INTEGRATION BY PARTS

Product rule:

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$$

Integrate both sides and rearrange terms:

$$\int u'(x)\,v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

<u>Example</u> $\int x^2 e^x \, dx = I$

$$\int \underbrace{x^2}_{v}\,\underbrace{e^x}_{u'}\,dx = \underbrace{x^2}_{v}\,\underbrace{e^x}_{u} - \underbrace{\int \underbrace{2x}_{v'}\,\underbrace{e^x}_{u}\,dx}_{2xe^x - \int 2e^x dx}$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C = I$$

Derivative: $e^x x^2 + \underbrace{2xe^x - 2xe^x}\,\underbrace{-\,2e^x + 2e^x}$

$$= x^2 e^x$$

<u>Example</u> $\int \ln x \, dx = I$

$$\int \ln x \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx$$

$$= x \ln x - \int \underbrace{\frac{1}{x}}_{v'} \underbrace{x}_{u} \, dx$$

$$= x \ln x - x + C$$

Let us check again by taking the derivative:

$$\underbrace{x \cdot \frac{1}{x}}_{=1} + \ln x - 1 = \ln x$$

Theorem Every rational function can be integrated in closed form.

$$H(x) = \frac{P(x)}{Q(x)}, \quad P(x),\ Q(x) \text{ polynomials.}$$

Example $$\int \frac{dx}{x(x^6+1)^2} = I$$

Substitution: $x^6 = t$, $6x^5 dx = dt$

$$\int \frac{dx}{x(x^6+1)^2} = \frac{1}{6}\int \frac{6x^5 dx}{x^6(x^6+1)^2}$$

$$= \frac{1}{6}\int \frac{dt}{t(t+1)^2}$$

Partial fraction decomposition:

$$\frac{1}{t(t+1)^2} = \frac{A}{t} + \frac{B}{(t+1)^2} + \frac{C}{t+1}$$

$$= \frac{A(t+1)^2 + Bt + Ct(t+1)}{t(t+1)^2}$$

Set the numerators to be equal:

$$t^2: \quad A + C = 0$$

$$t^1: \quad 2A + B + C = 0$$

$$t^0: \quad A = 1$$

$$\Rightarrow \quad A = 1, \quad C = -1, \quad B = -1$$

Now we can integrate:

$$I = \frac{1}{6}\left(\ln|t| + \frac{1}{t+1} - \ln|t+1|\right) + C$$

$$= \frac{1}{6}\left(\ln\frac{x^6}{x^6+1} + \frac{1}{x^6+1}\right) + C$$

Example

$$\int \frac{x^4+1}{x^3-x^2+x+1}\,dx = \int \left(x+1 + \frac{2}{x^3-x^2+x-1} \right) dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{2}{(x-1)(x^2+1)}\,dx$$

$$\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Solve:

$$A(x^2+1) + (Bx+C)(x-1) = 2$$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ -B + C = 0 \\ A - C = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$

Integrate:

$$\int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

Everything together:

$$I = \frac{1}{2}x^2 + x + \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 + x + \ln|x-1| - \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

For rational functions the integral function always is a sum of polynomials, logarithms, or arctans.

<u>Example</u> $I_n = \int x^n e^x\,dx \quad (n \in \mathbb{N})$

Recursion:

$$\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx$$

$$I_n = x^n e^x - n I_{n-1}$$

It terminates, since $I_0 = \int e^x dx = e^x + C$

Example

$$I = \int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx$$

$$= -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

$$= -\sin x \cos x + \int dx - I$$

We get : $I = \dfrac{x - \sin x \cos x}{2} + C$

Example $\quad I = \int e^{kx} \sin nx \, dx \quad (k \neq 0)$

$$I = \frac{1}{k} e^{kx} \sin nx - \int \frac{n}{k} e^{kx} \cos nx \, dx$$

$$\left[\int e^{kx} \cos nx \, dx = \frac{1}{k} e^{kx} \cos nx + \int \frac{n}{k} e^{kx} \sin nx \, dx \right]$$

$$= \frac{1}{k} e^{kx} \sin nx - \frac{n}{k^2} e^{kx} \cos nx - \frac{n^2}{k^2} I$$

Again, solve for I !

ORDINARY DIFFERENTIAL EQUATIONS (ODE)

General 1^{st} order ODE: $\frac{dy(x)}{dx} = f(x, y(x))$

the solution curve: $y(x)$

At every point (x, y) the slope of the solution curve $y(x)$ is $f(x, y(x))$.

Equation: $y' = f(x, y)$

→ we have information only up to a constant

Terminology:

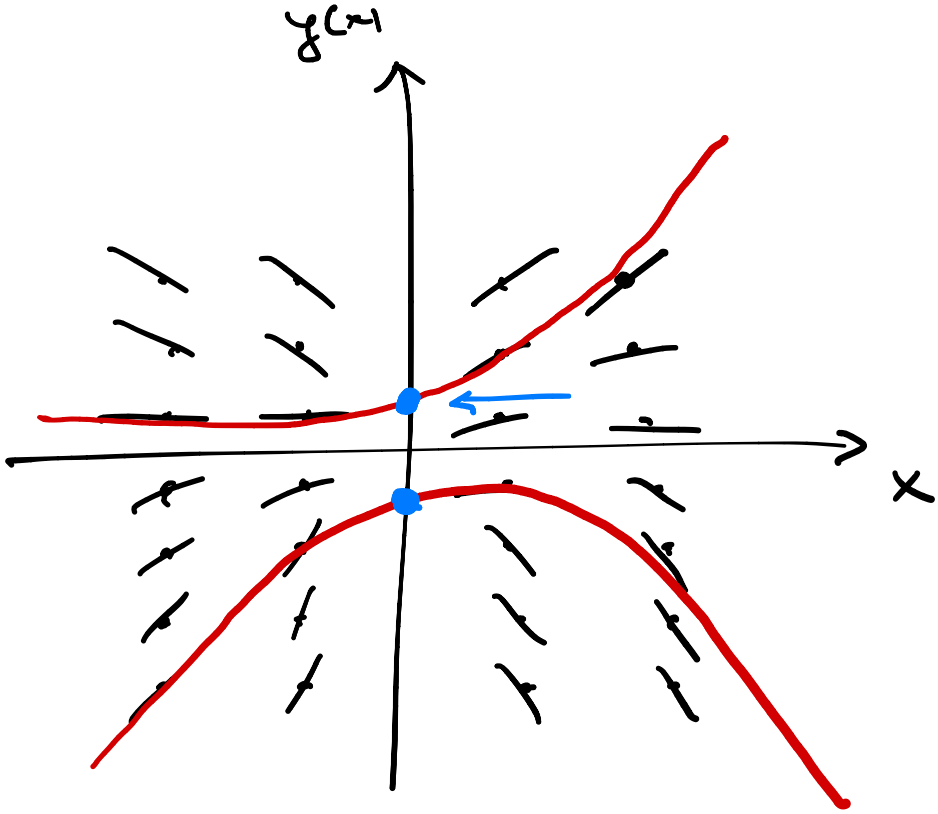
- the general solution includes all possible solutions
- the initial conditions lead to particular solutions
- the order of the ODE: the value of the highest derivative in the equation

Newton's Law: $F = ma$

$\Longleftrightarrow F = m \frac{d^2 s}{dt^2}$

is a 2^{nd} order ODE.

PHASE PORTRAIT



The ODE connects every x to some y(x).

At every point (x,y) we can indicate the slope of y(x) since we know how to compute the derivative.

Initial conditions fix particular solutions.

SEPARABLE EQUATIONS : $\frac{dy}{dx} = f(x)\, g(y)$

Formal equation :

$$\frac{dy}{g(y)} = f(x)\, dx$$

By integrating

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C$$

<u>Example</u>

$$\frac{dy}{dx} = \frac{x}{y}$$

Here $f(x) = x$, $g(y) = \frac{1}{y}$

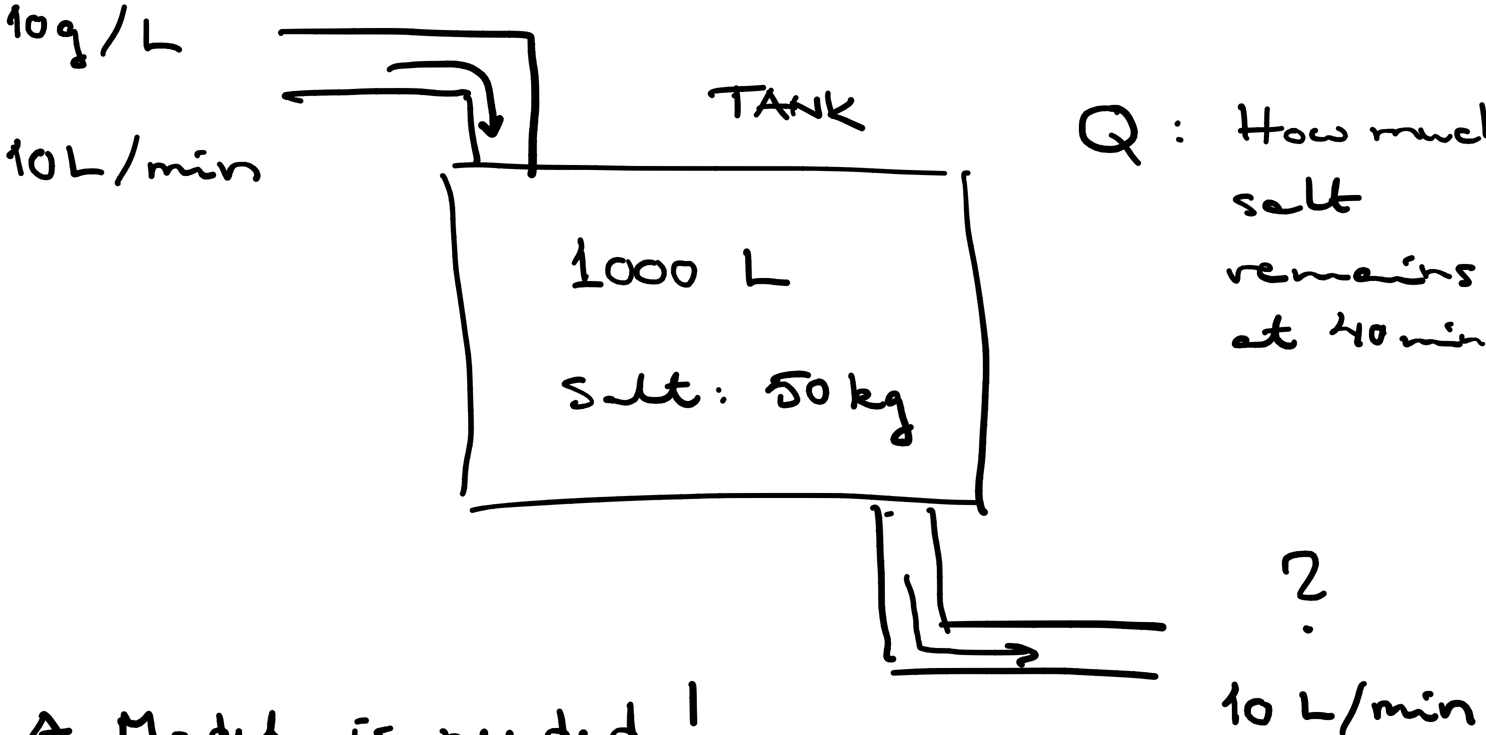
Thus, $\int y\, dy = \int x\, dx + \tilde{C}$

$$\Rightarrow \frac{1}{2} y^2 = \frac{1}{2} x^2 + \tilde{C}$$

Setting $2\tilde{C} = C$ we get $y^2 - x^2 = C$.

Hyperbolae, with asymptotes : $y = x$, $y = -x$ corresponding to $C = 0$.

A SOLUTION CONCENTRATION PROBLEM



Q: How much salt remains at 40 min?

A <u>Model</u> is needed!

$x(t)$ is the amount of salt ; $x(0) = 50$ kg

Salt entering the system :

$$10\,g/L \cdot 10\,L/min = \frac{1}{10}\,\frac{kg}{min}$$

Exiting :

$$\frac{x}{1000}\,\frac{kg}{L} \cdot 10\,L/min = \frac{x}{100}\,\frac{kg}{min}$$

The rate of change : $\frac{dx}{dt}$ = rate in − rate out

$$= \frac{1}{10} - \frac{x}{100} = \frac{10 - x}{100}$$

$$\frac{dx}{10-x} = \frac{dt}{100}$$

$$\Rightarrow \; -\ln|\underbrace{x-10}_{x>10}| = \frac{t}{100} + C$$

$$\Rightarrow \; \ln(x-10) = -\frac{t}{100} - C$$

$$\Rightarrow \; x(0) = 50 \quad \text{i.e.} \; -C = \ln 40$$

We get : $x = x(t) = 10 + 40e^{-t/100}$

After 40 minutes :

$$x(40) = 10 + 40e^{-0.4} \simeq 36.8 \text{ kg}$$

LINEAR 1ST ORDER ODE :

$$\frac{dy}{dx} + p(x)\, y = q(x)$$

If $q(x) = 0$, homogeneous,
$q(x) \neq 0$, non homogeneous.

$$\frac{dy}{dx} + p(x)\, y = 0 \quad \text{is separable :}$$

$$y = K e^{-\mu(x)}, \quad \mu(x) = \int p(x)\, dx ;$$

$$\frac{d\mu}{dx} = p(x)$$

Formally : $L = \dfrac{d(\cdot)}{dx} + p(x)(\cdot)$

so that

$$L(y) = q(x)$$

If $L(y_h) = 0$, then surely

$$L(y) + L(y_h) = q(x)$$

or $\quad L(y + y_h) = q(x)$

A: Integrating factor

$$\frac{d}{dx}\left(e^{\mu(x)} y(x)\right) = e^{\mu(x)} \frac{dy}{dx} + e^{\mu(x)} \frac{d\mu}{dx} y(x)$$

$$= e^{\mu(x)} \left(\frac{dy}{dx} + p(x) y(x) \right) = e^{\mu(x)} q(x)$$

Integrate: $e^{\mu(x)} y(x) = \int e^{\mu(x)} q(x)\, dx$

$$\Rightarrow y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x)\, dx$$

B: Variation of the parameter: $K = K(x)$

$$\frac{d}{dx}\left(K(x) e^{-\mu(x)}\right) + p(x) K(x) e^{-\mu(x)} = q(x)$$

$$\Rightarrow K'(x) e^{-\mu(x)} - K(x) \mu'(x) e^{-\mu(x)} + K(x) p(x) e^{-\mu(x)} = q(x)$$

$$\Rightarrow K'(x) = e^{\mu(x)} q(x)$$