4. Theoretical exercises

Demo exercises

Throughout these exercises, assume that $\mathbb{E}[x_{t-v}\epsilon_t] = 0$ for all $v \geq 1$. In addition, assume that $(\epsilon_t)_{t\in T} \sim$ i.i.d.$(0, \sigma^2)$, such that $\sigma^2 < +\infty$.

4.1 Consider the following AR(1) processes:

$$x_t = 0.7x_{t-1} + \epsilon_t \tag{1}$$
$$x_t = -0.5x_{t-1} + \epsilon_t \tag{2}$$

(a) Show that both processes are (weakly) stationary.

(b) Using pen and paper, draw the auto- and partial autocorrelation functions that correspond to the process (2).

Solution.

(a) An AR(1) autoregressive process has the following lag polynomial representation: $(1-\phi_1 L)x_t = \epsilon_t$. An ARMA process is stationary, if the zeros of the lag polynomial of the autoregressive part lie outside the closed unit disk. The lag polynomials are,

$$(1 - 0.7L) = 0 \quad \Rightarrow \quad L = 10/7$$
$$(1 + 0.5L) = 0 \quad \Rightarrow \quad L = -2$$

Hence, both AR(1) processes are stationary.

(b) In the previous homework assignment, we derived the autocorrelation function for the AR(1) process:

$$\rho(\tau) = \phi_1^\tau.$$

Use this formula to draw the autocorrelation function. Note that the autocorrelation function decays exponentially. For example, when $\phi_1 = -0.5$:

$$\rho_0 = 1, \quad \rho_1 = -1/2, \quad \rho_2 = 1/4, \quad \rho_3 = -1/8, \ldots$$

Recall that the partial autocorrelation function of an AR(1) process cuts of after lag 1. Thus, it suffices to determine only the first partial autocorrelation. Hereby, the Yule-Walker equations give us directly that, $\alpha_{11} = \rho_1$.

4.2 Solve the partial autocorrelation α_2 by using the Yule-Walker equations.

Solution. Denote $\alpha_{21} = \alpha_1$ ja $\alpha_{22} = \alpha_2$. The Yule-Walker equations give,

$$\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}.$$

The matrix equations produce the following,

$$\begin{cases} \alpha_1 + \alpha_2\rho_1 = \rho_1 \\ \alpha_1\rho_1 + \alpha_2 = \rho_2. \end{cases}$$

By solving the upper equation for α_1 and substituting the solution into the lower equation, we get that,

$$\begin{aligned} &(\rho_1 - \alpha_2\rho_1)\rho_1 + \alpha_2 = \rho_2 \\ &\Rightarrow \alpha_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}. \end{aligned}$$

4.3 Derive the spectral density functions of MA(1) and SMA$(1)_{12}$ processes.

Solution. The processes are of the form

$$\begin{aligned} x_t &= \varepsilon_t + \theta_1\varepsilon_{t-1} \quad (\text{MA}(1)), \\ x_t &= \varepsilon_t + \Theta_1\varepsilon_{t-12} \quad (\text{SMA}(1)_{12}), \end{aligned}$$

where $\varepsilon_t \sim \text{i.i.d}(0, \sigma^2)$ in both processes for every $t \in T$. The spectral density function $f(\lambda)$ of a stationary process is

$$f(\lambda) = \frac{1}{2\pi}\left(\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k\cos(\lambda k)\right), \qquad \lambda \in [0, \pi],$$

where γ_k is the k:th autocovariance and λ is the frequency. By the previous homework assignment, we have that the autocovariance function of a MA(1) process is,

$$\gamma_k = \begin{cases} (1 + \theta_1^2)\sigma^2, & k = 0 \\ \theta_1\sigma^2, & |k| = 1 \\ 0, & |k| > 1. \end{cases}$$

Hence, the spectral density function of MA(1) process is

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + \theta_1^2 + 2\theta_1\cos(\lambda)\right).$$

Similarly as in the case of MA(1) process, one can derive the autocovariance function for SMA$(1)_{12}$ process. The result is given below.

$$\gamma_k = \begin{cases} (1 + \Theta_1^2)\sigma^2, & k = 0 \\ \Theta_1\sigma^2, & k = \pm 12 \\ 0, & k \in \mathbb{Z}\setminus\{-12, 0, 12\}. \end{cases}$$

Hence, the spectral density function of SMA$(1)_{12}$ process is

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + \Theta_1^2 + 2\Theta_1\cos(12\lambda)\right).$$

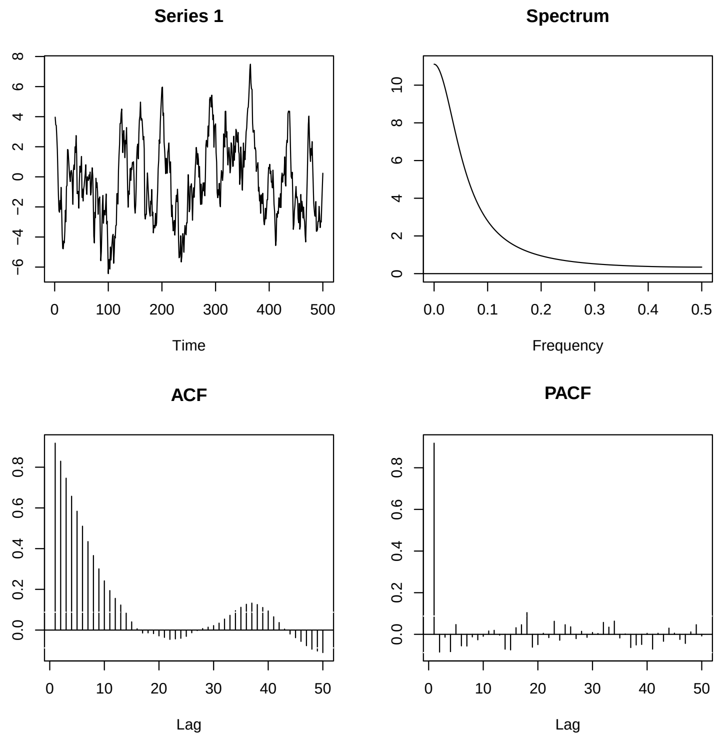

Figure 1: Time series 1 and the corresponding spectral, auto- and partial autocorrelation functions.

Homework

4.4 We have simulated four time series using R. Figures 1–4 contain the trajectories, spectrum, autocovariance function and partial autocovariance function of the corresponding time series. Using Figures 1–4, choose the correct model from the choices given in Table 1. Justify your selection!

Table 1: Choose the correct process.

Time series	**Model candidates**
1	MA(1), AR(1)
2	AR(2), MA(2), ARMA(2,2)
3	$SMA(1)_{12}$, AR(12), $SAR(1)_{12}$
4	$SMA(1)_{12}$, MA(12), $SAR(1)_{12}$

In Figures 1-4, the spectral density functions are calculated from the theoretical stochastic process. The corresponding autocorrelation functions and the partial autocorrelation functions are estimated from the observed time series.

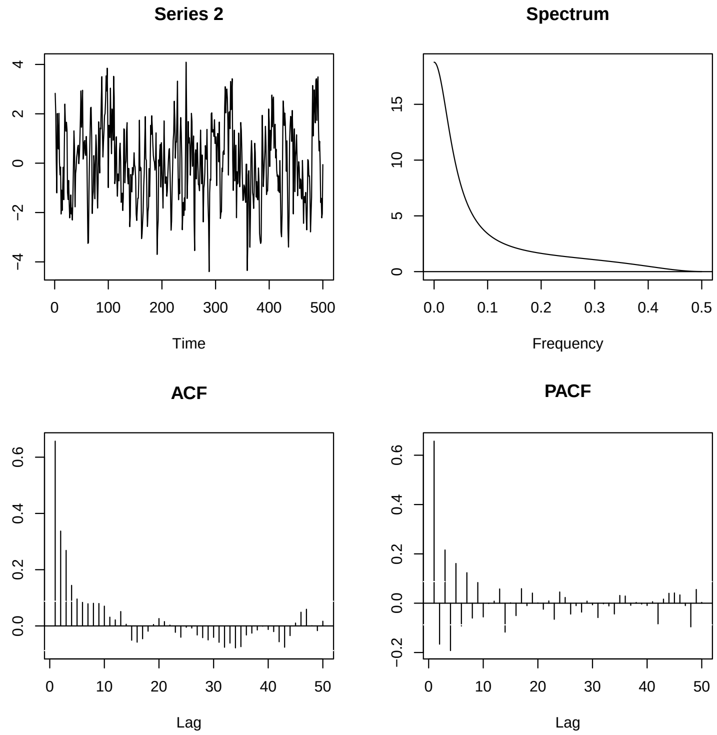

Figure 2: Time series 2 and the corresponding spectral, auto- and partial autocorrelation functions.

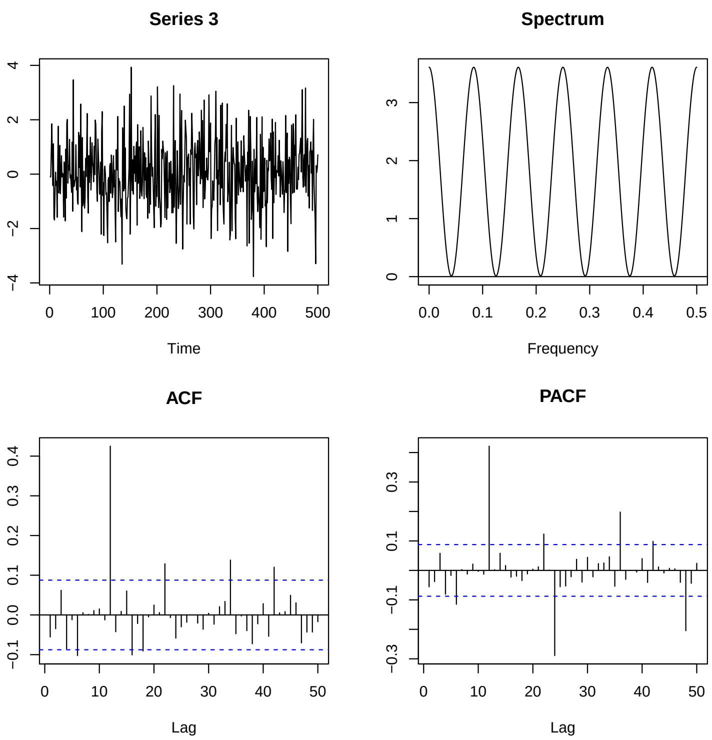

Figure 3: Time series 3 and the corresponding spectral, auto- and partial autocorrelation functions.

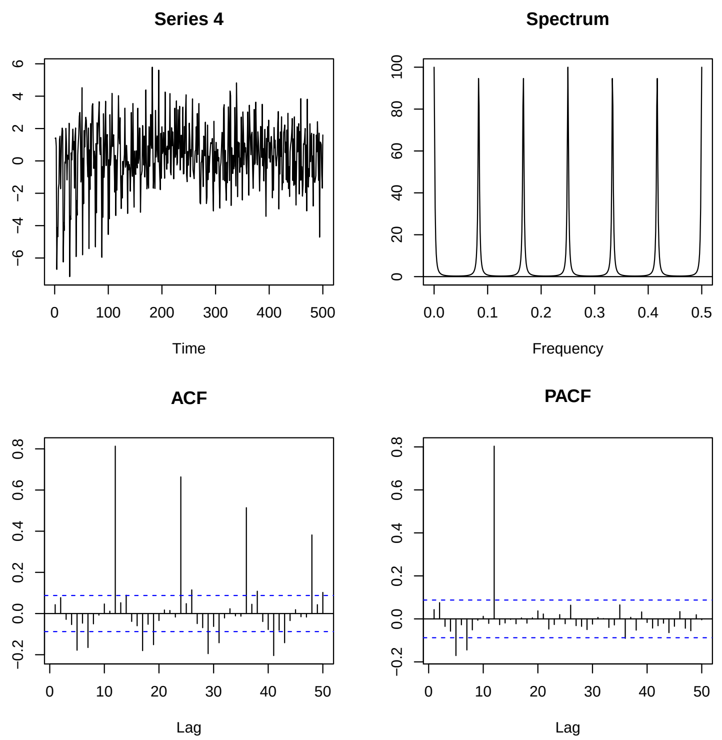

Figure 4: Time series 4 and the corresponding spectral, auto- and partial autocorrelation functions.