Numerical Solution of ODEs

Idea: Let us approximate the solution curve $y = y(x)$.

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

; As in the case of numerical quadratures we fix a step length h:
Points are then x_0, $x_0 + h$, $x_0 + 2h$, ...

<u>Definition</u> Euler's Method (Explicit)

$$x_{n+1} = x_n + h \quad ; \quad y_{n+1} = y_n + h f(x_n, y_n)$$

<u>Example</u>

$$\begin{cases} \dfrac{dy}{dx} = x - y \\ y(0) = 1 \end{cases}$$

; Over the interval $[0,1]$
$h = 1/5$

Exact solution: $y = x - 1 + 2e^{-x}$

Euler : $x_0 = 0$, $y_0 = 1$; $x_n = \dfrac{n}{5}$; $y_{n+1} = y_n + \frac{1}{5}(x_n - y_n)$

At $x_n = 1$; Error $e_n = y(x_n) - y_n \sim 0.08$

<u>Definition</u> Modified Euler's Method

$$x_{n+1} = x_n + h$$

$$u_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h\big(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\big)/2$$

This is an example of so-called predictor - corrector methods. u_{n+1} is a prediction which is corrected in y_{n+1}.

__Definition__ Euler's Method (Implicit)

$$x_{n+1} = x_n + h \; ; \; y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

Notice: Every step requires a solution of an equation.

__Rule of thumb__: As $h \to 0$ Euler's method becomes convergent.

Conversely, Implicit Euler is stable for all h.

There are many methods available and even under development.

2^{nd} ORDER ODEs

General case : $\phi(x, y, y', y'') = 0$ (implicit)

or $y'' = f(x, y, y')$ (explicit)

The solution has the form : $y = \varphi(x, C_1, C_2)$

The particular solution includes two constants and thus two conditions have to be defined.

Either : An initial value : $y(x_0) = y_0$, $y'(x_0) = p_0$

or : A boundary value problem : $y(x_1) = y_1$, $y(x_2) = y_2$

Theorem Explicit equation $y'' = f(x, y, y')$ is equivalent with a normal group

$$\begin{cases} y' = z \\ z' = f(x, y, z) . \end{cases}$$

Proof 1) $y'' = f(x, y, y')$ and the normal group have the same solution.

2) $y'(x) = z(x) \Rightarrow y''(x) = z'(x) = f(x, y(x), z(x))$

$\Rightarrow y''(x) = f(x, y(x), y'(x))$ $\square$

2^{nd} Order Linear ODE with Constant Coefficients

Consider $y'' + ay' + by = 0$.

The solution is likely to have a form $y = e^{rx}$, let us see what happens !

So : $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

Substituting we get an auxiliary equation

$$r^2 + ar + b = 0 ,$$

with roots $r = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$.

Three different cases :

A) $a^2 - 4b > 0$: Two distinct real roots r_1, r_2

B) $a^2 - 4b = 0$: Double root $r_{1,2} = -\frac{a}{2}$

C) $a^2 - 4b < 0$: Complex conjugate pair $r_{1,2} = \alpha \pm i\beta$

The general solution has the form given by the roots :

A) $y(x) = y_1(x) + y_2(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

B) $y(x) = (C_1 + C_2 x)\, e^{-a/2\, x}$

C) $y(x) = e^{\alpha x} (C_1 \sin \beta x + C_2 \cos \beta x)$

The equation $y'' + ay' + by = R(x)$

can always be solved with two applications of quadrature rules.