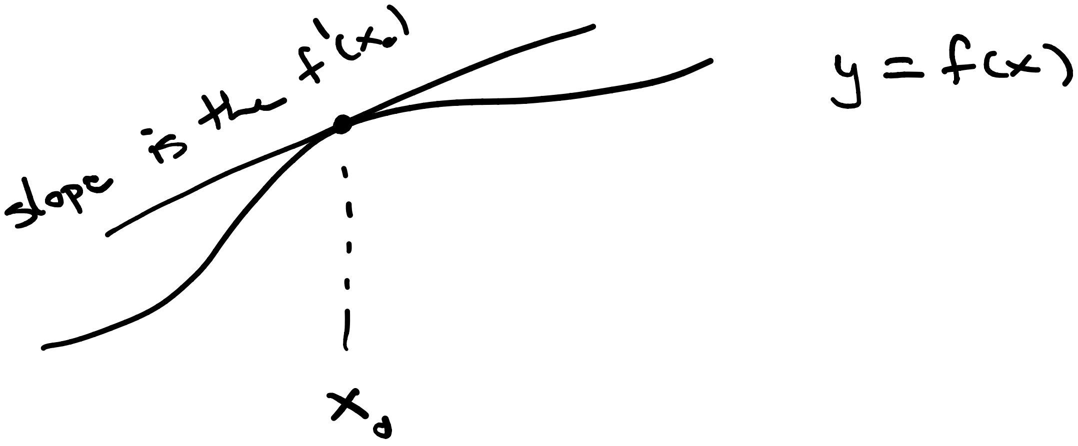
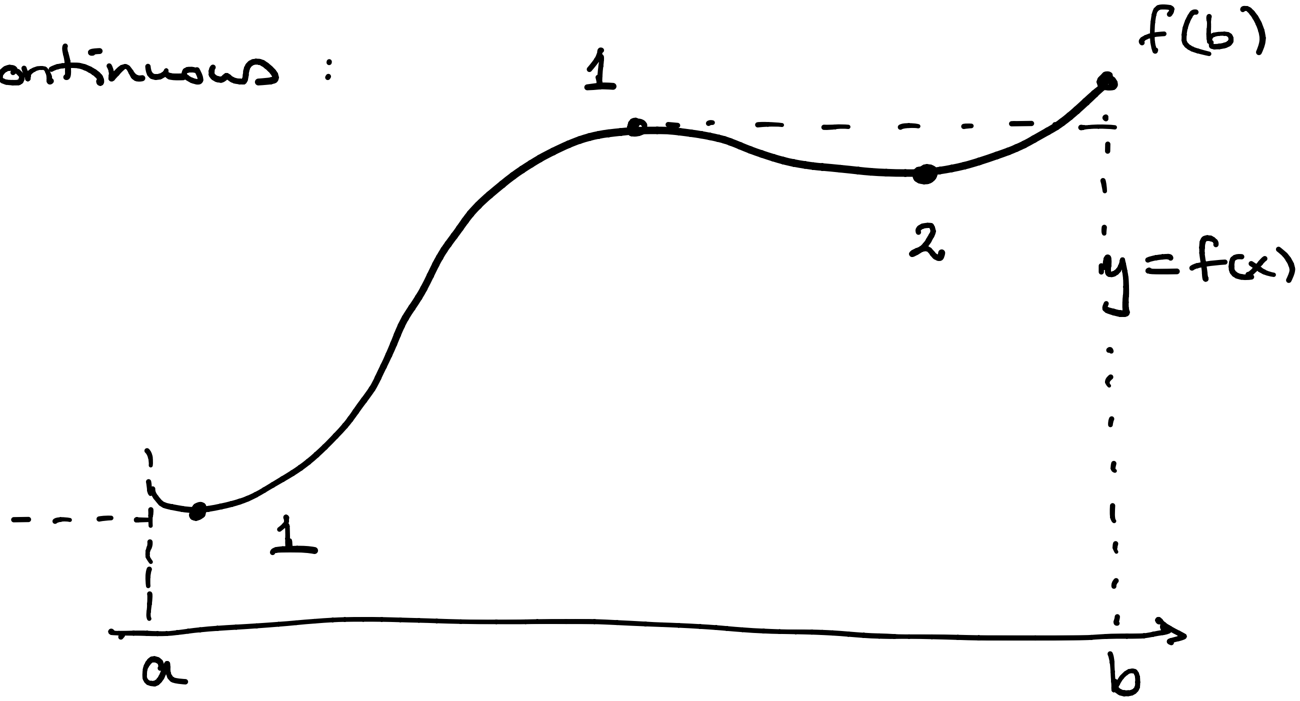
Applications of Derivatives



<u>Theorem</u> If $f'(x_0)$ exists and is > 0 (< 0), then f is increasing (decreasing) at x_0.

<u>Theorem</u> If f has a local extremal value at x_0 and if $f'(x_0)$ exists, then $f'(x_0) = 0$.

f continuous :

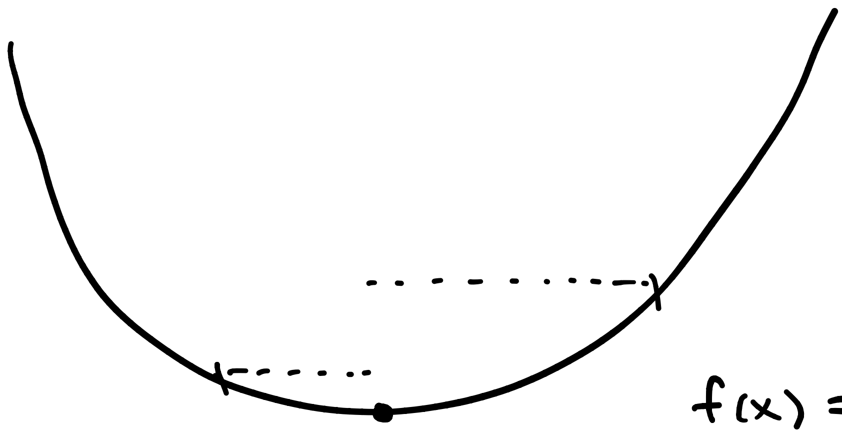


Weierstrass:

Every range of a continuous function over a closed interval has a maximum and a minimum.

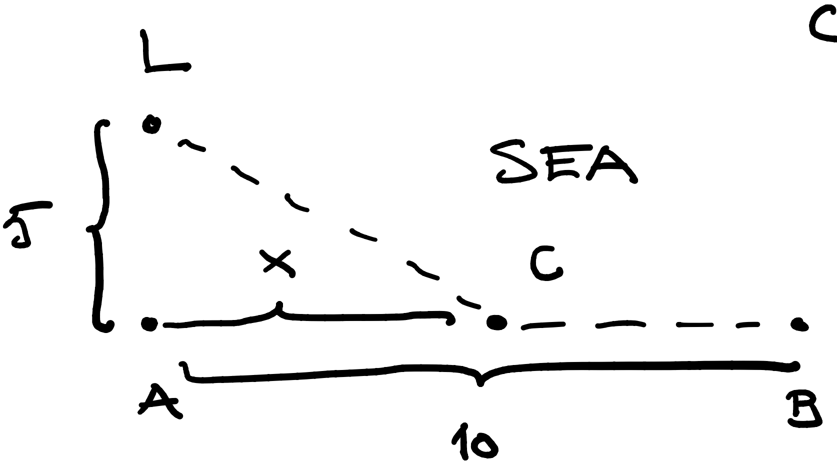
Classification of extreme values:

<u>Theorem</u> If $f'(x_0) = 0$ and $f''(x_0) > 0$ (< 0) then $f(x_0)$ is a local minimum (maximum).



$$f(x) = x^2$$
$$f'(x) = 2x$$
$$f''(x) = 2 > 0$$

Example "Cable"



Cost: LC : 5000 / unit
BC : 3000 / unit

Total cost: $T = T(x) = 5000\sqrt{25 + x^2} + 3000\,(10 - x)$

$$\frac{dT}{dx} = \frac{5000x}{\sqrt{25 + x^2}} - 3000 = 0$$

$$\Rightarrow \quad x = \frac{15}{4} = 3.75$$

To find the minimum cost, we must also consider the end points:

$$T(0) = 55000, \quad T(10) \cong 55900$$

$$T\left(\frac{15}{4}\right) = 50000$$

What if there is no cost differential between sea & shore?

$$\frac{dT}{dx} = \frac{x}{\sqrt{25 + x^2}} - 1 = 0 \iff x^2 = 25 + x^2$$

$\Longrightarrow$ solution is LB

Exponential and Logarithmic Functions

Exponential function : $y = a^x$,

$a \in (a, \infty) \setminus \{1\}$

Natural exponential : $y = e^x$, where

e is the Euler's number

In fact : $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$, $x \in \mathbb{R}$

Definition The inverse of $y = e^x$ is the natural logarithmic function :

$$\ln : \mathbb{R}_+ \longrightarrow \mathbb{R},$$

$$y = \ln x \iff x = e^y.$$

Rules :

$$\ln xy = \ln x + \ln y, \quad x > 0, \; y > 0$$

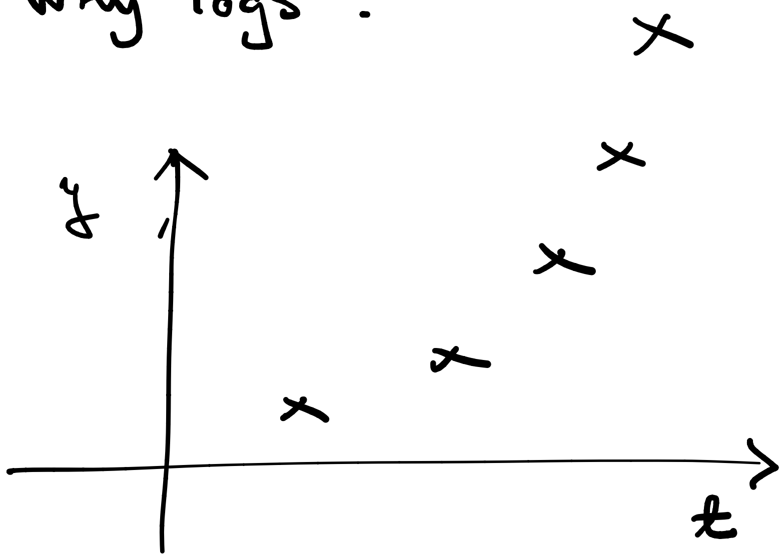
$$\ln x^y = y \ln x, \quad x > 0, \; y \in \mathbb{R}$$

Notice :

$$e^{\ln xy} = xy = e^{\ln x} e^{\ln y}$$

$$= e^{\ln x + \ln y}$$

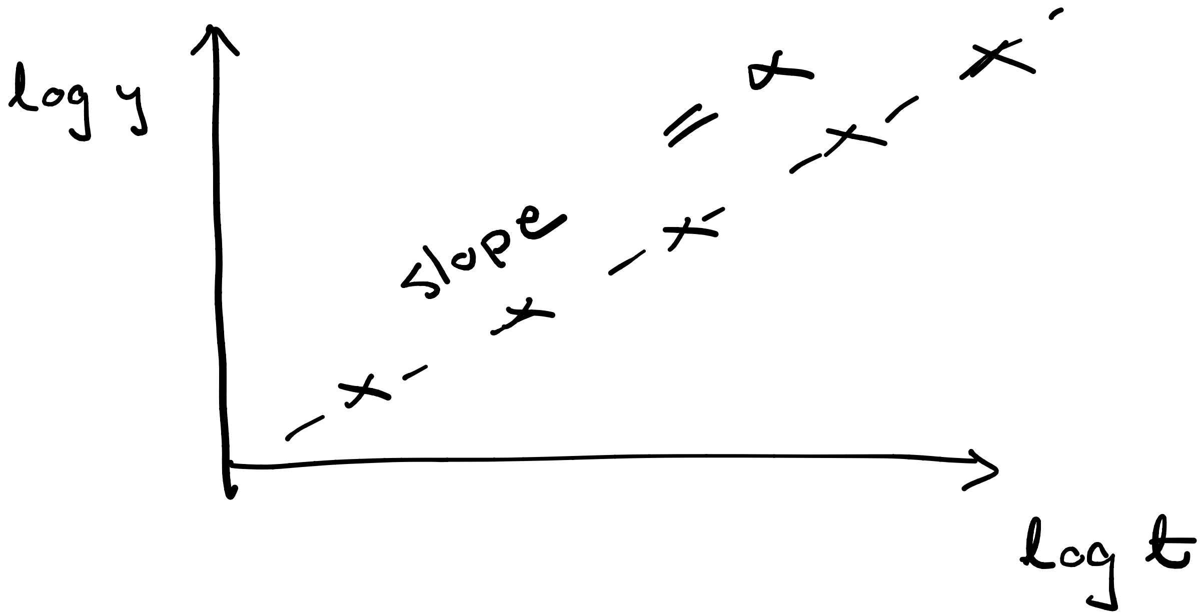
Why logs ?



$$y = f(t) \;?$$
$$= ct^{\alpha}$$

loglog - plots :

$$y = f(t)$$
$$\log y = \log f(t)$$
$$= \log (ct^{\alpha})$$
$$= \log c + \log t^{\alpha} = \log c + \alpha \log t$$



Derivatives: $\frac{de^x}{dx} = e^x$; $\frac{d\ln x}{dx} = \frac{1}{x}$

Newton Quotient: $\frac{e^{x+h} - e^x}{h} = \frac{e^x(e^h - 1)}{h}$

Next stop: Exponential function

<u>Definition</u> $a^x = e^{x\ln a}$, $a > 0$, $x \in \mathbb{R}$

Derivative: $\frac{da^x}{dx} = \frac{d}{dx} e^{x\ln a}$

$$= e^{x\ln a} \ln a$$

$$= a^x \ln a$$

Logarithmic function: $y = \log_a x \iff x = a^y$

Implicit differentiation:

Assume that $y = y(x)$ and differentiate on both sides:

$$x = a^y \overset{D}{\Longrightarrow} 1 = a^y \ln a \frac{dy}{dx} = x \ln a \frac{dy}{dx}$$

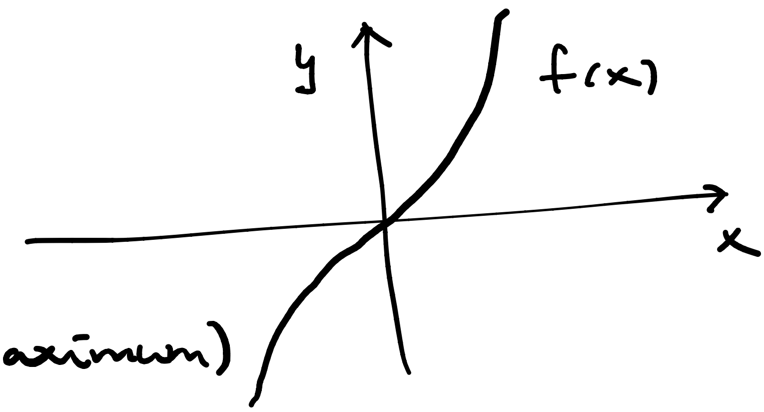
$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \log_a x = \frac{1}{x\ln a}$$

However: $\log_a x = \frac{\ln x}{\ln a}$

Quiz $\underline{1}$: $f(x) = x^3$

$f'(x) = 0$

However, it is not
a minimum (nor a maximum)



Inverse Trigonometric Functions

Trigonometric functions are periodic, therefore inverse functions can only be considered over special intervals or branches.

Sine: $f(x) = \sin x \quad , \quad -\pi/2 \leq x \leq \pi/2$

Definition arcsin x is the inverse function of the sine function.

$$y = \arcsin x \iff x = \sin y, \quad y \in [-\pi/2, \pi/2]$$

Derivative $x = \sin y \overset{D}{\Rightarrow} 1 = \cos y \frac{dy}{dx}$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

<u>Tangent</u>: $f(x) = \tan x$, $-\pi/2 < x < \pi/2$.

<u>Definition</u> $\arctan x$ is the inverse of the tangent function.

$$y = \arctan x \iff x = \tan y, \quad -\pi/2 < y < \pi/2.$$

<u>Derivative</u>

$$x = \tan y$$

$$\overset{D}{\Longrightarrow} \quad 1 = \frac{1}{\cos^2 y} \frac{dy}{dx} = (1 + \tan^2 y) \frac{dy}{dx}$$

$$\left[\text{Why?} \quad \begin{aligned} 1 &= \cos^2 y + \sin^2 y \\ \frac{1}{\cos^2 y} &= 1 + \tan^2 y \end{aligned} \right]$$

$$= (1 + x^2) \frac{dy}{dx}$$

$$\Longrightarrow \quad \frac{d}{dx} \arctan x = \frac{1}{x^2 + 1}$$

Comments on notation:

$$\arctan x = y \quad ; \quad y = \tan^{-1} x$$

The main branch can be emphasised:

$$\overline{\arctan} x = y$$

Finally: We do not sec or cosec in our class.