Taylor Polynomials

Real problems can only rarely be solved exactly. We must approximate and accept error. This is not admission of defeat, however, every practical approach must be accompanied by an error estimate.

Definition The linearisation of the function f about a is the function L defined by

$$L(x) = f(a) + f'(a)(x-a).$$

Example $\sqrt{26} \approx ?$

Obviously $f(x) = \sqrt{x}$. We know that $\sqrt{25} = 5$, which suggest a linearisation about $x = 25$: $\quad f'(x) = \dfrac{1}{2\sqrt{x}}$

$$L(x) = 5 + \frac{1}{10}(x - 25) \quad \Rightarrow L(26) = 5.1$$

Compare with your calculator! (Which provides another approximation ...)

Error estimation Let $E(t) = f(t) - f(a) - f'(a)(a-t)$

and thus $E'(t) = f'(t) - f'(a)$. The latter suggest the second derivative:

Generalised Mean Value theorem: $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$

We choose $E(t)$ and $(t-a)^2$ on $[a,x]$:

$$\frac{E(x) - E(a)}{(x-a)^2 - (a-a)^2} = \frac{E'(\xi)}{2(\xi - a)} = \frac{f'(\xi) - f(a)}{2(\xi - a)} = \frac{1}{2} f''(\eta)$$

$$\Rightarrow E(x) = \frac{1}{2} f''(\eta)(x-a)^2$$

Let us next extend the idea to higher order polynomial approximations.

Let us assume that $f(a), f'(a), \dots, f^{(n-1)}(a)$ exist.
We want to approximate f about a with a polynomial T_{n-1} with a maximal degree $n-1$ such that its value and derivatives at a are exact.

We write:

$$T_{n-1}(x,a) = c_0 + c_1(x-a) + \dots + c_{n-1}(x-a)^{n-1}$$

$$T'_{n-1}(x,a) = c_1 + 2c_2(x-a) + \dots + (n-1)c_{n-1}(x-a)^{n-2}$$

$$\vdots$$

$$T^{(k)}_{n-1}(x,a) = k!\,c_k + (x-a)\underbrace{P(x)}_{\text{some polynomial}}, \quad k = 1, 2, \dots, n-2$$

$$\vdots$$

$$T^{(n-1)}_{n-1}(x,a) = (n-1)!\,c_{n-1}$$

Condition: $T^{(k)}_{n-1}(a,a) = f^{(k)}(a), \quad k = 0, 1, \dots, n-1,$

leads to a unique set of coefficients:

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, \dots, n-1.$$

<u>Definition</u> Taylor Polynomial

$$T_{n-1}(x,a) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

<u>Theorem</u> Lagrange Remainder

If $f^{(n)}(x)$ is continuous over $[a,x]$, then

$$f(x) = T_{n-1}(x,a) + \frac{f^{(n)}(\xi)}{n!}(x-a)^n,$$

where $\xi \in [a,x]$.

Example Maclaurin polynomial ; $a = 0$

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned} \qquad\qquad \begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \\ g'''(x) &= \sin x \\ g^{(4)}(x) &= \cos x \end{aligned}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \Theta(x^7) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \Theta(x^{2n+3})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \Theta(x^6) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \Theta(x^{2n+2})$$

Theorem Alternative formulation

If $f(x) = Q_n(x) + \Theta((x-a)^{n+1})$ as $x \to a$, where Q_n is a polynomial of degree at most n, then $Q_n(x) = T_n(x)$, that is, Q_n is the Taylor polynomial for $f(x)$ at $x = a$.

Example $T_3(x, 1)$ for e^{2x}. We know $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \Theta(x^{n+1})$

Writing $x = 1 + (x-1)$ we have

$$\begin{aligned} e^{2x} = e^{2 + 2(x-1)} &= e^2 e^{2(x-1)} \\ &= e^2 \left[1 + 2(x-1) + \frac{2^2 (x-1)^2}{2!} + \frac{2^3 (x-1)^3}{3!} + \Theta((x-1)^4) \right] \end{aligned}$$

as $x \to 1$.

Therefore:

$$T_3(x, 1) = e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2}{3}(x-1)^3$$

Taylor polynomial unlocks the general extreme value classification problem:

Theorem Let $f^{(n)}$ be continuous in the neighbourhood of $x = a$ and $f'(a) = f''(a) = \dots = f^{(n-1)}(a) = 0$ with $f^{(n)}(a) \neq 0$. If n is even and $f^{(n)}(a) > 0$ (< 0), then $f(a)$ is a local minimum (maximum). If n is odd, $f(a)$ is not an extreme value.

Taylor also brings joy to limits:

Example

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{2e^x - 2 - 2x - x^2}$$

Let us use 3^{rd} order polynomials and ignore error terms.

$$= \lim_{x \to 0} \frac{2\left(x - \frac{x^3}{3!}\right) - \left(2x - \frac{2^3 x^3}{3!}\right)}{2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) - 2 - 2x - x^2}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3} + \frac{4x^3}{3}}{\frac{x^3}{3}} = 3$$