A0111

Lecture 1 : Sequences

Notation: Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

Real numbers $\mathbb{R}$

Definition A sequence is an infinite sequence of reals $a_n \in \mathbb{R}$, where $n \in \mathbb{N}$.

There are many different notations : $(a_n)_{n\in\mathbb{N}} = (a_n)_{n=1}^{\infty}$

$= (a_1, a_2, a_3, \ldots)$

For our purposes it is convenient to interpret a sequence as a function :

$$f : \mathbb{N} \to \mathbb{R}, \quad f(n) = a_n, \quad n = 1, 2, 3, \ldots$$

This means that it is possible to visualise the sequence as a graph of a function.

Notice that indexing can start with any integer if so desired.

Sequences are natural, they are produced for instance by observations (idealised) or algorithms.

Example Fibonacci

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_n + f_{n+1}$$

This was originally a simple population model.

Limit of a Sequence

Definition A sequence (a_n) converges to a limit $a \in \mathbb{R}$, if $|a_n - a| \xrightarrow[n \to \infty]{} 0$.

Formal: (ε, δ) - version

For every $\varepsilon > 0$ there exists an index $n = n(\varepsilon)$, such that : $|a_n - a| < \varepsilon$ for every $n \geq n(\varepsilon)$.

We write : $\lim\limits_{n \to \infty} a_n = a$.

Example $\lim\limits_{n \to \infty} \frac{1}{n^2} = 0$

It follows immediately : $\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} < \varepsilon$,

if $n > 1/\sqrt{\varepsilon}$.

We can now choose $n(\varepsilon)$ as the next integer following $1/\sqrt{\varepsilon}$, that is, $n(\varepsilon) = \lceil 1/\sqrt{\varepsilon} \rceil$, where $\lceil \cdot \rceil$ is the ceiling - function.

With this construction $\varepsilon > 0$ can be taken to be arbitrarily small.

Existence of e

Axiom of Real Numbers: Every increasing and bounded (from above) sequence converges.

In other words: If $a_{n+1} \geq a_n$ and $a_n \leq M$ (constant), for all $n \in \mathbb{N}$, then the limit exists

$$a = \lim_{n \to \infty} a_n .$$

Euler's number, e: $\quad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \,?$

Our task is to show that the limit exists.

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!\,n^k}$$

$$= \sum_{k=0}^{n} \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)}{k!\;\underbrace{n\cdot n\cdot\ldots\cdot n}_{k \text{ times}}}$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\, 1\cdot \underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}_{n > k-1}$$

Increasing, since $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$.

Consider $\quad \dfrac{(n+1)\cdot n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{(n+1)^k} - \dfrac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}$

$$\Rightarrow (n+1)\,n^k - (n+1)^k > 0 \quad \text{always.}$$

Let us denote:

$$u_n = \left(1 + \frac{1}{n}\right)^n \quad ; \quad u_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}$$

We should be able to show that $u_{n+1}/u_n \geq 1$.

$$\frac{u_{n+1}}{u_n} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \Bigg(\underbrace{\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}}_{(*)}\Bigg)^n \left(1 + \frac{1}{n+1}\right)$$

$$(*) = \frac{n(n+2)}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Insert and use Bernoulli: For all $-1 < x \in \mathbb{R}$,
$(1+x)^n \geq 1 + nx$.

$$\geq \left(1 - \frac{n}{(n+1)^2}\right)\left(1 + \frac{1}{n+1}\right) = 1 + \frac{1}{(n+1)^3} \geq 1$$

This is just one way to show this property.

We shall return to this later.

Is it bounded from above?

$$\left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!} < 1+1+\sum_{k=2}^{n} \underbrace{\frac{1}{k(k-1)}}_{\frac{1}{k-1}-\frac{1}{k}}$$

$$= 1+1+\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ \dots\ +\left(\frac{1}{n-1}-\frac{1}{n}\right)\right)$$

$$= 3-\frac{1}{n} < 3 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the limit exists : $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Functions

Definition Function f is a mapping from a domain to its range.

$$f : A \to B$$

Domain : A

Range : B

Often : $f_A = \{ f(a) \mid a \in A \} \subset B$ is the image.

Here we focus on functions of a single real variable ;

$A \subseteq \mathbb{R}$

Let us extend the discussion on sequences to functions.

Definition Continuity

Let $A \subset \mathbb{R}$; $f : A \to \mathbb{R}$

The function f is continuous at point $a \in A$, if :

Always when $a_n \in A$ and $\lim_{n \to \infty} a_n = a$, then

$$\lim_{n \to \infty} f(a_n) = f(a).$$

If f is continuous at every point $a \in A$, then we say that f is continuous on the set A.

Given the definition above, it is clear that the concept of a limit of a function merits some further attention.

Let $f: A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that a sequence (a_n) exists, where $a_n \in A \setminus \{x_0\}$ for all n and further $\lim_{n \to \infty} a_n = x_0$.

Is this overly complicated? Let A be an open interval and x_0 its end point. A function can have a limit $\lim_{x \to x_0} f(x)$ even if the value $f(x_0)$ is defined.

<u>Definition</u> Limit

Let f be as above. The limit exists, if

$$\lim_{n \to \infty} f(a_n) = L \quad \text{always when } a_n \in A \setminus \{x_0\}$$
$$\text{and } \lim_{n \to \infty} a_n = x_0.$$

Notation: $\lim_{x \to x_0} f(x) = L$, where L is the limit.

Two important limits in this course are the derivative and the definite integral.

<u>Comment on the abstract definitions</u>: Why are we stating "obvious" facts in such a complicated way?

Good question!

When the analysis moves to higher dimensions the geometric complexity increases dramatically and these abstract definitions become absolutely necessary.