Techniques for Integration

Methods and identities :

Indefinite integral $F(x)$: $F'(x) = f(x)$

$$F(x) = \int_a^x f(t)\,dt$$ over closed interval $[a, x]$; f continuous

Since the derivative of a constant function is zero, we have

$$\int f(x)\,dx = F(x) + C, \quad C \text{ constant}.$$

Method of Substitution

The idea is simple, but the result is truly powerful.

Consider

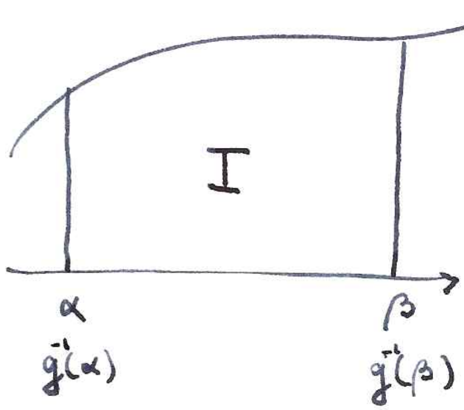
$$\int f(x)\,dx = \int F'(x)\,dx = F(x) + C, \quad \text{set } x = g(t)$$

$$= F(g(t)) + C = \int \frac{d}{dt} F(g(t))\,dt$$

$$= \int F'(g(t))\,g'(t)\,dt = \int f(g(t))\,g'(t)\,dt$$

But, what about definite integrals ?

the change of a variable can be interpreted as the change of units we are measuring.



The area under the curve cannot change, therefore the "unit" has to change : $dx = g'(t)\,dt$

$x = g(t) \Rightarrow \alpha = g(t_1)$

$\beta = g(t_2)$

What are the benefits?

If we know what we are doing, we can <u>transform</u> the original integral to something that is easier to work with.

<u>Example</u> $\displaystyle\int \frac{dx}{x^2+a^2}$, $a > 0$

It looks like arctan, but that $a^2 \neq 1$ and we are stuck.

What if we let $x = at$? Remember: $dx = a\,dt$

We get

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\,dt}{a^2t^2+a^2} = \frac{1}{a}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C$$

For the definite integral: $\alpha = at \Rightarrow t = \alpha/a$

$$\int_{\alpha}^{\beta} \frac{dx}{x^2+a^2} = \frac{1}{a}\int_{\alpha/a}^{\beta/a} \frac{dt}{t^2+1}$$; and the results agree!

Question: How do I determine the right change of variables in the general case?

Answer: This is based on pattern recognition, the more patterns you recognise, the more substitutions you can apply!

Hyperbolic Functions

Definition $\quad \cosh x = \dfrac{e^{x} + e^{-x}}{2} \quad , \quad \sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$$D \cosh x = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

$$D \sinh x = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

Naturally, there are many more identities analogous to the trigonometric identities.

What about their inverses? No surprises here:

Example $\quad \operatorname{arsinh} x = y \quad \Longleftrightarrow \quad x = \sinh y$

$$x = \sinh y = \frac{e^{y} - e^{-y}}{2} = \frac{(e^{y})^2 - 1}{2e^{y}}$$

We get: $(e^{y})^2 - 2x e^{y} - 1 = 0$

$$\Rightarrow \quad e^{y} = x \pm \underbrace{\sqrt{x^2+1}}_{>0} \quad , \text{ where } \pm = + \text{ since } e^{y} > 0 .$$

$$\Rightarrow \quad y = \ln\left(x + \sqrt{x^2+1}\right) = \operatorname{arsinh} x$$

The hyperbolic functions are used in engineering, it is good to know that they exist. For us the benefit is that we get still more patterns.

More examples:

$$\int \frac{dx}{x^2-a^2}, \quad a \neq 0. \quad \text{Let } x = at \Rightarrow \frac{1}{a}\int \frac{dt}{t^2-1} = \frac{1}{2a}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{x\,dx}{x^4+1} = ? \quad \text{Let } x^2 = t, \quad 2x\,dx = dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\overline{\arctan}\, t + C = \frac{1}{2}\overline{\arctan}(x^2) + C$$

$$\int_0^a \sqrt{a^2-x^2}\,dx = ? \quad \text{Let } x = a\sin t, \quad dx = a\cos t\,dt;$$

with $x \in [0, a]$, choose $t \in [0, \pi/2]$.

$$= \int_0^{\pi/2} a^2\cos^2 t\,dt = a^2\int_0^{\pi/2} \frac{1+\sin t\cos t}{2} = \frac{1}{4}\pi a^2$$

Useful identities:

f even: $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

f odd: $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

f ω-periodic:

$$\int_a^b f(x)\,dx = \int_{a+\omega}^{b+\omega} f(x)\,dx$$