INTEGRATION BY PARTS

Product rule:

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$$

Integrate both sides and rearrange terms:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

<u>Example</u> $\int x^2 e^x\,dx = I$

$$\int \underbrace{x^2}_{v}\,\underbrace{e^x}_{u'}\,dx = \underbrace{x^2}_{v}\,\underbrace{e^x}_{u} - \underbrace{\int \underbrace{2x}_{v'}\,\underbrace{e^x}_{u}\,dx}_{2xe^x - \int 2e^x dx}$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C = I$$

Derivative: $e^x x^2 + \underbrace{2xe^x - 2xe^x}\, \underbrace{- 2e^x + 2e^x}$

$$= x^2e^x$$

<u>Example</u> $\int \ln x \, dx = I$

$$\int \ln x \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx$$

$$= x \ln x - \int \underbrace{\frac{1}{x}}_{v'} \underbrace{x}_{u} \, dx$$

$$= x \ln x - x + C$$

Let us check again by taking the derivative:

$$\underbrace{x \cdot \frac{1}{x}}_{=1} + \ln x - 1 = \ln x$$

<u>Theorem</u> Every rational function can be integrated in closed form.

$$H(x) = \frac{P(x)}{Q(x)}, \quad P(x), Q(x) \text{ polynomials.}$$

<u>Example</u> $$\int \frac{dx}{x(x^6+1)^2} = I$$

Substitution: $x^6 = t$, $6x^5 dx = dt$

$$\int \frac{dx}{x(x^6+1)^2} = \frac{1}{6} \int \frac{6x^5 dx}{x^6(x^6+1)^2}$$

$$= \frac{1}{6} \int \frac{dt}{t(t+1)^2}$$

Partial fraction decomposition:

$$\frac{1}{t(t+1)^2} = \frac{A}{t} + \frac{B}{(t+1)^2} + \frac{C}{t+1}$$

$$= \frac{A(t+1)^2 + Bt + Ct(t+1)}{t(t+1)^2}$$

Set the numerators to be equal:

$$t^2 : \quad A + \qquad C = 0$$

$$t^1 : \quad 2A + B + C = 0$$

$$t^0 : \quad A \qquad = 1$$

$$\Rightarrow \quad A = 1 \,, \quad C = -1 \,, \quad B = -1$$

Now we can integrate:

$$I = \frac{1}{6}\left(\ln|t| + \frac{1}{t+1} - \ln|t+1| \right) + C$$

$$= \frac{1}{6}\left(\ln \frac{x^6}{x^6+1} + \frac{1}{x^6+1} \right) + C$$

Example

$$\int \frac{x^4+1}{x^3-x^2+x+1}\,dx = \int \left(x+1+\frac{2}{x^3-x^2+x-1} \right) dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{2}{(x-1)(x^2+1)}\,dx$$

$$\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Solve:

$$A(x^2+1) + (Bx+C)(x-1) = 2$$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ -B + C = 0 \\ A - C = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$

Integrate:

$$\int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

Everything together:

$$I = \frac{1}{2}x^2 + x + \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 + x + \ln|x-1| - \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

For rational functions the integral function always is a sum of polynomials, logarithms, or arctans.

<u>Example</u> $I_n = \int x^n e^x\,dx \quad (n \in \mathbb{N})$

Recursion:

$$\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx$$

$$I_n = x^n e^x - n I_{n-1}$$

It terminates, since $I_0 = \int e^x dx = e^x + C$

Example

$$I = \int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx$$

$$= -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

$$= -\sin x \cos x + \int dx - I$$

We get : $I = \dfrac{x - \sin x \cos x}{2} + C$

Example $\quad I = \int e^{kx} \sin nx \, dx \quad (k \neq 0)$

$$I = \frac{1}{k} e^{kx} \sin nx - \int \frac{n}{k} e^{kx} \cos nx \, dx$$

$$\left[\int e^{kx} \cos nx \, dx = \frac{1}{k} e^{kx} \cos nx + \int \frac{n}{k} e^{kx} \sin nx \, dx \right]$$

$$= \frac{1}{k} e^{kx} \sin nx - \frac{n}{k^2} e^{kx} \cos nx - \frac{n^2}{k^2} I$$

Again, solve for I !