ORDINARY DIFFERENTIAL EQUATIONS (ODE)

General 1^{st} Order ODE : $\dfrac{dy}{dx} = f(x, y(x))$

The solution curve : $y(x)$

The equation connects every x to some $y(x)$.
Precisely at (x, y) the slope of the solution curve is $f(x, y(x))$.

Why this is not straightforward ?

- Since we have taken the derivative, we can only know the solution up to a constant.

Hence, the general solution includes all possible solutions. Initial conditions lead to particular solutions.

This geometric interpretation can be used to sketch solutions via so-called phase portraits (or diagrams).

The order of the ODE is the highest derivative in the equation. For instance, Newton's Law

$$F = ma \iff F = m\frac{d^2 s}{dt^2}$$

is a 2^{nd} order ODE.

It is clear that analytic solution techniques are limited by our ability to integrate and any numerical method must have an underlying quadrature rule associated with it.

Separable Equation : $\frac{dy}{dx} = f(x)\,g(y)$

We integrate both sides of a formal equation

$$\frac{dy}{g(y)} = f(x)\,dx$$

and arrive at

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C .$$

Example $\frac{dy}{dx} = \frac{x}{y}$; Here $f(x) = x$, $g(y) = \frac{1}{y}$

Thus $\int y\,dy = \int x\,dx + \tilde{C} \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + \tilde{C}$

Setting $2\tilde{C} = C$ we get $y^2 - x^2 = C$.

The solution curves are hyperbolae with asymptotes $y = x$, $y = -x$, corresponding the choice $C = 0$.

A Solution Concentration Problem

Initially a tank contains 1000 litres of brine with 50 kg of dissolved salt. Brine containing 10 g per litre is flowing into the tank at a constant rate of 10 litres per minute. If the contents of the tank are kept thoroughly mixed at all times, and if the solution also flows out at 10 litres per minute, how much salt remains in the tank at the end of 40 minutes?

We need a <u>model</u> : Let $x(t)$ be the amount of salt in the system ; $x(0) = 50$ is given.

Salt entering the system : $10 \text{ g/L} \cdot 10 \text{ L/min} = \frac{1}{10} \frac{\text{kg}}{\text{min}}$

exiting : $x/1000 \text{ kg/L} \cdot 10 \text{ L/min} = \frac{x}{100} \frac{\text{kg}}{\text{min}}$

The rate of change:

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= \frac{1}{10} - \frac{x}{100} = \frac{10 - x}{100}$$

Notice that constant solution $x = 10$ does not satisfy the initial conditions. The ODE is separable

$$\frac{dx}{100 - x} = \frac{dt}{100}$$

$$\Rightarrow \quad -\ln|x - 10| = \frac{t}{100} + C \quad ; \quad x - 10 > 0 \text{ always}$$

$$\Rightarrow \quad \ln(x - 10) = -\frac{t}{100} - C \quad ; \quad x(0) = 50$$

$-C = \ln 40$ and $x = x(t) = 10 + 40e^{-t/100}$

After 40 minutes : $x(10) = 10 + 40e^{-0.4} \simeq 36.8$ kg

Linear 1^{st} Order ODE : $\frac{dy}{dx} + p(x)\, y = q(x)$

Homogeneous, if $q(x) = 0$, otherwise non-homogeneous.

$\frac{dy}{dx} + p(x)\, y = 0$ is separable :

$$y = K e^{-\mu(x)}, \quad \mu(x) = \int p(x)\, dx \; ; \; \frac{d\mu}{dx} = p(x)$$

The solution of a homogeneous equation can always be added to any solution of the non-homogeneous equation.

Formally : Let us denote $L = \frac{d}{dx} + p(x)$ so that the problem is simply $L(y) = q(x)$. If $L(y_h) = 0$, then surely $L(y) + L(y_h) = q(x)$.

Two approaches :

A: Integrating factor : Multiply by $e^{\mu(x)}$: (!)

$$\frac{d}{dx}\left(e^{\mu(x)} y(x)\right) = e^{\mu(x)} \frac{dy(x)}{dx} + e^{\mu(x)} \frac{d\mu(x)}{dx} y(x)$$

$$= e^{\mu(x)} \left(\frac{dy(x)}{dx} + p(x)\, y(x) \right) = e^{\mu(x)} q(x)$$

Integrate:
$$e^{\mu(x)} y(x) = \int e^{\mu(x)} q(x)\, dx$$

$$\Rightarrow \quad y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x)\, dx$$

B: Variation of the parameter: Set $K = K(x)$:

$$\frac{d}{dx}\left(K(x)\,e^{-\mu(x)}\right) + p(x)\,K(x)\,e^{-\mu(x)} = q(x)$$

$$\Rightarrow K'(x)\,e^{-\mu(x)} - \underbrace{\mu'(x)}_{p(x)}\,K(x)\,e^{-\mu(x)} + p(x)\,K(x)\,e^{-\mu(x)} = q(x)$$

$\Rightarrow K'(x) = e^{\mu(x)}\,q(x)$ and the solution is exactly as before.

Example: $\frac{dy}{dx} + \frac{y}{x} = 1, \quad x > 0$

$p(x) = \frac{1}{x}, \quad \mu(x) = \int \frac{dx}{x} = \ln x \ (x > 0); \quad e^{\mu(x)} = x$

So $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x$

$$\Rightarrow xy = \int x\,dx = \frac{1}{2}x^2 + C \quad \Rightarrow y = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right)$$
$$= \frac{x}{2} + \frac{C}{x}$$

Alternative: $K = K(x)$; $\frac{dy}{dx} + \frac{y}{x} = 0 \Rightarrow y = Ke^{-\mu(x)} = \frac{K}{x}$

$$\frac{1}{x}K'(x) - \frac{1}{x^2}K(x) + \frac{1}{x^2}K(x) = 1$$

$\Rightarrow K'(x) = x \quad \Rightarrow K(x) = \frac{1}{2}x^2 + C$ (Hurrah!)