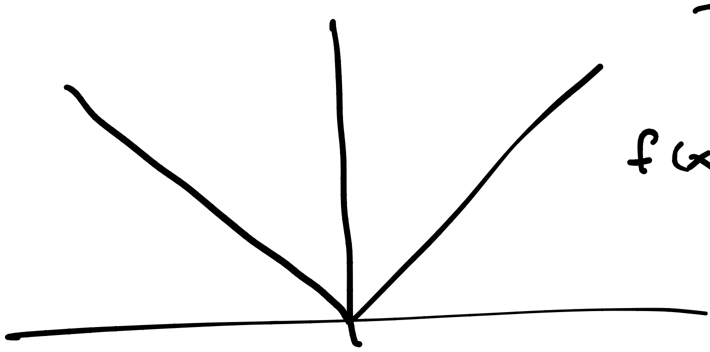
Quiz Review 1:

$$\frac{2n^2}{n^2+n+1} \xrightarrow[n\to\infty]{} 2 \qquad \text{Scale with } n^2 : \quad \frac{2}{1+1/n+1/n^2}$$

$$\begin{array}{r|l}
 & x^2 + 3x + 9 \\
\hline
x^2 - 3x + 2 & x^4 + 2x^2 - 3 \\
 & x^4 - 3x^3 + 2x^2 \\
\hline
 & 3x^3 \\
 & 3x^3 - 9x^2 + 6x \\
\hline
 & 9x^2 - 6x - 3 \\
 & 9x^2 - 27x + 18 \\
\hline
 & 21x - 21
\end{array}$$

$$\frac{21(x-1)}{x^2-3x+2} = \frac{21(x-1)}{(x-1)(x-2)} = \frac{21}{x-2}$$

$$\lim_{x\to 1} \quad 1 + 3 + 9 - 21 = -8$$



$f(x) = |x|$

continuous

Return to 2: $D(x^4 + 2x^2 - 3) = 4x^3 + 4x$

$D(x^2 - 3x + 2) = 2x - 3$

$x \to 1 \quad : \quad 8/-1 = -8$

Existence of e

Axiom of Real Numbers:

If a sequence is increasing and it is bounded above, then it converges.

If $a_{n+1} \geq a_n$ and $a_n \leq M$ (constant) for all $n \in \mathbb{N}$

$$\lim_{n \to \infty} a_n = a.$$

Euler's number, e: $\displaystyle \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = ?$

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!\,n^k}$$

$$= \sum_{k=0}^{n} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!\ \underbrace{n \cdot n \cdot \ \ldots \cdot n}_{k \text{ times}}}$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\, 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right)$$

$$u_n = \left(1+\frac{1}{n}\right)^n \quad ; \quad u_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}$$

$$= \underbrace{\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)}_{(*)}{}^{n} \left(1+\frac{1}{n+1}\right)$$

$$(*) = \frac{n(n+2)}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Bernoulli : For all $-1 < x \in \mathbb{R}$,

$$(1+x)^n \geq 1 + nx$$

$$\geq \left(1 - \frac{n}{(n+1)^2}\right)\left(1+\frac{1}{n+1}\right) =$$

$$= 1 + \frac{1}{(n+1)^3} \geq 1$$

Is it bounded from above?

$$\left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!} < 1+1+\sum_{k=2}^{n} \frac{1}{k(k-1)}$$

$$= 1+1+\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\right.$$

$$\left. \dots + \left(\frac{1}{n-1}-\frac{1}{n}\right)\right)$$

$$= 3-\frac{1}{n} < 3 \text{ for all } n \in \mathbb{N}.$$

Therefore, the limit exists!

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Derivative

Newton's Quotient

Definition

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

if this limit exists ($\in \mathbb{R}$)
we say that f is differentiable at a,
and the limit is the derivative of f at a.

$$f'(a) = Df(a) = \left.\frac{df}{dx}\right|_{x=a}$$

Example $\frac{d}{dx} \sin x = \cos x$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \overbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}^{=0}$$

$$+ \lim_{h \to 0} \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=1}$$

$$= \cos x$$

<u>Rules</u>

$$D(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$D\,\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad g(x) \neq 0$$

$$D\,f(g(x)) = g'(x)f'(g(x)) \quad \text{(chain rule)}$$

l'Hospital's Rule

Intermediate value theorem:

Theorem Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on the open interval (a, b) $(=]a, b[)$.

Then there exists a point $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

or alternatively $f(b) - f(a) =$

$$f'(\xi)(b - a).$$