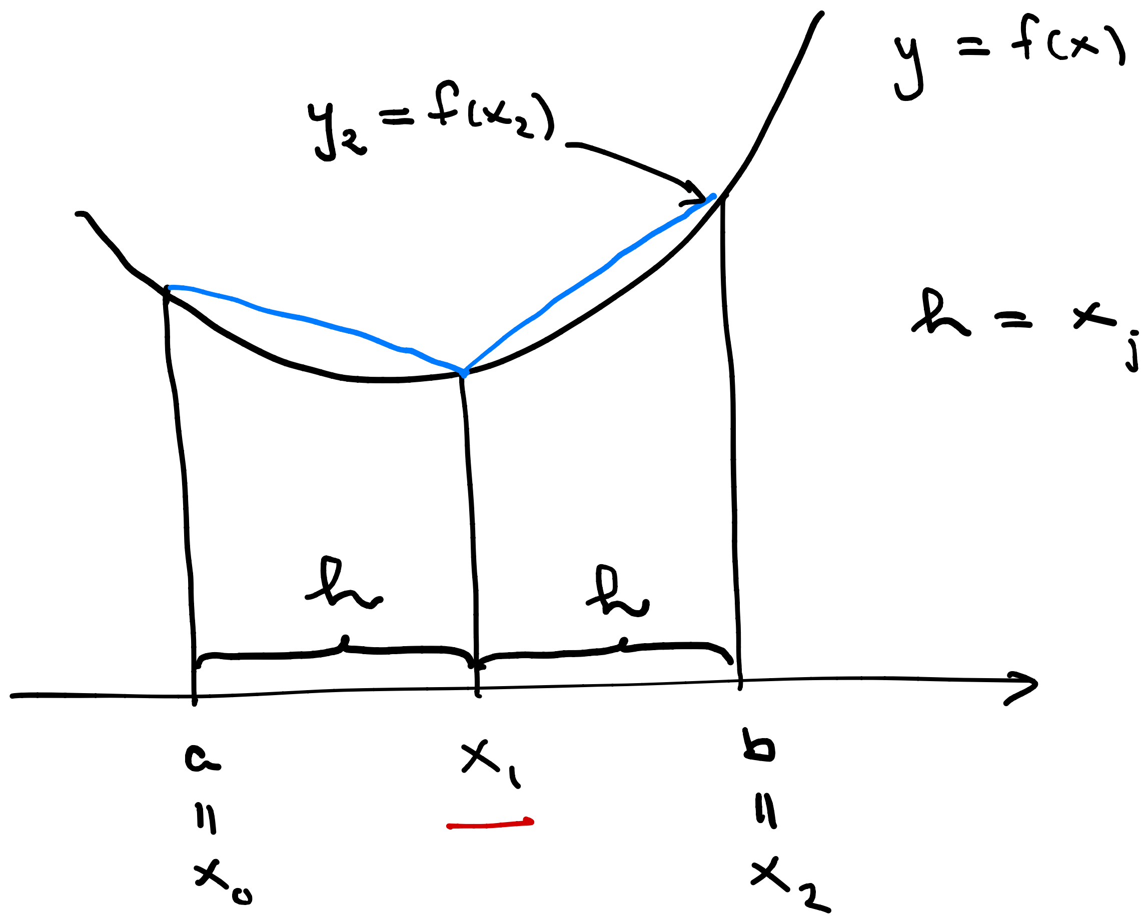
$T_n[f; a, b]$: THE TRAPEZOIDAL RULE



$$h = x_j - x_{j-1}$$

The idea: Linearise the function f over each subinterval seperately.

One interval:
$$\int_{x_{j-1}}^{x_j} f(x)\,dx \simeq h\,\frac{y_{j-1} + y_j}{2}, \quad 1 \le j \le n.$$

Definition

$$T_n[f; a, b] = h\left(\frac{1}{2} y_0 + y_1 + y_2 + \dots + y_{n-1} + \frac{1}{2} y_n\right)$$

$$= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n \right)$$

Weights: $\frac{h}{2} + \frac{h}{2} + (n-1)\, 2\, \frac{h}{2} = nh = b - a$

Example $\quad I = \int_1^2 \frac{dx}{x} \;;\; T_4 = ?$

$$h = \frac{2-1}{4} = \frac{1}{4}$$

$$T_4 = \frac{1}{4}\left(\frac{1}{2}\cdot 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\cdot\frac{1}{2} \right)$$

$$\simeq 0.697$$

In detail: the quadrature points x_i:

$T_4\left[\frac{1}{x} ; 1, 2\right]$, $\quad x_0 = a = 1$

$\quad x_4 = b = 2$

$x_1 = \frac{5}{4}$, $\quad x_2 = \frac{6}{4} = \frac{3}{2}$, $\quad x_3 = \frac{7}{4}$

Data: $y_j = f(x_j)$

$y_0 = 1 \quad y_1 = \frac{1}{x_1} = \frac{4}{5}$, $y_2 = \frac{2}{3}$, $y_3 = \frac{4}{7}$

$y_4 = \frac{1}{2}$

$M_n[f; a, b]$: THE MIDPOINT RULE

Let $h = \dfrac{b-a}{n}$.

Points $m_j = a + (j - \tfrac{1}{2})h, \quad 1 \leq j \leq n$

<u>Definition</u> $\quad M_n[f; a, b] = h \displaystyle\sum_{j=1}^{n} f(m_j)$

(Special Riemann sum! $\xi_k = m_k$)

Example $\quad I = \displaystyle\int_1^2 f(x)\,dx \;; \quad M_4 = ?$

$$M_4 = \underbrace{\frac{1}{4}}_{= h} \left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15} \right]$$

$$m_1 = 1 + (1 - \tfrac{1}{2}) \frac{1}{4} = \frac{9}{8}$$

$M_4 \simeq 0.691$

$I = \ln 2 \simeq 0.693$

THEOREM The error estimates

Let f^4 be continuous and bounded over $[a,b]$, that is, $|f^4(x)| \leq K$ (= constant).

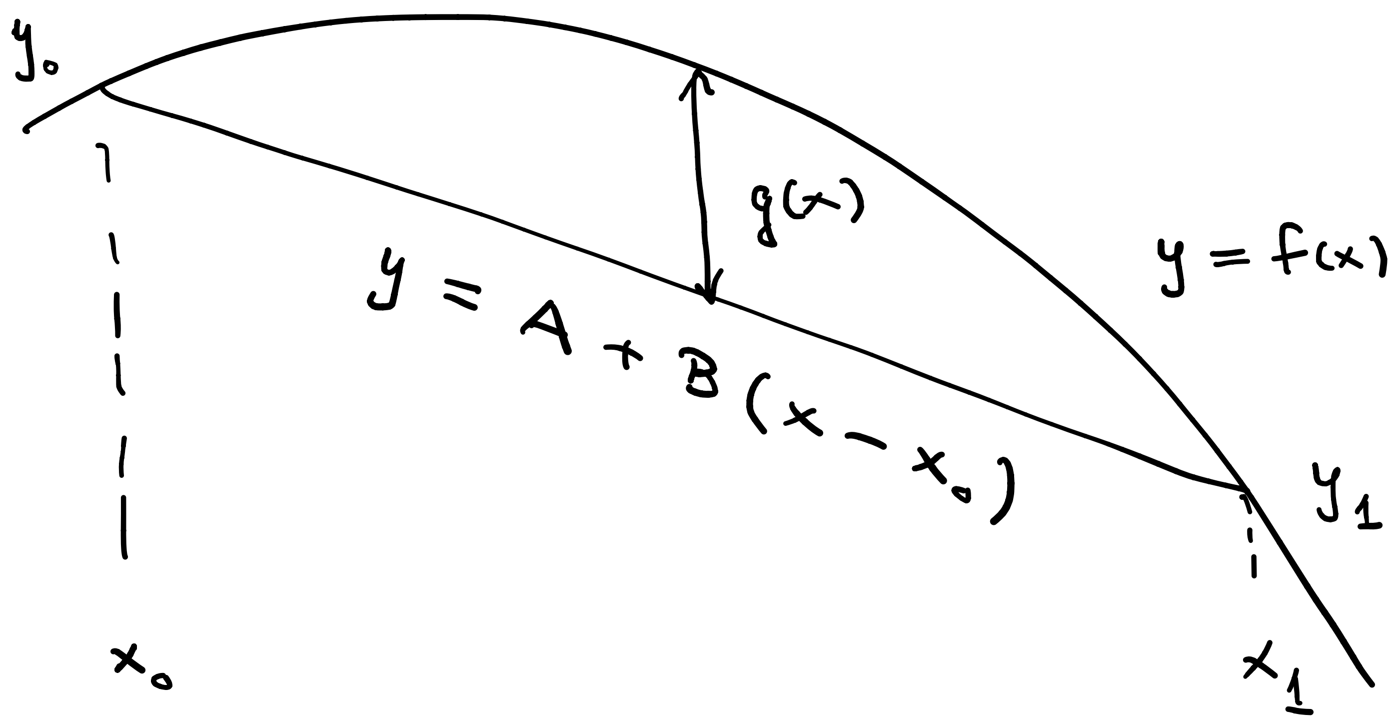
then, with $h = \frac{b-a}{n}$, we have

$$\left| \int_a^b f(x)\,dx - T_n \right| \leq \frac{K(b-a)}{12} h^2 = \frac{K(b-a)^3}{12n^2}$$

$$\left| \int_a^b f(x)\,dx - M_n \right| \leq \frac{K(b-a)}{24} h^2 = \frac{K(b-a)^3}{24n^2}$$

Both methods are quadratic, i.e., for the error $\sim \Theta\left(\frac{1}{n^2}\right)$.

PROOF (TRAPEZOID)



the error $g(x) = f(x) - A - B(x - x_0)$

$$= f(x) - y_0 - \frac{1}{h}(y_1 - y_0)(x - x_0)$$

Now : $$\int_{x_0}^{x_1} g(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx - h\,\frac{y_0 + y_1}{2}$$

$$g''(x) = f''(x), \quad g(x_0) = g(x_1) = 0$$

$$\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,g''(x)\,dx = -2\int_{x_0}^{x_1} g(x)\,dx$$

Triangle inequality :

$$\left| \int_{x_0}^{x_1} f(x)\,dx - h\,\frac{y_0+y_1}{2} \right| = \left| \int_{x_0}^{x_1} g(x)\,dx \right|$$

$$= \left| \frac{1}{2} \int_{x_0}^{x_1} (x-x_0)(x_1-x)\, f''(x)\,dx \right|$$

$$\leq \frac{1}{2} \int_{x_0}^{x_1} (x-x_0)(x_1-x)\, |f''(x)|\,dx$$

$$\leq \frac{K}{2} \int_{x_0}^{x_1} \left(-x^2 + (x_0+x_1)x - x_0x_1 \right) dx$$

$$= \frac{K}{12} (x_1-x_0)^3 = \frac{K}{12} h^3$$

The whole interval :

$$\left| \int_a^b f(x)\,dx - T_n \right| = \left| \sum_{j=1}^{n} \left(\int_{x_{j-1}}^{x_j} f(x)\,dx - h\,\frac{y_{j-1}+y_j}{2} \right) \right|$$

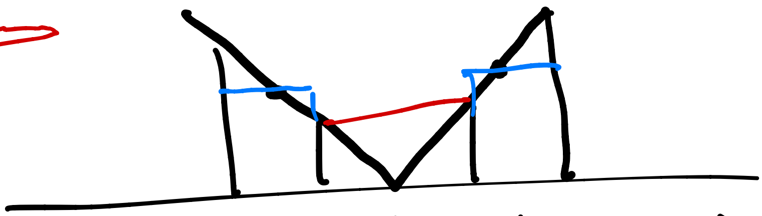
$$\le \sum_{j=1}^{n} \left| \int_{x_{j-1}}^{x_j} f(x)\,dx - h \frac{y_{j-1}+y_j}{2} \right|$$

$$= \sum_{j=1}^{n} \frac{K}{12} h^3 = \frac{K(b-a)}{12} h^2$$

$\square$

Special case

$|x| = y$



$\hookrightarrow |0| = 0$

The triangle inequality for sums extends to definite integrals:

If $a \le b$, then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$$