The Definite Integral

Upper and lower Riemann sums

Let $f : [a,b] \to \mathbb{R}$ be bounded from above, i.e., $|f(x)| \leq M$ for all $x \in [a,b]$. Notice that f need not be continuous.

Let $p = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ be partition of $[a,b]$; $a = x_0 < x_1 < x_2 < \dots < x_n = b$; For the intervals we get

$$I_k = [x_{k-1}, x_k], \quad \Delta x = x_k - x_{k-1}, \quad k = 1, 2, \dots, n.$$

Since f is bounded it has a smallest upper bound (supremum) and a greatest lower bound (infimum).

Note: Adams uses "a least upper bound" for supremum.

We define:

$$G = \sup_{x \in [a,b]} f(x), \qquad g = \inf_{x \in [a,b]} f(x)$$

$$G_k = \sup_{x \in I_k} f(x), \qquad g_k = \inf_{x \in I_k} f(x)$$

Consequently, for every partition p of $[a,b]$ we can define

upper sum : $\overline{S}_p = \sum_{k=1}^{n} G_k \Delta x_k$, lower sum : $\underline{S}_p = \sum_{k=1}^{n} g_k \Delta x_k$

Obviously $g \leq g_k \leq G_k \leq G$ and $\sum_{k=1}^{n} \Delta x_k = b - a$.

Thus $$g \cdot (b-a) \leq \underline{S}_p \leq \overline{S}_p \leq G \cdot (b-a)$$

Definition $$\sup_p \underline{S}_p = \underline{\int_{[a,b]}} f, \qquad \inf_p \overline{S}_p = \overline{\int_{[a,b]}} f$$

lower integral upper integral

<u>Definition</u> If $\underline{\int_{[a,b]}} f = \overline{\int_{[a,b]}} f$, f is integrable and its integral is the value.

Notation: $\int_{[a,b]} f = \int_a^b f = \int_a^b f(x)\,dx$.

<u>Riemann Sum</u> : $S_p = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$, $\xi_k \in I_k$

Let the norm of the partition $|p| = \max_{1 \le k \le n} \Delta x_k$.

We can now give an equivalent definition for a definite integral:

<u>Definition</u> S_p has a limit A as $|p| \to 0$, if for every $\varepsilon > 0$ there exist $\delta > 0$ such that

$|p| < \delta \Rightarrow |S_p - A| < \varepsilon$ independent of the choice of ξ_k.

We write: $\lim_{|p| \to 0} S_p = A$.

<u>Theorem</u> If f is integrable, then $\lim_{|p| \to 0} S_p = \int_a^b f$.

Question: Is any of this actually useful?

<u>Example</u> Consider $\int_0^1 e^x\,dx$. $f(x) = e^x$ is continuous and hence integrable over $[0,1]$.

Let p be a uniform partition: $x_k = \frac{k}{n}$, $\Delta x_k = \frac{1}{n}$

Take $\xi_k = \frac{k-1}{n}$ (lower end of the interval): we get

$$S_p = \sum_{k=1}^{n} e^{k-1/n} \cdot \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} \left(e^{1/n}\right)^{k-1} = \frac{1}{n} \frac{1-e}{1-e^{1/n}}$$

$$= (e-1) \cdot \frac{1/n}{e^{1/n}-1} = e - 1 \text{ letting } n \to \infty$$

__Example__

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \cdot \frac{1}{n}$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln 2$$

__Properties__ : For all integrable functions:

$$\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$$

$$\int_a^b f = \int_a^c f + \int_c^b f, \quad a < c < b$$

__Theorem__ Let f be integrable over $[a,b]$ and continuous at $x_0 \in [a,b]$. Then $f:[a,b] \to \mathbb{R}$,

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable at x_0 and $F'(x_0) = f(x_0)$.

__Theorem__ Fundamental Theorem of Calculus

Let f be continuous over $[a,b]$, and G such that $G'(x) = f(x)$ for all $x \in [a,b]$.

Then $\displaystyle\int_a^b f(x)\,dx = G(b) - G(a)$.

__Example__ $$\int_0^1 \frac{dx}{1+x} = \Big/_0^1 \ln(1+x) = \ln 2 - \ln 1 = \ln 2$$