MARGINAL MATTERS

Improper Integrals : Two cases :

a) the interval is infinite

b) the function f is not bounded over the whole interval

Example

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b \frac{dx}{1+x^2}$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \Big/_a^b \arctan x = \lim_{b \to \infty} \arctan b - \lim_{a \to -\infty} \arctan a$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Similarly at jumps, set the proper limits and proceed.

Series

Definition 1 $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$ (Infinite Sum)

Definition 2 A series converges (if it does!) to some value s:

$$\sum_{n=1}^{\infty} a_n = s, \quad \text{if } \lim_{n \to \infty} s_n = s, \text{ where}$$

s_n is a partial sum.

If a series does not converge, it diverges.

Definition 3 Geometric series : $\sum_{n=1}^{\infty} a r^{n-1}$; $\frac{a_{n+1}}{a_n} = r$

It is known that
$$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} 0, & \text{if } a = 0 \\ \frac{a}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{otherwise} \end{cases}$$

The radius of convergence here is $|r| < 1$.

In the general case it is not at all clear to verify convergence. There are many tests available in the literature.

Definition 4 Taylor polynomial leads to Taylor series.

Definition 5 A function f is analytic at some point c if its Taylor series converges at c.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \quad ; \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ for all } x$$

Complex Numbers

$$\mathbb{C} = \{ (x,y) \mid x, y \in \mathbb{R} \}$$

$$\left.\begin{aligned} z_1 &= (x_1, y_1) \\ z_2 &= (x_2, y_2) \end{aligned}\right\} \quad \begin{aligned} z_1 + z_2 &= (x_1 + x_2 , y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2 , x_1 y_2 + x_2 y_1) \end{aligned}$$

For instance:

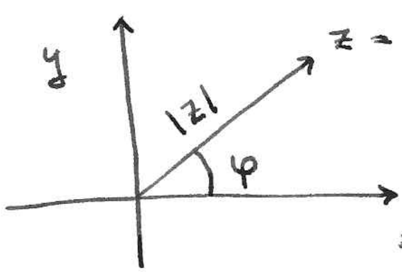
$$(0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1 , 0\cdot 1 + 0\cdot 1)$$
$$= (-1, 0)$$

Notation: $i = (0,1)$:

$$z = (x,y) = (x,0) + (0,y)$$
$$= (x,0) + (0,1)\cdot(y,0)$$
$$= x + iy , \quad x, y \in \mathbb{R}$$

That is: $i^2 = -1$.

Polar coordinates:



$$z = (x,y) = x + iy$$
$$= |z| (\cos\varphi + i \sin\varphi)$$
$$= |z| e^{i\varphi} = r e^{i\varphi}$$

Multiplication with a complex number has a geometric interpretation: it means scaling and rotation on the plane.