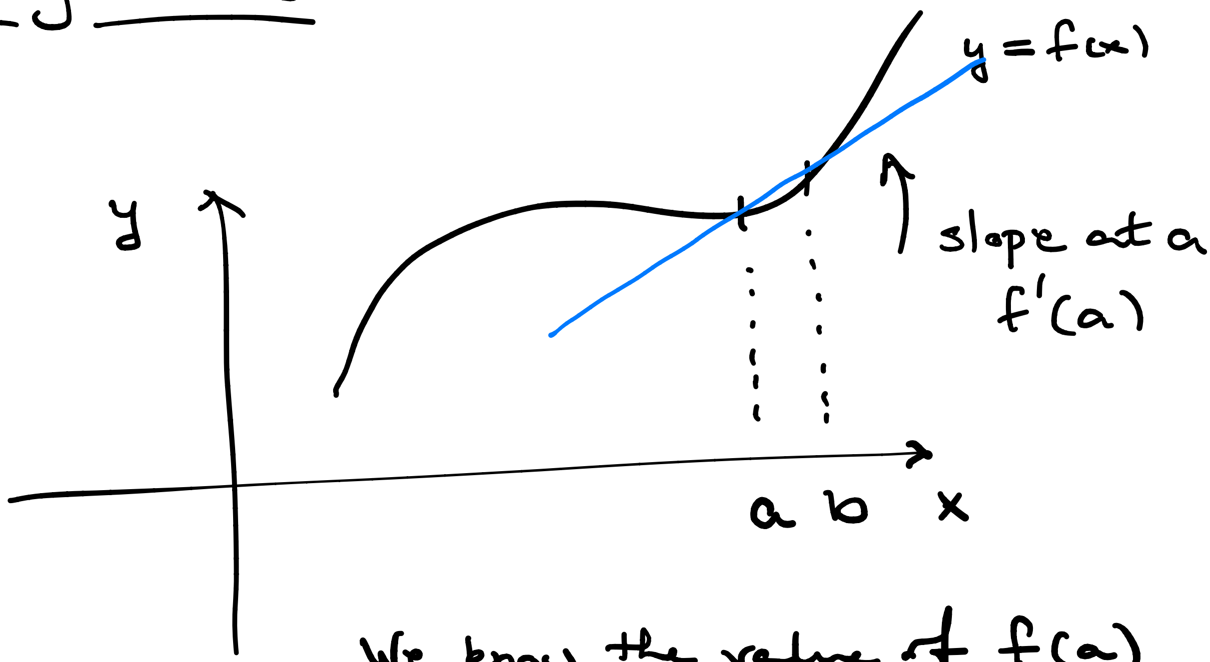
Taylor Polynomials

Example



We know the value of $f(a)$.

- $b = a + h$
- $f(b)$?

The idea is to approximate the value at b using some information on f.

$$f(b) \simeq f(a) + f'(a)(b-a)$$

Definition The Linearisation of the function f about a is the function L defined by

$$L(x) = f(a) + f'(a)(x-a).$$

Example $\sqrt{26} \approx$? $\quad f(x) = \sqrt{x}$

$\sqrt{25} = 5$; $a = 25$; $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$

$f'(a) = \frac{1}{10}$ $\Rightarrow L(x) = 5 + \frac{1}{10}(x-25)$

$\Rightarrow L(26) = 5.1$

Error Estimation $\quad E(t) = f(t) - f(a) - f'(a)(t-a)$

$$E'(t) = f'(t) - f'(a)$$

Generalised Mean Value Theorem :

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

On $[a, x]$:

$$\frac{E(x) - E(a)}{(x-a)^2 - (a-a)^2} \qquad \left(= \frac{E(x)}{(x-a)^2} \right)$$

$$= \frac{E'(\xi)}{2(\xi - a)} = \frac{f'(\xi) - f'(a)}{2(\xi - a)}$$

$$= \frac{1}{2} f''(\eta)$$

$$\Rightarrow \quad E(x) = \frac{1}{2} f''(\eta)(x-a)^2$$

$$\Rightarrow \quad |E(x)| \leq \frac{1}{2} \max_{\eta} |f''(\eta)| \, (x-a)^2$$

Now we extend this process to higher orders.

Assumptions: $f(a), f'(a), \ldots, f^{(n-1)}(a)$ exist

Goal:

We want to approximate f with a polynomial T_{n-1} with a maximal degree $n-1$ such that its value and derivatives at a are exact.

$$T_{n-1}(x,a) = c_0 + c_1(x-a) + \ldots + c_{n-1}(x-a)^{n-1}$$

$$T'_{n-1}(x,a) = c_1 + 2c_2(x-a) + \ldots + (n-1)c_{n-1}(x-a)^{n-2}$$

$$\vdots$$

$$T^{(k)}_{n-1}(x,a) = k!\,c_k + (x-a)\underbrace{P(x)}_{\text{some polynomial}}, \quad k=1,2,\ldots,n-2$$

$$\vdots$$

$$T^{(n-1)}_{n-1}(x,a) = (n-1)!\,c_{n-1}$$

Condition: $T^{(k)}_{n-1}(a,a) = f^{(k)}(a)$,

$k=0,1,\ldots,n-1$

$$\Rightarrow \quad c_k = \frac{f^{(k)}(a)}{k!}, \quad k=0,\ldots,n-1$$

Definition Taylor Polynomial

$$T_{n-1}(x,a) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem Lagrange Remainder

If $f^{(n)}(x)$ is continuous over $[a,x]$, then

$$f(x) = T_{n-1}(x,a) + \frac{f^{(n)}(\xi)}{n!}(x-a)^n,$$

where $\xi \in [a,x]$.

Example : Maclaurin polynomial ; $a=0$

$$\begin{aligned} f(x) &= \sin x & g(x) &= \cos x \\ f'(x) &= \cos x & g'(x) &= -\sin x \\ f''(x) &= -\sin x & g''(x) &= -\cos x \\ f'''(x) &= -\cos x & g'''(x) &= \sin x \\ f^{(4)}(x) &= \sin x & g^{(4)}(x) &= \cos x \end{aligned}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \Theta(x^7)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \Theta(x^6)$$

Theorem (Alternative formulation)

If $f(x) = Q_n(x) + \Theta((x-a)^{n+1})$ as $x \to a$, where Q_n is a polynomial of degree at most n, then $Q_n(x) = T_n(x)$.

Example $T_3(x, 1)$ for e^{2x}.

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \Theta(x^{n+1})$$

Writing: $x = 1 + (x-1)$

$$e^{2x} = e^{2+2(x-1)} = e^2 e^{2(x-1)}$$

$$= e^2 \left[1 + 2(x-1) + \frac{2^2 (x-1)^2}{2!} + \frac{2^3 (x-1)^3}{3!} + \Theta((x-1)^4) \right]$$

as $x \to 1$.

$$T_3(x, 1) = e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2}{3}(x-1)^3$$

Big-Oh: $\Theta(x^4)$ e.g. $K(x) \leq C x^4$

Theorem Let $f^{(n)}$ be continuous in the neighbourhood of $x = a$, and

$$f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$$

with $f^{(n)}(a) \neq 0$.

If n is even and $f^{(n)}(a) > 0$ (< 0), then $f(a)$ is a local minimum (maximum). If n is odd, $f(a)$ is not an extreme value.

Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \Theta(x^7)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \Theta(x^{2n+3})$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$