omega t - 2\pi/3) \qquad u_\mathrm{c} = \sqrt{2}U \sin(\omega t - 4\pi/3)$$

where U is the phase-to-neutral rms voltage. The phase currents are

$$i_\mathrm{a} = \sqrt{2}I \sin(\omega t - \varphi) \qquad i_\mathrm{b} = \sqrt{2}I \sin(\omega t - \varphi - 2\pi/3) \qquad i_\mathrm{c} = \sqrt{2}I \sin(\omega t - \varphi - 4\pi/3)$$

Instantaneous powers in each phase are

$$p_a = u_a i_a = UI \cos \varphi - UI \cos(2\omega t - \varphi)$$
$$p_b = u_b i_b = UI \cos \varphi - UI \cos(2\omega t - \varphi - 4\pi/3)$$
$$p_c = u_c i_c = UI \cos \varphi - UI \cos(2\omega t - \varphi - 2\pi/3)$$

The total instantaneous power

$$p = p_a + p_b + p_c = 3UI \cos \varphi = P$$

equals the average power P, i.e., the second-harmonic components cancel out due their 120° phase difference. The power flow of a balanced three-phase system is smooth (unlike that of the single-phase system).

Problem 4: Transmission losses

The figure below shows two different configurations to feed the same power to the resistive load:

(a) The balanced three-phase supply feeds the three resistors R_{load}. The resistance of the transmission wires is R.

(b) The single-phase supply, taken from one phase of the previous three-phase supply, feeds the load resistor $R_{\text{load}}/3$. The same transmission wires are also used as in the previous case.

The resistance R of the wires can be assumed to be much smaller than the resistance R_{load}. Compare the transmission losses in these two cases.
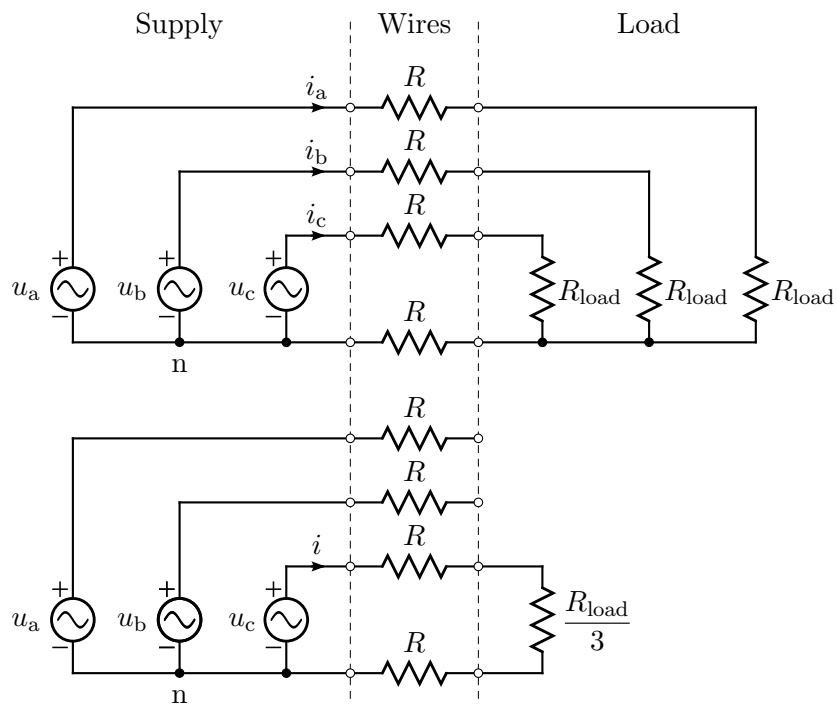
Solution

(a) Let us fist consider the three-phase system. Since the system is balanced, the sum of the phase currents is zero, and, therefore, the current in the neutral wire is also zero.

We can consider just one phase. The rms phase current is

$$I = \frac{U}{R + R_{\text{load}}} \approx \frac{U}{R_{\text{load}}}$$

where U is the rms phase voltage. The total load power in the three phases is

$$P_{\text{load}} = 3R_{\text{load}}I^2 = \frac{3U^2}{R_{\text{load}}}$$

The total transmission losses are

$$P_{\text{d}} = 3RI^2 = 3R\frac{U^2}{R_{\text{load}}^2} \tag{5}$$

(b) Let us next consider the single-phase system. The rms current is

$$I = \frac{U}{2R + R_{\text{load}}/3} \approx \frac{3U}{R_{\text{load}}}$$

The load power equals that in the tree-phase system

$$P_{\text{load}} = \frac{R_{\text{load}}}{3}I^2 = \frac{3U^2}{R_{\text{load}}}$$

The transmission losses are

$$P_{\text{d}} = 2RI^2 = 2R\frac{9U^2}{R_{\text{load}}^2} = 18R\frac{U^2}{R_{\text{load}}^2} \tag{6}$$

Comparing (5) and (6) shows that the transmission losses in the single-phase system are six times those in the three-phase system.