Integration by Parts

Product rule: $\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$

Let us integrate on both sides and rearrange the terms:

$$\int u'(x)\,v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

This is a very powerful transformation!

Example $\int x^2 e^x\,dx = I$

$$\int \underbrace{x^2}_{v}\underbrace{e^x}_{u'}\,dx = \underbrace{x^2}_{v}\underbrace{e^x}_{u} - \underbrace{\int \underbrace{2x}_{v'}\,\underbrace{e^x}_{u}\,dx}_{2xe^x - \int 2e^x dx}$$

$$= x^2e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$$

Example $\int \ln x\,dx = I$

$$\int \underbrace{1}_{u'}\cdot\underbrace{\ln x}_{v}\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - x + C$$

Let us check this: Take the derivative

$$x\cdot\frac{1}{x} + \ln x - 1 = \ln x$$

Theorem Every rational function can be integrated in closed form.

Example
$$\int \frac{dx}{x(x^6+1)^2} = I$$

Substitution: $x^6 = t$, $6x^5\,dx = dt$.

Why?
$$\int \frac{dx}{x(x^6+1)^2} = \frac{1}{6}\int \frac{6x^5\,dx}{x^6(x^6+1)^2}$$
$$= \frac{1}{6}\int \frac{dt}{t(t+1)^2}$$

Partial fraction decomposition:

$$\frac{1}{t(t+1)^2} = \frac{A}{t} + \frac{B}{(t+1)^2} + \frac{C}{t+1}$$
$$= \frac{A(t+1)^2 + Bt + Ct(t+1)}{t(t+1)^2}$$

Set the numerators to be equal (comparing the coefficients)

$$\begin{array}{lll} t^2: & A + C = 0 & \\ t^1: & 2A + B + C = 0 & \Rightarrow \\ t^0: & A = 1 & \end{array} \quad \begin{array}{l} A = 1 \\ B = -1 \\ C = -1 \end{array}$$

Now we can integrate:

$$I = \frac{1}{6}\left(\ln|t| + \frac{1}{t+1} - \ln|t+1|\right) + C$$
$$= \frac{1}{6}\left(\ln\frac{x^6}{x^6+1} + \frac{1}{x^6+1}\right) + C$$

Example

$$\int \frac{x^4+1}{x^3-x^2+x-1}\,dx = \int \left(x + 1 + \frac{2}{x^3-x^2+x-1} \right) dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{2}{(x-1)(x^2+1)}\,dx$$

$$\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (Bx+C)(x-1) = 2$$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ -B + C = 0 \\ A - C = 2 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$

$$I = \frac{1}{2}x^2 + x + \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 + x + \ln|x-1| - \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

Example $I_n = \int x^n e^x \, dx \quad (n \in \mathbb{N})$

Recursion:

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$

$$= x^n e^x - n I_{n-1}$$

This terminates, since $I_0 = \int e^x \, dx = e^x + C$

Example Mathematical amusement:

$$I = \int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx$$

$$= -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

$$= -\sin x \cos x + \int dx - I$$

$$I = \frac{x - \sin x \cos x}{2} + C$$

Example $I = \int e^{kx} \sin nx \, dx \quad (k \neq 0)$

$$I = \frac{1}{k} e^{kx} \sin nx - \int \frac{n}{k} e^{kx} \cos nx \, dx$$

$$= \frac{1}{k} e^{kx} \sin nx - \frac{n}{k} \left[\frac{1}{k} e^{kx} \cos nx + \int \frac{n}{k} e^{kx} \sin nx \, dx \right]$$

$$= \frac{1}{k} e^{kx} \sin nx - \frac{n}{k^2} e^{kx} \cos nx - \frac{n^2}{k^2} I$$

$$I = \int e^{kx} \sin nx \, dx = \frac{k \sin nx - n \cos nx}{k^2 + n^2} e^{kx} + C$$