3. Theoretical exercises

Demo exercises

3.1 Let u and v be random variables with the following properties,

$$\begin{aligned}\mathbb{E}(u) &= \mathbb{E}(v) = 0\\ \mathrm{Var}(u) &= \mathrm{Var}(v) = \sigma^2 < \infty\\ \mathrm{Cov}(u,v) &= \mathbb{E}(uv) = 0\end{aligned}$$

Let $\lambda \in \mathbb{R}$ be a nonrandom constant. Show that the process,

$$x_t = u \cdot \cos(\lambda t) + v \cdot \sin(\lambda t), \quad t \in T,$$

is stationary.

Solution.

i) The expectation of x_t is,

$$\begin{aligned}\mathbb{E}(x_t) &= \mathbb{E}(u \cdot \cos(\lambda t) + v \cdot \sin(\lambda t))\\ &= E(u)\cos(\lambda t) + \mathbb{E}(v)\sin(\lambda t)\\ &= 0 \cdot \cos(\lambda t) + 0 \cdot \sin(\lambda t)\\ &= 0, \quad \text{for all } t \in T.\end{aligned}$$

ii) The variance of x_t is,

$$\begin{aligned}\mathrm{Var}(x_t) &= \mathbb{E}(x_t^2) - [\mathbb{E}(x_t)]^2\\ &= \mathbb{E}(u^2\cos^2(\lambda t) + v^2\sin^2(\lambda t) + 2uv\cos(\lambda t)\sin(\lambda t))\\ &= \mathbb{E}(u^2)\cos^2(\lambda t) + \mathbb{E}(v^2)\sin^2(\lambda t) + 2\mathbb{E}(uv)\cos(\lambda t)\sin(\lambda t)\\ &= \sigma^2\cos^2(\lambda t) + \sigma^2\sin^2(\lambda t)\\ &= \sigma^2, \quad \text{for all } t \in T.\end{aligned}$$

iii) The autocovariance of the process x_t is given by,

$$\begin{aligned}\mathrm{Cov}(x_t, x_{t-\tau}) &= \mathbb{E}(x_t x_{t-\tau})\\ &= \mathbb{E}(u^2\cos(\lambda t)\cos(\lambda(t-\tau)) + v^2\sin(\lambda t)\sin(\lambda(t-\tau))\\ &+ uv\cos(\lambda t)\sin(\lambda(t-\tau)) + uv\sin(\lambda t)\cos(\lambda(t-\tau))]\\ &= \mathbb{E}(u^2)\cos(\lambda t)\cos(\lambda(t-\tau)) + \mathbb{E}(v^2)\sin(\lambda t)\sin(\lambda(t-\tau))\\ &+ \mathbb{E}(uv)\cos(\lambda t)\sin(\lambda(t-\tau)) + \mathbb{E}(uv)\sin(\lambda t)\cos(\lambda(t-\tau))\\ &= \sigma^2\cos(\lambda t)\cos(\lambda(t-\tau)) + \sigma^2\sin(\lambda t)\sin(\lambda(t-\tau))\\ &= \sigma^2\cos(\lambda t - \lambda(t-\tau))\\ &= \sigma^2\cos(\lambda\tau), \quad \text{for all } t \in T.\end{aligned}$$

Note that $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. By i), ii) and iii), it follows that x_t is stationary.

3.2 Consider the following stationary AR(1) process,

$$x_t - \phi_1 x_{t-1} = \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0, \hat{\sigma}^2). \tag{1}$$

Show that, in the case of AR(1)-processes, the assumption of weak stationarity implies that $|\phi_1| < 1$. In the proof, assume that,

$$\mathbb{E}\left[x_{t-1}\varepsilon_t\right] = 0, \quad \text{for every } t \in T. \tag{2}$$

Solution. A stochastic process x_t is weakly stationary if the following conditions are satisfied.

(i) $\mathbb{E}(x_t) = \mu, \quad \forall t \in T,$

(ii) $\text{Var}(x_t) = \sigma^2 < \infty, \quad \forall t \in T,$

(iii) $\text{Cov}(x_t, x_{t-\tau}) = \gamma_\tau, \quad \forall t, \tau \in T.$

By condition (ii), the variance of the process is not time dependent. Then, by Assumption (2),

$$\text{Var}(x_t) = \text{Var}(\phi_1 x_{t-1} + \varepsilon_t) = \phi_1^2 \text{Var}(x_{t-1}) + \text{Var}(\varepsilon_t) = \phi_1^2 \text{Var}(x_t) + \hat{\sigma}^2, \tag{3}$$

As we have

$$Var(x+y) = E[x+y-E(x+y)]^2$$

Substitute by x_t

$$\begin{aligned}
\text{Var}(x_t) &= \text{Var}(\phi_1 x_{t-1} + \varepsilon_t) \\
&= E[(\phi_1 x_{t-1} + \varepsilon_t) - E(\phi_1 x_{t-1} + \varepsilon_t)]^2 \\
&= E[(\phi_1 x_{t-1} + \varepsilon_t) - E(\phi_1 x_{t-1}) - E(\varepsilon_t)]^2 \\
&= E[(\phi_1 x_{t-1} + \varepsilon_t) - E(\phi_1 x_{t-1})]^2 \\
&= E[\phi_1^2 x_{t-1}^2] + 2E[\phi_1 x_{t-1}\varepsilon_t] + E[\varepsilon_t^2] - 2E[(\phi_1 x_{t-1} + \varepsilon_t)]E[\phi_1 x_{t-1}] + (E[\phi_1 x_{t-1}])^2 \\
&= \phi_1^2 E[x_{t-1}^2] + 2\phi_1 E[x_{t-1}\varepsilon_t] + E[\epsilon_t^2] - \phi_1^2 (E[x_{t-1}])^2 \\
&= \phi_1^2 E[x_{t-1}^2] - \phi_1^2 (E[x_{t-1}])^2 + E[\epsilon_t^2]
\end{aligned}$$

Therefore,

$$\text{Var}(x_t) = \phi_1^2 \text{Var}(x_{t-1}) + \text{Var}(\varepsilon_t)$$

As the variance of weakly stationary process does not depend on the time t.

$$\text{Var}(x_t) = \text{Var}(x_{t-1})$$

Then

$$\text{Var}(x_t) = \phi_1^2 \text{Var}(x_t) + \text{Var}(\varepsilon_t)$$

And since $\varepsilon_t \sim \text{i.i.d.}(0, \hat{\sigma}^2)$. By solving the equation above for $\text{Var}(x_t)$, we obtain,

$$\text{Var}(x_t) = \frac{\hat{\sigma}^2}{1 - \phi_1^2}.$$

Condition (ii) gives us that the variance is finite. Furthermore, by definition, variance cannot be negative. Thus, $|\phi_1| < 1$.

Homework

3.3 Show that MA(1) process

$$x_t = \varepsilon_t + \theta\varepsilon_{t-1}, \qquad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2),$$

is always stationary.

3.4 a) Derive the autocorrelation function of a MA(1) process.

b) Derive the autocorrelation function of a stationary AR(1) process:

$$x_t = \phi x_{t-1} + \varepsilon_t, \quad (\varepsilon_t)_{t\in\mathbb{Z}} \sim \text{i.i.d.}(0, \sigma^2).$$

Use the MA(∞) representation,

$$x_t = \Psi(L)\varepsilon_t = \sum_{i=0}^{\infty} \phi^i L^i \varepsilon_t,$$

where the series $\sum_{i=0}^{\infty} \phi^i$ converges absolutely.
Hint: If $\sum_{i=0}^{\infty} a_i$ converges absolutely, and $\mathbb{E}[|x_i|]$ is an index invariant finite constant, then $\mathbb{E}[\sum_{i=0}^{\infty} a_i x_i] = \sum_{i=0}^{\infty} a_i \mathbb{E}[x_i]$.