Random graphs and network statistics

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Appendix A

Probability

Here are some miscellaneous facts from probability theory that are used in the text.

A.1 Inequalities

Proposition A.1.1 (Markov's inequality)**.** *For any random number* $X \geq 0$ *and any* $a > 0$*,*

$$\mathbb{P}(X \geq a) \;\leq\; a^{-1}\mathbb{E}X.$$

Proof. First, note that $\mathbb{P}(X \geq a) = \mathbb{E}1(X \geq a)$ where $1(A)$ in general denotes the *indicator* of the event A. Hence by the linearity of expectation,

$$a\mathbb{P}(X \geq a) \;=\; a\mathbb{E}1(X \geq a) \;=\; \mathbb{E}a1(X \geq a).$$

Next, the inequalities

$$a1(X \geq a) \;\leq\; X1(X \geq a) \;\leq\; X$$

which are valid for any realization of X, and the monotonicity of the expectation imply that

$$\mathbb{E}a1(X \geq a) \;\leq\; \mathbb{E}X.$$

Hence $a\mathbb{P}(X \geq a) \leq \mathbb{E}X$, and the claim follows. □

Proposition A.1.2 (Chebyshev's inequality)**.** *For any random number* X *with a finite mean* $\mu = \mathbb{E}X$ *and any* $a > 0$*,*

$$\mathbb{P}(|X - \mu| \geq a) \;\leq\; a^{-2}\,\mathrm{Var}(X).$$

Proof. By applying Markov's inequality for $Y = (X-\mu)^2$, we find that

$$\begin{aligned}\mathbb{P}(|X-\mu| \ge a) &= \mathbb{P}((X-\mu)^2 \ge a^2) \\ &\le (a^2)^{-1}\mathbb{E}(X-\mu)^2 = a^{-2}\operatorname{Var}(X).\end{aligned}$$

□

The following inequality is due to the Finnish-born Wassily Hoeffding.

Proposition A.1.3. *Let $S_n = \sum_{i=1}^n X_i$ where the summands are independent and bounded by $a_i \le X_i \le b_i$. Then for any $t > 0$,*

$$\mathbb{P}(S_n \ge \mathbb{E}S_n + t) \le e^{-\frac{2t^2}{\sum_i (b_i-a_i)^2}},$$

$$\mathbb{P}(S_n \le \mathbb{E}S_n - t) \le e^{-\frac{2t^2}{\sum_i (b_i-a_i)^2}},$$

and

$$\mathbb{P}(|S_n - \mathbb{E}S_n| \ge t) \le 2e^{-\frac{2t^2}{\sum_i (b_i-a_i)^2}}.$$

Proof. A well-written proof of the first inequality, based on an extremality property related to convex stochastic orders, is available in the original research article Hoeffding [5]. The second inequality follows by applying the first inequality to $\tilde{S}_n = -S_n$ and the third inequality follows from the first two by the union bound. □

A.2 Weak convergence of probability measures

Let $\mu, \mu_1, \mu_2, \dots$ be probability distributions on $\mathbb{R}$. We say that $\mu_n \to \mu$ *weakly* if $\int \phi(x)\mu_n(dx) \to \int \phi(x)\mu(dx)$ for every bounded continuous function $\phi : \mathbb{R} \to \mathbb{R}$. We say that $\mu_n \to \mu$ *weakly and with k-th moments*, if in addition μ_n and μ have finite k-th moments and $\int |x|^k \mu_n(dx) \to \int |x|^k \mu(dx)$. The sequence (μ_n) is called *uniformly integrable* if $\sup_n \int |x| \mu_n(dx) 1(|x| > K) \to 0$ as $K \to \infty$. Let $X, X_1, X_2, \dots$ be real-valued random variables. We say that $X_n \to X$ weakly (resp. with weakly with k-th moments) if the corresponding probability distributions converge weakly (resp. weakly with k-th moments). We say that (X_n) is uniformly integrable if the collection of corresponding probability distributions is uniformly integrable.

Lemma A.2.1. *Let X_n and X be random numbers such that $X_n \to X$ weakly with 1st moments. Then the sequence (X_n) is uniformly integrable.*

Proof. Given $\epsilon > 0$, by Lebesgue's dominated convergence we may choose $K > 0$ such that $EX1(X > K) \leq \epsilon/3$. Then let ϕ_K be a continuous bounded function such that $\phi_K(x) = x$ for $x \leq K$ and $\phi_K = 0$ for $x \geq K + 1$. Then

$$x1(x \leq K) \ \leq \ \phi_K(x) \ \leq \ x1(x \leq K+1),$$

so that

$$\begin{aligned}
\mathbb{E}X_n 1(X_n > K+1) &= \mathbb{E}X_n - \mathbb{E}X_n 1(X_n \leq K+1) \\
&\leq \mathbb{E}X_n - \mathbb{E}\phi_K(X_n) \\
&= \mathbb{E}X_n - \mathbb{E}\phi_K(X) + \mathbb{E}\phi_K(X) - \mathbb{E}\phi_K(X_n) \\
&\leq \mathbb{E}X_n - \mathbb{E}X1(X \leq K) + \mathbb{E}\phi_K(X) - \mathbb{E}\phi_K(X_n) \\
&= \mathbb{E}X1(X > K) + \mathbb{E}X_n - \mathbb{E}X + \mathbb{E}\phi_K(X) - \mathbb{E}\phi_K(X_n) \\
&\leq \epsilon/3 + |\mathbb{E}X_n - \mathbb{E}X| + |\mathbb{E}\phi_K(X_n) - \mathbb{E}\phi_K(X)|.
\end{aligned}$$

Then we may choose n_0 so large that $|\mathbb{E}X_n - \mathbb{E}X| \leq \epsilon/3$ and $|\mathbb{E}\phi_K(X_n) - \mathbb{E}\phi_K(X)| \leq \epsilon/3$ for all $n > n_0$. Hence $\mathbb{E}X_n 1(X_n > K+1) \leq \epsilon$ for all $n > n_0$. Furthermore, for every $1 \leq m \leq n_0$ we may choose, again by Lebesgue's dominated convergence, K_m such that $\mathbb{E}X_m 1(X_m > K_m) \leq \epsilon$. Now if we choose $L = \max\{K+1, K_1, \ldots, K_{n_0}\}$, it follows that $\sup_n \mathbb{E}X_n 1(X_n > L) \leq \epsilon$. □

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