2^{ND} ORDER LINEAR ODE WITH CONSTANT COEFFICIENTS

Consider:

$$y'' + ay' + by = 0 \quad , \quad a, b \text{ constants}$$

Educated guess : $y = e^{rx}$

So: $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

Substitute:

$$r^2 e^{rx} + a r e^{rx} + b e^{rx} = 0$$

$$\Longleftrightarrow \quad \underbrace{r^2 + ar + b = 0}_{\text{AUXILIARY EQUATION}}$$

with roots $\quad r = -\frac{a}{2} \pm \sqrt{\underbrace{\frac{a^2}{4} - b}_{\text{discriminant}}}$

three different cases:

A) $a^2 - 4b > 0$: Two distinct real roots r_1, r_2

B) $a^2 - 4b = 0$: Double root $r_{1,2} = -\frac{a}{2}$

C) $a^2 - 4b < 0$: $r_{1,2} = \alpha \pm i\beta$ (conjugate pair) the solution has the form:

A) $y(x) = y_1(x) + y_2(x)$

$$= c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

B) $y(x) = (c_1 + c_2 x) e^{-a/2 x}$

c) $y(x) = e^{\alpha x} (c_1 \sin \beta x + c_2 \cos \beta x)$

General form:

$$y'' + ay' + by = R(x)$$

1) $R(x)$ = polynomial of degree n

$y'' + ay' + by = x^2 + x + 1$

Candidate: $y_0(x)$ = polynomial of degree 2

Use the method of undetermined coefficients!

$$y'' + ay' + by = x^2 + x + 1$$

Try: $y_0(x) = a_2 x^2 + a_1 x^1 + a_0 x^0$

$$y_0'(x) = 2a_2 x + a_1$$

$$y_0''(x) = 2a_2$$

$$2a_2 + a(2a_2 x + a_1) + ba_2 x^2 + ba_1 x^2 + ba_0$$
$$= x^2 + x + 1$$

$$\left.\begin{aligned} x^2 &: \quad ba_2 = 1 \\ x^1 &: \quad a(2a_2) + ba_1 = 1 \\ x^0 &: \quad 2a_2 + aa_1 + ba_0 = 1 \end{aligned}\right\} \text{Solve } a_0, a_1, a_2$$

2) $R(x) = Ae^{\lambda x}$

Try: $y_0(x) = Ke^{\lambda x}$

Get: $y_0(x) = \dfrac{A}{\lambda^2 + a\lambda + b} e^{\lambda x}$

If λ is a root of the auxiliary equation, we need to change our educated guess :

Try : $y_0(x) = K x^m e^{\lambda x}$, where m is the order of the root.

<u>Example</u> $y'' + 2y' + y = e^{-x}$

Homog. $y_H = (C_1 + C_2 x) e^{-x}$

In terms of λ : auxiliary equation is

$(\lambda + 1)^2 \longrightarrow \lambda$ is a double root ($\lambda = -1$)

Try : $y_0 = K x^2 e^{-x}$

$y_0' = K(2x - x^2) e^{-x}$

$y_0'' = K(2 - 4x + x^2) e^{-x}$

$\Rightarrow K = 1/2$

The general solution:

$$y = y_H + y_0 = \left(C_1 + C_2 x + \frac{1}{2} x^2\right) e^{-x}$$

3) $R(x) = A \sin \omega x + B \cos \omega x, \quad \omega \neq 0$

Try : $y_0(x) = K \sin \omega x + L \cos \omega x$

Special case : $a = 0$, $b = \omega^2$, i.e.

$$y'' + \omega^2 y = A \sin \omega x + B \cos \omega x$$

Resonance :

Try : $y_0(x) = K\underline{x} \sin \omega x + L\underline{x} \cos \omega x$

<u>Example</u> $\quad y'' + 4y = \sin 2t$

$$y_0 = Kt \sin 2t + Lt \cos 2t$$

$$y_0' = (K - 2Lt) \sin 2t + (K + 2Lt) \cos 2t$$

$$y_0'' = -4(L + Kt) \sin 2t + 4(K - Lt) \cos 2t$$

We get :

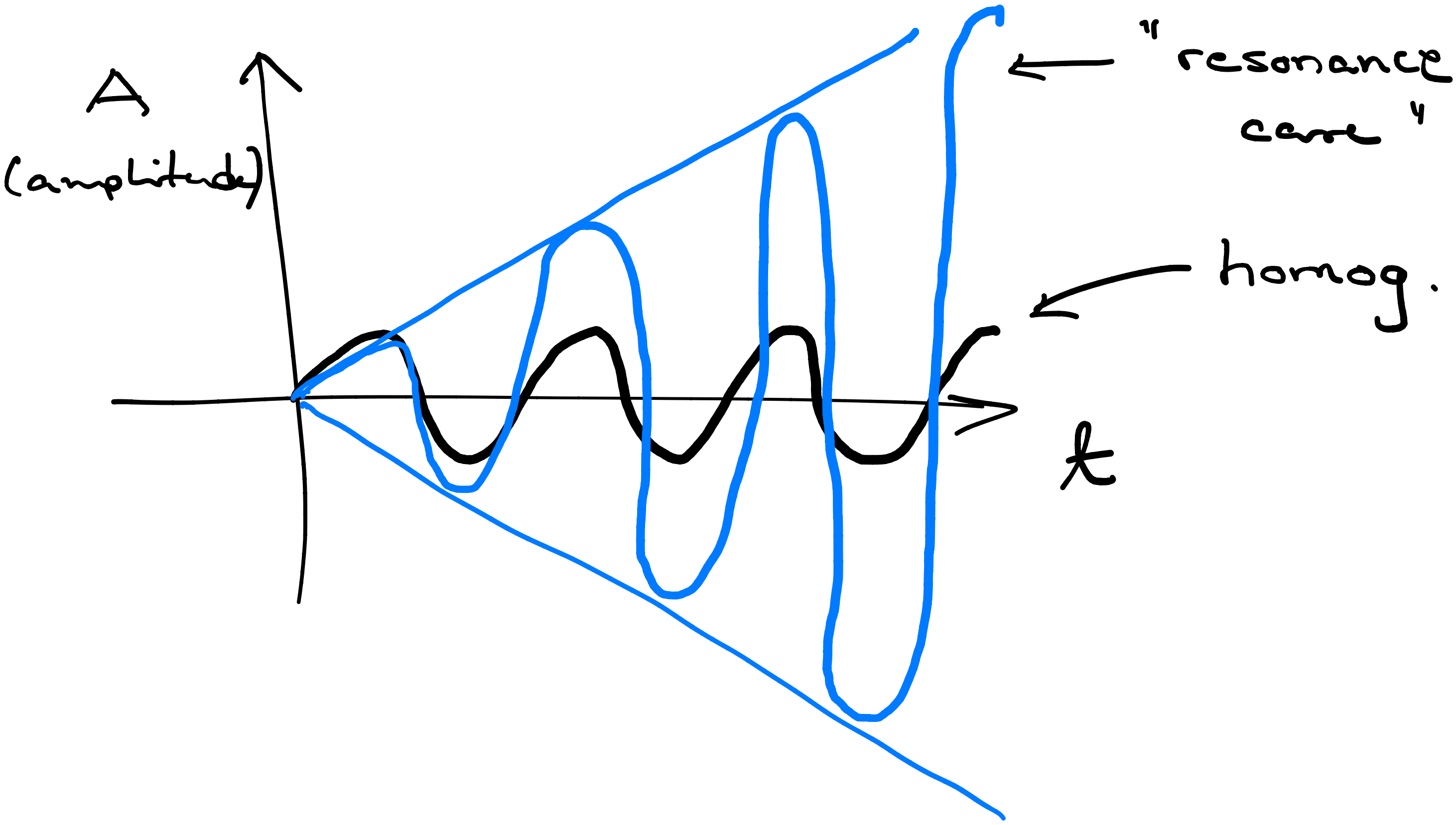
$$-4(L + Kt) \sin 2t + 4(K - Lt) \cos 2t + 4Kt \sin 2t + 4Lt \cos 2t = \sin 2t$$

$$\Longleftrightarrow \quad -4L \sin 2t + 4K \cos 2t = \sin 2t$$

$$\Longrightarrow \quad K = 0 \,, \quad L = -\frac{1}{4}$$

General solution: $y = y_H + y_0$

$$= c_1 \sin 2t + c_2 \cos 2t - \frac{1}{4} t \cos 2t$$



PRACTICAL RESONANCE

→ system with damping

$$y_H(x) = e^{-\frac{a}{2}x}\left(c_1 \sin\omega x + c_2 \cos\omega x\right)$$

→ $R(x)$ at frequency ω