2^{nd} Order Linear ODE with Constant Coefficients

Consider $y'' + ay' + by = 0$.

The solution is likely to have a form $y = e^{rx}$, let us see what happens!

So: $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

Substituting we get an auxiliary equation

$$r^2 + ar + b = 0,$$

with roots $r = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$.

Three different cases:

A) $a^2 - 4b > 0$: Two distinct real roots r_1, r_2

B) $a^2 - 4b = 0$: Double root $r_{1,2} = -\frac{a}{2}$

C) $a^2 - 4b < 0$: Complex conjugate pair $r_{1,2} = \alpha \pm i\beta$

The general solution has the form given by the roots:

A) $y(x) = y_1(x) + y_2(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

B) $y(x) = (C_1 + C_2 x)\, e^{-a/2\, x}$

C) $y(x) = e^{\alpha x}(C_1 \sin\beta x + C_2 \cos\beta x)$

The equation $y'' + ay' + by = R(x)$

can always be solved with two applications of quadrature rules.

Let us next examine common types of problems by testing with different RHSs.

1) $R(x)$ = polynomial of degree n

Method of undetermined coefficients should be used just as in the case of partial fractions before.

Substitute a polynomial $y_0(x)$ of degree n and solve for the coefficients.

2) $R(x) = Ae^{\lambda x}$ (exponential)

Try: $y_0(x) = Ke^{\lambda x}$

Get: $y_0(x) = \dfrac{A}{\lambda^2 + a\lambda + b} e^{\lambda x}$

If λ is a root of the auxiliary equation, try

$y_0(x) = Kx^m e^{\lambda x}$, where m is the order of the root.

<u>Example</u> $y'' + 2y' + y = e^{-x}$

Homog: $y_H = (C_1 + C_2 x)e^{-x}$

Particular: The right-hand-side matches the solution of the homogeneous problem.

Auxiliary equation: $(1+\lambda)^2 = 0$

$$\left.\begin{aligned} \text{Try: } y_0 &= Kx^2 e^{-x}, \\ y_0' &= K(2x - x^2)e^{-x} \\ y_0'' &= K(2 - 4x + x^2)e^{-x} \end{aligned}\right\} \Rightarrow K = 1/2$$

The general solution is $y = y_H + y_0 = (C_1 + C_2 x + \frac{1}{2}x^2)e^{-x}$

3) $R(x) = A \sin \omega x + B \cos \omega x \quad , \quad \omega \neq 0$

Try: $y_0(x) = K \sin \omega x + L \cos \omega x$

Special case: $a = 0$ and $b = \omega^2$ gives

$$y'' + \omega^2 y = A \sin \omega x + B \cos \omega x ,$$

where the frequency ω is the same on both sides.

This is known as resonance.

Try: $y_0(x) = K x \sin \omega x + L x \cos \omega x$.

Example $\quad y'' + 4y = \sin \omega t \; ; \; \omega = 2$

$$y_0 = Kt \sin 2t + Lt \cos 2t$$

$$y_0' = (K - 2Lt) \sin 2t + (K + 2Lt) \cos 2t$$

$$y_0'' = -4(L + Kt) \sin 2t + 4(K - Lt) \cos 2t$$

Substituting:

$$-4(L + Kt) \sin 2t + 4(K - Lt) \cos 2t$$
$$+ 4Kt \sin 2t + 4Lt \cos 2t = \sin 2t$$

$$\Longleftrightarrow \quad -4L \sin 2t + 4K \cos 2t = \sin 2t$$

$$\Longrightarrow \quad K = 0 \; , \; L = -\frac{1}{4}$$

General solution: $y = y_H + y_0$

$$= C_1 \cos 2t + C_2 \sin 2t - \frac{1}{4} t \cos 2t$$

Why resonance? The amplitude grows as $t \to \infty$.