Derivative

Newton's Quotient (Difference Quotient, Newton Quotient)

Definition If the limit of the Newton quotient

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

exists ($\in \mathbb{R}$), we say that f is differentiable at a, and the limit is the derivative of f at a.

Many different notations are used: $f'(a) = Df(a) = \left.\frac{df}{dx}\right|_{x=a}$

Continuity and differentiability are related:

Theorem If f is differentiable at $x=a$, it is continuous at $x=a$.

Proof $f'(a)$ exists $\Rightarrow \dfrac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a) + \varepsilon(\Delta x)$,

where $\varepsilon(\Delta x) \to 0$ as $\Delta x \to 0$. Hence,

$$f(a+\Delta x) = f(a) + f'(a)\Delta x + \Delta x\, \varepsilon(\Delta x)$$

$$\Rightarrow \lim_{\Delta x \to 0} f(a+\Delta x) = f(a), \text{ i.e., } \lim_{x \to a} f(x) = f(a).$$

$\square$

What about the "obvious" fact that the derivative of a constant function is identically zero?

Theorem $f(x) = c$ for all $x \in \mathbb{R}$ $\Rightarrow f'(x) = 0$

Proof $\Delta f = 0$ for all $\Delta x \neq 0$ $\Rightarrow f'(x) = 0$ for all $x \in \mathbb{R}$.

$$\frac{dc}{dx} \equiv 0.$$

$\square$

Geometric interpretation : for $f : \mathbb{R} \to \mathbb{R}$ the derivative is the slope of the curve.

Rules Using the definition one can derive differentiation rules.

$$D(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$D\,\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\ , \quad g(x) \neq 0$$

$$D\,f(g(x)) = g'(x)f'(g(x))$$

An exotic one is for the inverse function f^{-1} :

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Example : $f(x) = x + x^3$; $f(1) = 2 \Rightarrow f^{-1}(2) = 1$

Then $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{1+3\cdot 1^2} = \dfrac{1}{4}$

Polynomial rule : $f'(x) = 1 + 3x^2$

Example $\dfrac{d}{dx}\sin x = \cos x$ using only the definition :

$$\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$

$$= \lim_{h\to 0}\sin x\,\underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{=0} + \lim_{h\to 0}\cos x\,\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{=1}$$

$$= \cos x$$

Clearly the two limits require some further explanations.

l' Hospital's Rule

An important building block of analysis is the concept of an intermediate value.

<u>Theorem</u> Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on the open interval (a,b) $(=]a,b[ )$. Then there exists a point $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b-a}$$

or alternatively $f(b) - f(a) = f'(\xi)(b-a)$.

This is foundation of the following:

<u>Theorem</u> l' Hospital

Let $f(x_0) = 0 = g(x_0)$ and both f and g be differentiable in the neighbourhood of x_0. If the limit $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} .$$

<u>Not a formal proof</u>: Assume that the derivatives are continuous and $g'(x_0) \neq 0$.

IVT:
$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi_1)(x - x_0)}{g'(\xi_2)(x - x_0)}$$

$$= \frac{f'(\xi_1)}{g'(\xi_2)} \longrightarrow \frac{f'(x_0)}{g'(x_0)} \quad \text{as } x \to x_0$$

Notice, that as the interval tends to a point x_0, also $\xi_1, \xi_2 \longrightarrow x_0$.