Applications of Derivatives

We have already mentioned the geometric interpretation of a derivative of a function. This leads to optimisation.

<u>Theorem</u> If $f'(x_0)$ exists and is > 0 (< 0), then f is increasing (decreasing) at x_0.

There is a powerful result by Weierstrass:

Every range of a continuous function over a closed interval has a maximum and a minimum.

Every function can have local extremal values, of course. The following holds:

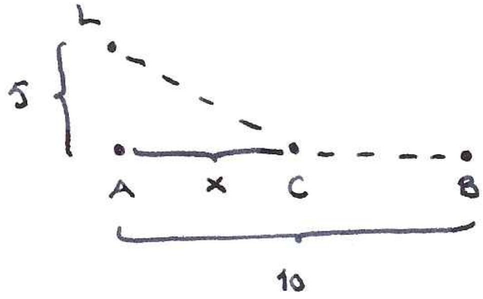
<u>Theorem</u> If f has a local extremal value at x_0 and if $f'(x_0)$ exists, then $f'(x_0) = 0$.

Extreme values can be classified using the second derivatives:

<u>Theorem</u> If $f'(x_0) = 0$ and $f''(x_0) > 0$ (< 0), then $f(x_0)$ is a local minimum (maximum).

Example "Cable laying problem" : Cost LC : 5000 / unit

BC : 3000 / unit



Total cost :

$$T = T(x)$$

$$= 5000\sqrt{25 + x^2} + 3000\,(10 - x)$$

$T(x)$ is continuous for all $x \in [0, 10]$, thus its minimum value is either at the end points or at a critical point on the interval.

$$\frac{dT}{dx} = \frac{5000\,x}{\sqrt{25 + x^2}} - 3000 = 0 \quad \Rightarrow \quad x = \frac{15}{4} = 3.75$$

$$T(0) = 55000\,, \quad T\left(\tfrac{15}{4}\right) = 50000\,, \quad T(10) \simeq 55900$$

The location of C should be at $x = 15/4$.

Exponential and Logarithmic Functions

Exponential function: $y = a^x$, $a \in (0, \infty) \setminus \{1\}$

Natural exponential: $y = e^x$, where e is the Euler's number.

In fact: $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$, $x \in \mathbb{R}$.

Definition The inverse of $y = e^x$ is the natural logarithmic function: $\ln : \mathbb{R}_+ \to \mathbb{R}$,

$$y = \ln x \iff x = e^y.$$

Rules: $\ln xy = \ln x + \ln y$, $x > 0$, $y > 0$

$\ln x^y = y \ln x$, $x > 0$, $y \in \mathbb{R}$

Notice: $e^{\ln xy} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$

Derivatives: $\frac{de^x}{dx} = e^x$, $\frac{d \ln x}{dx} = \frac{1}{x}$

Newton's quotient: $\frac{e^{x+h} - e^x}{h} = \frac{e^x(e^h - 1)}{h}$

(We cannot find this limit yet.)

Interestingly, we can now do the exponential:

Definition $a^x = e^{x \ln a}$, $a > 0$, $x \in \mathbb{R}$

Derivatives: $\frac{da^x}{dx} = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$

Logarithmic function : $y = \log_a x \iff x = a^y$

How to find the derivative? Solution: Implicit differentiation

Assume that $y = y(x)$ and differentiate on both sides :

$$x = a^y \overset{D}{\Longrightarrow} 1 = a^y \ln a \frac{dy}{dx} = x \ln a \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

Of course: $\log_a x = \frac{\ln x}{\ln a}$ leads to the same conclusion.

<u>Inverse Trigonometric Functions</u>

Trigonometric functions are periodic, therefore inverse functions can only be considered over specific intervals, or branches.

<u>Sine</u> : $f(x) = \sin x , \ -\pi/2 \leq x \leq \pi/2 .$

<u>Definition</u> arcsin x is the inverse function of the sine function

$$y = \arcsin x \iff x = \sin y , \ y \in [-\pi/2 , \pi/2]$$

<u>Derivative</u> $x = \sin y \overset{D}{\Longrightarrow} 1 = \cos y \frac{dy}{dx}$ [cos y does not change its sign]

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\Rightarrow \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

Tangent : $f(x) = \tan x \ , \quad -\pi/2 < x < \pi/2$

Definition arctan x is the inverse of the tangent function.

$$y = \arctan x \iff x = \tan y \ , \quad -\pi/2 < y < \pi/2$$

Derivative $x = \tan y \overset{D}{\Rightarrow} 1 = \frac{1}{\cos^2 y} \frac{dy}{dx} = (1 + \tan^2 y) \frac{dy}{dx}$

$$= (1 + x^2) \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$