MARGINAL MATTERS

Improper Integrals :

Two cases :

a) the interval is infinite

b) the function f is not bounded over the whole interval

Example :
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b \frac{dx}{1+x^2}$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \Big/_a^b \arctan x =$$

$$= \lim_{b \to \infty} \arctan b - \lim_{a \to -\infty} \arctan a$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Similarly at jumps, set the proper limits and proceed !

Series

Definition 1 Infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$$

Definition 2 A series converges to some value S:

$$\sum_{n=1}^{\infty} a_n = S, \quad \text{if } \lim_{n \to \infty} s_n = S, \quad \text{where}$$

s_n is a partial sum.

If a series does not converge, it diverges.

Definition 3 Geometric series

$$\sum_{n=1}^{\infty} a r^{n-1} \quad ; \quad \frac{a_{n+1}}{a_n} = r$$

$$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} 0, & \text{if } a = 0 \\ \frac{a}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{otherwise} \end{cases}$$

The <u>radius of convergence</u>: Here $|r| < 1$.

In the general case it is <u>not</u> straight-forward to determine convergence.

→ One must apply some convergence test

<u>Definition 4</u> Taylor series is a Taylor polynomial extended to a series.

<u>Definition 5</u>

A function f is <u>analytic</u> at some point c if its <u>Taylor series</u> converges at c.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

COMPLEX NUMBERS

$$\mathbb{C} = \{ (x,y) \mid x,y \in \mathbb{R} \}$$

$$\left.\begin{aligned} z_1 &= (x_1, y_1) \\ z_2 &= (x_2, y_2) \end{aligned}\right| \quad \begin{aligned} z_1 + z_2 &= (x_1 + x_2, y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$

For instance:

$$(0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 0\cdot 1)$$

$$= (-1, 0)$$

Notation: $i = (0,1)$

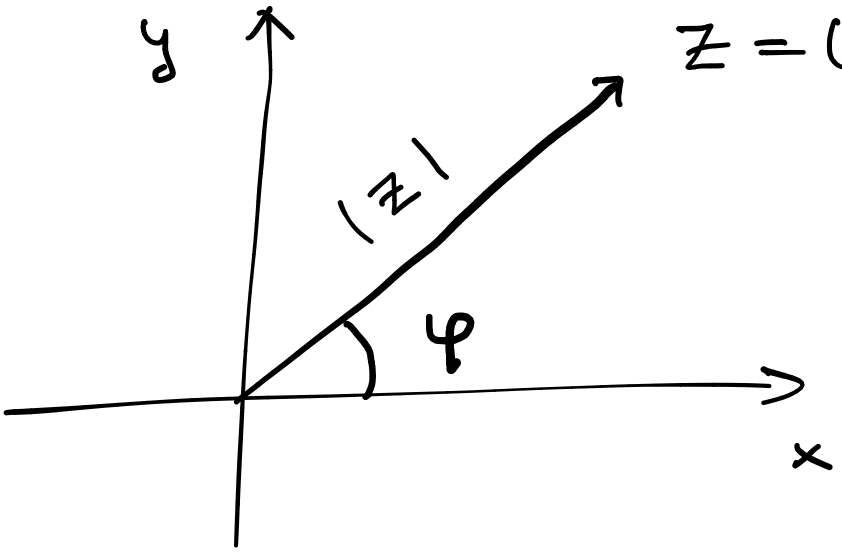
$$z = (x,y) = (x,0) + (0,y)$$

$$= (x,0) + (0,1)\cdot(y,0)$$

$$= x + iy, \quad x,y \in \mathbb{R}$$

That is: $$i^2 = -1$$

Polar coordinates:



$z = (x, y) = x + iy$

$$z = |z|(\cos\varphi + i\sin\varphi)$$
$$= |z|e^{i\varphi} = re^{i\varphi} \quad \text{(Euler notation)}$$

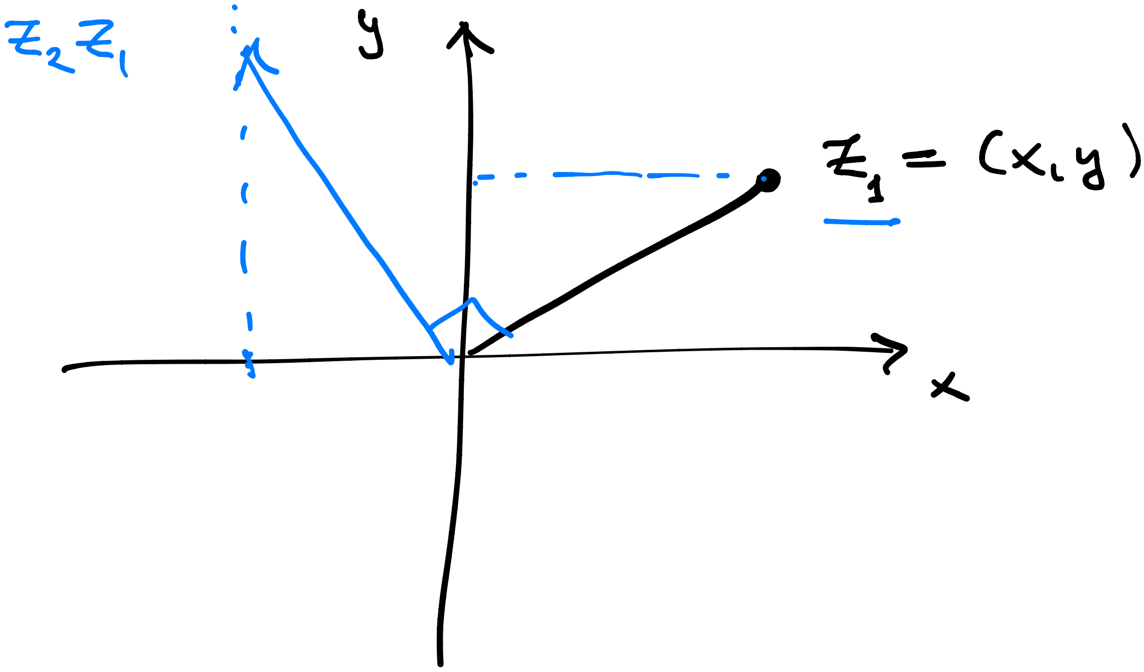
Multiplication:

Geometric interpretation:

scaling & rotation on the plane

Notice: $(0, y) = (0, 1) \cdot (y, 0)$
$= iy$

In other words i rotates by $\pi/2$.



$$z_1 = x + iy \quad ; \quad z_2 = i$$

$$z_2 z_1 = i(x + iy) = -y + ix$$

Euler notation:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$i\sin\varphi$

$$e^{i\varphi} = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!} = 1 + i\varphi - \frac{\varphi^2}{2} - \frac{i\varphi^3}{3!} + \dots$$

$\cos\varphi$

EXAM :

→ remote, open book, timed

→ no proctoring ← no monitoring

Structure of the exam :

Normal exercises (5 problems)

→ Learning objectives

Three Pillars :

→ Taylor polynomial

→ integration by parts

→ 2^{nd} order linear with constant coefficients

Basic Principles :

→ error estimates

→ "Where do things come from?"