Techniques for Integration

Indefinite integral : $F(x)$ s.t. $F'(x) = f(x)$

$$F(x) = \int_a^x f(t)\,dt$$ over closed interval $[a, x]$; f continuous

Since the derivative of a constant function is zero, we have

$$\int f(x)\,dx = F(x) + C, \quad C \text{ is constant}$$

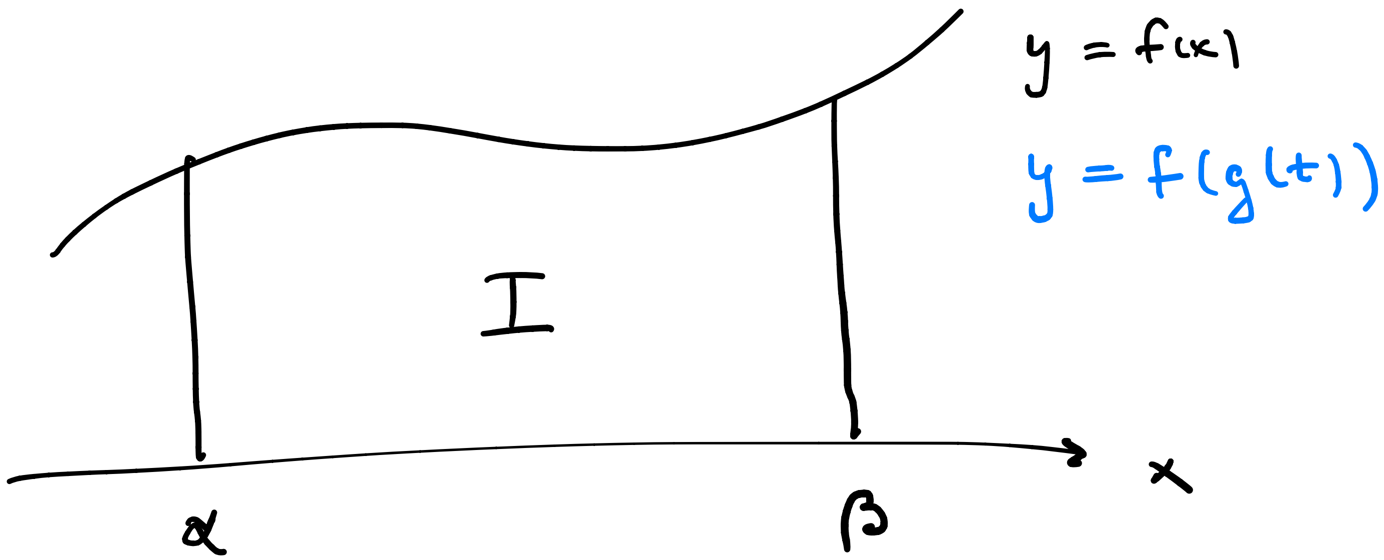
Method of substitution

$$\int f(x)\,dx = \int F'(x)\,dx = F(x) + C$$

set $x = g(t)$

$$= F(g(t)) + C = \int \frac{d}{dt} F(g(t))\,dt$$

$$= \int F'(g(t))\,g'(t)\,dt = \int f(g(t))\,g'(t)\,dt$$



$\alpha = g(t_1)$ $\qquad\qquad$ $\beta = g(t_2)$

$x = g(t)$; the "unit" changes $\Rightarrow dx = g'(t)\,dt$

<u>Example</u> $\displaystyle\int \frac{dx}{x^2+a^2}$, $a > 0$

Let $x = at$; $dx = a\,dt$

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\,dt}{a^2t^2+a^2} = \frac{1}{a}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{a}\arctan t + C$$

$$= \frac{1}{a}\arctan \frac{x}{a} + C$$

For the definite integral : $x \in [\alpha, \beta]$

$x = at \implies \alpha = at \implies t = \alpha/a$

$\qquad\qquad\quad \beta = at \implies t = \beta/a$

$$\int_{\alpha}^{\beta} \frac{dx}{x^2+a^2} = \frac{1}{a} \int_{\alpha/a}^{\beta/a} \frac{dt}{t^2+1}$$

$$= \frac{1}{a}\left(\overline{\arctan} \frac{\beta}{a} - \overline{\arctan} \frac{\alpha}{a} \right)$$

HYPERBOLIC FUNCTIONS

Definition

Hyperbolic cosine : $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

Hyperbolic sine : $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

Properties :

$$D \cosh x = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

$$D \sinh x = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\cosh^2 x = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\cosh^2 x - \sinh^2 x = 1$$

Example (Inverses)

$$\operatorname{arsinh} x = y \iff x = \sinh y$$

$$\underline{x = \sinh y} = \frac{e^y - e^{-y}}{2} = \frac{(e^y)^2 - 1}{2e^y}$$

We get:

$$(e^y)^2 - 2x e^y - 1 = 0$$

$$\Rightarrow \quad e^y = x \pm \sqrt{\underbrace{x^2+1}_{>0}} \quad (\pm \to +)$$

$$\Rightarrow \quad y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \operatorname{arsinh} x$$

More Examples

$$\int \frac{dx}{x^2 - a^2}, \quad a \neq 0. \quad \text{Let } \begin{aligned} x &= at \\ dx &= a\,dt \end{aligned}$$

$$= \int \frac{a\,dt}{a^2 t^2 - a^2} = \frac{1}{a} \int \frac{dt}{t^2 - 1}$$

$$\left.\begin{aligned} &-\frac{1}{2}\,\frac{1}{t+1} + \frac{1}{2}\,\frac{1}{t-1} \\ &= -\frac{1}{2}\,\frac{t-1}{t^2-1} + \frac{1}{2}\,\frac{t+1}{t^2-1} = \frac{1}{t^2-1} \end{aligned}\right\} ?$$

$$= \frac{1}{a}\left(\frac{1}{2}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1}\right)$$

$$= \frac{1}{2a} \ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\overbrace{x\,dx}}{x^4+1} = I \qquad \text{Let } x^2 = t\ ;$$

$$\underbrace{2x\,dx} = dt$$

$$I = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\arctan t + C$$

$$= \frac{1}{2}\arctan x^2 + C$$

$$\int_0^a \sqrt{a^2 - x^2}\,dx = I$$

Let $x = a\sin t$; $dx = \underline{a\cos t\,dt}$

with $x \in [0, a]$, choose $t \in [0, \pi/2]$

$$I = \int_0^{\pi/2} a^2\cos^2 t\,dt = a^2 \int_0^{\pi/2} \frac{1 + \sin t \cos t}{2}$$

$$= \frac{1}{4}\pi a^2$$

$$\cos^2 t = |\cos t| \cos t$$

Identities :

f even : $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

f odd : $$\int_{-a}^{a} f(x)\,dx = 0$$

f ω-periodic :

$$\int_{a}^{b} f(x)\,dx = \int_{a+\omega}^{b+\omega} f(x)\,dx$$